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THE LÖWNER-HEINZ INEQUALITY IN BANACH *-ALGEBRAS

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Abstract. We prove the Löwner-Heinz inequality, via the Cordes inequality, for elements $a, b > 0$ of a unital hermitian Banach *-algebra A. Letting p be a real number in the interval (0,1], the former asserts that $a^p \leq b^p$ if $a \leq b$, $a^p < b^p$ if $a < b$, provided that the involution of A is continuous, and the latter that $s(a^p b^p) \leq s(ab)^p$, where $s(x) = r(x^* x)^{1/2}$ and $r(x)$ is the spectral radius of an element x.

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1. The Löwner-Heinz inequality (Heinz [**6**], Löwner [**8**]) asserts that bounded operators A, B on a Hilbert space such that $O \leq A \leq B$ necessarily satisfy $A^p \leq B^p$ for any $p \in (0, 1]$. This matter has received much attention from mathematicians because not only is it so beautiful in itself but also it plays a crucial role in various stages of operator and operator algebra theory.

It is known that some classes of Banach *-algebras have a canonical order and the *power* z^p operates at least to their positive elements with positive spectra. Therefore the question arises: whether the Löwner-Heinz inequality remains true for positive elements of such Banach *-algebras.

However, some care must be taken in view of the fact that the power z^p ($p \in (0, 1)$) may operate only on restricted positive elements of Banach algebras. Actually, Katznelson's square root theorem [**7**] asserts that, if A is a unital abelian semisimple Banach algebra, the complex conjugation z^- operates on A and the square root $z^{1/2}$ operates on any element $a \in A$ with $\sigma(a) \subset [0, +\infty)$, where $\sigma(a)$ is the spectrum of a, then A is isomorphic to $\hat{A}$, the Gelfand representation of A. Hatori [**5**] showed, further, that if A is a Banach function algebra on a compact Hausdorff space X and the power z^p ($p \in (0, 1)$) operates on any element $a \in A$ with $\sigma(a) \subset [0, +\infty)$, then A coincides with the Banach algebra $C(X)$ of all complex-valued continuous functions on X.

We shall give an answer to the question in Theorem 2 below, together with giving in Theorem 1 a generalized version of the Cordes inequality [**2**, Lemma 5.1] (cf. Furuta [**3**]). The method employed is essentially due to Pedersen [**9**].

A Banach *-algebra A is said to be *hermitian* if the spectrum of any self-adjoint element of A consists of real numbers, whereas an $a \in A$ is *self-adjoint* if and only if $a^* = a$. Hermitian Banach *-algebras have their own canonical order. Any C*-algebra is hermitian. Any group algebra of an abelian group, of a compact group, and any measure algebra of a discrete group is known to be hermitian.

We assume in what follows that a Banach *-algebra A is hermitian. We assume also that A is unital in order to simplify the discussion, the unit is denoted by e; while the involution on A may be discontinuous in norm.

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2. We start by recalling the following definitions: $a \geq 0$ means that a is self-adjoint and the spectrum of a consists of non-negative real numbers, while $a > 0$ means $a \geq 0$ and $0 \notin \sigma(a)$; $a \geq b$ means that $a - b \geq 0$, while $a > b$ means $a - b > 0$. a^α for $a \in A$ with $\sigma(a) \subset (0, +\infty)$ means $\exp(\alpha \log a)$, where log is the principal branch of the complex logarithm.

It is known that, if $a, \ b \in A$, then $a, \ b \geq 0$ implies $a + b \geq 0$ ([**1**, Lemma 41.4]), and $a \geq 0, \quad \alpha \geq 0$ implies that $\alpha a \geq 0$. In addition, we have the following facts.

Remark 1. *If $a, \ b \in A$, then $a > 0, \ b \geq 0$ implies that $a + b > 0$.*

Proof. By the assumption there exists an $\lambda > 0$ such that $a - \lambda e \geq 0$ and so $a + b - \lambda e = (a - \lambda e) + b \geq 0$. Hence, $a + b \geq \lambda e$, which implies $a + b > 0$. QED

Remark 2. *If $a, \ b \in A$, then either $0 < a \leq b$ or $0 \leq a < b$ implies $b > 0$.*

Proof. This is immediate from the preceding remark. QED

It is known, by the Shirali-Ford theorem [**11**], that A is necessarily *symmetric*; namely, for any $a \in A$, the spectrum of a^*a consists of non-negative real numbers. (See [**1**, Theorem 41.5] and [**4**], [**10**].)

Let $a \in A$. Define

$$r(a) = \inf ||a^n||^{1/n} \quad \text{and} \quad s(a) = r(a^*a)^{1/2};$$

the former is the *spectral radius* of a. Then we have

$$r(a) \leq s(a)$$

by [**1**, Lemma 41.2]; s is a B^*-semi-norm (in fact a maximal B^*-semi-norm) on A by [**1**, Theorem 41.7, Corollary 41.8]; and so, it is continuous in norm [**1**, Theorem 39.3].

For convenience' sake, we put for real r, and for $\lambda > 0$ such that $\sigma(\lambda a) \subset (0, 1]$,

$$a_n^{(r,\lambda)} = \lambda^{-r}\left(e + \sum_{k=1}^{n} \binom{r}{k}(e - \lambda a)^k\right), \quad (n = 1, 2, \cdots).$$

If a is self-adjoint, then $a_n^{(r,\lambda)}$ is self-adjoint, $a_m^{(r,\lambda)}$ and $a_n^{(r,\lambda)}$ commute, $\{a_n^{(r,\lambda)}\}$ converges to a^r in norm and so, by the spectral mapping theorem, $a_n^{(p,\lambda)} > 0$, for any sufficiently large n, while a^r may not be self-adjoint.

Theorem 1. *Let $a, \ b \in A$. If $a > 0, \ b > 0$ and $p \in (0, 1]$, then*

$$s(a^p b^p) \leq s(ab)^p.$$

Proof. The inequality above is true when $p = 1$. Next, let $\lambda > 0$ be chosen sufficiently small. We put for any integer $n > 0$,

$$a_n = a_n^{(1/2,\lambda)} \quad \text{and} \quad b_n = b_n^{(1/2,\lambda)}.$$

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Then,

$$a_n \longrightarrow a^{1/2}, \quad b_n \longrightarrow b^{1/2} \quad \text{as} \quad n \longrightarrow \infty$$

in norm. Hence

$$s(a_n b_n) \longrightarrow s(a^{1/2} b^{1/2}), \quad s(a_n^2 b_n^2) \longrightarrow s(ab) \quad \text{as} \quad n \longrightarrow \infty.$$

Therefore, since

$$s(a_n b_n) = r((a_n b_n)^*(a_n b_n))^{1/2} = r(b_n a_n^2 b_n)^{1/2} = r(a_n^2 b_n^2)^{1/2} \leq s(a_n^2 b_n^2)^{1/2},$$

it follows that

$$s(a^{1/2} b^{1/2}) \leq s(ab)^{1/2}.$$

Next we assume that for $p, \ q \in (0, 1]$,

$$s(a^p b^p) \leq s(ab)^p \quad \text{and} \quad s(a^q b^q) \leq s(ab)^q.$$

We put, for any integer $n > 0$ sufficiently large,

$$a_n = a_n^{(p/2, \lambda)}, \quad a'_n = a_n^{(q/2, \lambda)}, \quad b_n = b_n^{(p/2, \lambda)}, \quad \text{and} \quad b'_n = b_n^{(q/2, \lambda)}.$$

Then, a_m and a'_n commute, b_m and b'_n commute; also

$$a_n a'_n \longrightarrow a^{(p+q)/2}, \quad \text{and} \quad b_n b'_n \longrightarrow b^{(p+q)/2} \quad \text{as} \quad n \longrightarrow \infty$$

in norm. But we have

$$\begin{aligned} s(a_n a'_n b_n b'_n) &= r((a_n a'_n b_n b'_n)^*(a_n a'_n b_n b'_n))^{1/2} = r(b'_n b_n a'_n a_n^2 a'_n b_n b'_n)^{1/2} = r(b_n^2 a_n^2 a_n'^2 b_n'^2)^{1/2} \\ &\leq s(b_n^2 a_n^2 a_n'^2 b_n'^2)^{1/2} \leq s(b_n^2 a_n^2)^{1/2} s(a_n'^2 b_n'^2)^{1/2} = s(a_n^2 b_n^2)^{1/2} s(a_n'^2 b_n'^2)^{1/2}. \end{aligned}$$

so that

$$s(a^{(p+q)/2} b^{(p+q)/2}) \leq s(a^p b^p)^{1/2} s(a^q b^q)^{1/2}.$$

Therefore,

$$s(a^{(p+q)/2} b^{(p+q)/2}) \leq s(ab)^{(p+q)/2},$$

by the assumption. Thus, according to the norm continuity of s, we know that the inequality in Theorem 1 holds for any $p \in (0, 1]$. QED

3. We assume hereafter that the involution on A is continuous in norm.

LEMMA. *Let $a, \ b \in A$ and $p \in (0, 1]$. If $0 < a \leq b$, then $r(a^p b^{-p}) \leq 1$; if $0 < a < b$, then $r(a^p b^{-p}) < 1$.*

Proof. Assume first that $0 < a \leq b$. Then, by Remark 2 and the hermiticity of A, b is invertible and $0 \leq b^{-1/2} a b^{-1/2} \leq e$. This implies that

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$$s(a^{1/2}b^{-1/2}) = r((a^{1/2}b^{-1/2})^*(a^{1/2}b^{-1/2}))^{1/2} = r(b^{-1/2}ab^{-1/2}) \leq 1.$$

But by the spectral mapping theorem, $\sigma(a^{1/2})$ and $\sigma(b^{-1/2})$ lie in $(0, +\infty)$. Hence,

$$r(a^p b^{-p}) = r(b^{-p/2}a^p b^{-p/2}) = s(a^{p/2}b^{-p/2}) \leq s(a^{1/2}b^{-1/2})^p \leq 1.$$

Assume next that $0 < a < b$. Then, in a similar way we obtain

$$r(a^p b^{-p}) =< 1. \qquad \text{QED}$$

THEOREM 2. *Let* $a, b \in A$, *and* $p \in (0, 1]$. *If* $0 < a \leq b$, *then* $a^p \leq b^p$; *if* $0 < a < b$, *then* $a^p < b^p$.

Proof. Since the involution is continuous in norm, $b^{-p/2}a^p b^{-p/2}$ is self-adjoint and so, by the preceding lemma, $0 < a \leq b$ implies $e - b^{-p/2}a^p b^{-p/2} \geq 0$. Hence we have $a^p \leq b^p$. Again, by the preceding lemma, $0 < a < b$ implies $e - b^{-p/2}a^p b^{-p/2} > 0$. Hence we have $a^p < b^p$. QED

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