LOOPS WITH ADJOINTS

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Introduction. It is shown in **(6)** how to represent certain sets of orthogonal Latin squares as a group together with a set of permutations of the group elements. The correspondence between 3-nets and loops is well known; for example, see **(8)**. We shall consider a loop G together with a certain set of permutations on the elements of G and shall interpret such a structure as an incidence system in which the 3-net of the loop is embedded. Specifically, the permutations or "adjoints" will give rise to lines which may be adjoined to the 3-net of G in the sense of **(3)**. The group of autotopisms of the loop determines a group of automorphisms of its 3-net analogous to collineations in an affine plane. We shall study the problem of extending these incidence preserving mappings to the adjoined lines. By analogy with the study of loops with operators, we shall consider homomorphisms of loops with adjoints and examine geometric consequences. Particular attention will be paid to the case where G has the inverse property and the adjoints are "linear." The special case in which G is an abelian group is of geometric interest in that the corresponding incidence systems include the Veblen-Wedderburn affine planes.

1. Nets, loops, and adjoints. A *net* $\mathfrak{N}$ is a set of undefined objects called "points" and "lines," together with a symmetric incidence relation (point on line, line through point), such that the lines can be partitioned into non-empty, disjoint subsets called "parallel classes" and the following incidence axioms hold:

(i) Any point of $\mathfrak{N}$ lies on exactly one line of each parallel class.

(ii) Any pair of lines from distinct classes have exactly one point in common.

(iii) There are at least three distinct parallel classes.

An *affine plane* is a net which satisfies

(iv) Given two distinct points of $\mathfrak{N}$, there is a unique line containing both of them.

(v) There exists a set of four distinct points of $\mathfrak{N}$, no three of which lie on the same line.

If a net $\mathfrak{N}$ possesses a finite number k of parallel classes, one refers to $\mathfrak{N}$ as a k-net.

Suppose L is a set of points of the net $\mathfrak{N}$ such that if M is any line of $\mathfrak{N}$,

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then L contains exactly one point of M. We say that L may be "adjoined as a line to $\mathfrak{N}$."

Let $(G, +)$ be a loop. By the 3-net associated with G we mean the net $\mathfrak{N}(G)$ whose points are ordered pairs (x, y), x and y in G, and whose three classes of lines are given by $x = c$, $y = c$, and $y = x + c$ where c ranges over G and incidence means satisfaction of the equation.

Let ϵ be the identity map on the loop G. An *adjoint* of G is a permutation σ on G for which there exists a permutation τ on G such that $\epsilon + \tau = \sigma$, where addition of mappings is defined by adding images. The permutation τ is a "complete mapping" in the sense of **(6)** and will be called the *companion* of σ.

THEOREM 1. *A line L can be adjoined to $\mathfrak{N}(G)$ if and only if G possesses an adjoint σ.*

Proof. Suppose σ is an adjoint of G with companion τ. Define L to be the set of points $(x, x\sigma)$, $x \in G$. If $c \in G$ then L contains exactly the point $(c, c\sigma)$ of the line $x = c$, the point $(c\sigma^{-1}, c)$ of the line $y = c$, and the point $(c\tau^{-1}, c\tau^{-1}\sigma)$ of the line $y = x + c$. Thus L may be adjoined to $\mathfrak{N}(G)$.

Conversely, let L be a set of points which is adjoined as a line to $\mathfrak{N}(G)$. If $(a, b) \in L$, define $a\sigma = b$. Since L contains exactly one point from each line $x = c$ and each line $y = c$, we see that σ is a permutation on G. Define the mapping τ of G into G by the equation $a\sigma = a + a\tau$, $a \in G$. For each $c \in G$, the line $y = x + c$ passes through exactly one point $(a, a\sigma)$ of L. Hence τ is a permutation and σ is an adjoint of G.

The incidence system consisting of $\mathfrak{N}(G)$ together with the adjoined lines associated with a set Σ of adjoints of G will be called the *quasinet* $\mathfrak{Q}(G, \Sigma)$.

A set Σ of adjoints of G is said to be *compatible* if, for every distinct pair σ_1, σ_2 in Σ, there is at most one $x \in G$ such that $x\sigma_1 = x\sigma_2$; that is, the corresponding lines of $\mathfrak{Q}(G, \Sigma)$ share at most one point.

2. Adjoints under isotopy. The loops $(G, +)$ and $(G, \oplus)$ defined on the same set G are said to be *isotopic* if there exists an ordered triple (α, γ, β) of permutations of G such that $x\alpha \oplus y\gamma = (x + y)\beta$. We write $(G, +) = (\alpha, \gamma, \beta)(G, \oplus)$. If $\oplus$ is the same operation as $+$ then (α, γ, β) is called an *autotopism* of $(G, +)$. Isotopy is an equivalence relation on the set of all loops and the autotopisms of a loop form a group which contains the automorphism group of G. We refer the reader to **(2)** for a discussion of the algebraic properties of isotopy and autotopy.

A *homomorphism* of a net $\mathfrak{N}$ onto a net $\mathfrak{N}'$ is a mapping of points onto points and lines onto lines which preserves incidence and parallelism and is one-one on classes of lines. If the homomorphism is one-one on points and lines, it is called an *isomorphism*.

The proofs of the following two well-known theorems may essentially be found in **(8)**.

THEOREM 2. *The mapping*

$$
\begin{aligned}
(x, y) &\to (x\alpha, y\beta)\\
[x = c] &\to [x = c\alpha]\\
[y = c] &\to [y = c\beta]\\
[y = x + c] &\to [y = x \oplus c\gamma]
\end{aligned}
$$

is an isomorphism of $\mathfrak{N}(G, +)$ *onto* $\mathfrak{N}(G, \oplus)$ *if and only if* $(G, +) = (\alpha, \gamma, \beta)(G, \oplus)$.

THEOREM 3. *Let* $\mathfrak{N}$ *be an arbitrary* 3-*net and let* O *be a point of* $\mathfrak{N}$. *Let the designations* $x = c$, $y = c$, *and* $y = x + c$ *be assigned to the three classes. Then, for some appropriate loop* G, $\mathfrak{N} = \mathfrak{N}(G)$, O *is the point* $(0, 0)$, *and the three classes are as designated. Moreover, if a different point* O' *is chosen and the class designation remains the same, then* $\mathfrak{N} = \mathfrak{N}(G')$ *for some* G' *isotopic to* G.

It is clear that isomorphic nets have corresponding adjoined lines; thus isotopic loops have corresponding adjoints. The correspondence is given in the proof of

THEOREM 4. *Let* $(G, +)$ *be a loop with an adjoint* σ. *Suppose* $(G, +) = (\alpha, \gamma, \beta)(G, \oplus)$. *Then* $(G, \oplus)$ *possesses a unique adjoint* σ' *so that the isomorphism of Theorem* 2 *can be extended to an incidence preserving mapping of the associated adjoined lines.*

Proof. The set of images of the points $(x, x\sigma)$, $x \in G$ is a line adjoined to $\mathfrak{N}(G, \oplus)$ if and only if $(x\alpha, x\sigma\beta) = (x\alpha, x\alpha\sigma')$ for each x where σ' is an adjoint of $(G, \oplus)$. If the condition holds,then $\sigma' = \alpha^{-1}\sigma\beta$ and we show that this σ' is an adjoint for $(G, \oplus)$. Let σ have companion τ. Then, for all $x \in G$,

$$
\begin{aligned}
x(\epsilon \oplus \alpha^{-1}\tau\gamma) &= x\alpha^{-1}\alpha(\epsilon \oplus \alpha^{-1}\tau\gamma) = (x\alpha^{-1})\alpha \oplus (x\alpha^{-1}\tau)\gamma = \\
&\qquad (x\alpha^{-1} + x\alpha^{-1}\tau)\beta = x\alpha^{-1}(\epsilon + \tau)\beta = x\alpha^{-1}\sigma\beta.
\end{aligned}
$$

Thus the companion of $\alpha^{-1}\sigma\beta$ is $\alpha^{-1}\tau\gamma$.

If $V = (\alpha, \gamma, \beta)$ is an autotopism of a loop G and Σ is a set of adjoints of G, then we say that V is *extensible* relative to Σ if $\alpha^{-1}\Sigma\beta = \Sigma$. From the proof of Theorem 4, we see that the extensible autotopisms are exactly those for which the automorphism of $\mathfrak{N}(G)$ given by Theorem 2 induces a line onto line, incidence preserving mapping of $\mathfrak{L}(G, \Sigma)$.

An example of an extensible autotopism is furnished by a projective plane with a collineation which leaves fixed every point of some line $\mathscr{L}$. Let $\mathscr{L}$ be used as the line at infinity and construct a co-ordinate system as in (4). Take G to be the additive loop of the ternary and let Σ be the set of mappings $x \to x \,.\, m \circ b$ where m assumes some set of values exclusive of $0, 1, \infty$, and where b takes on all values from the ternary for each m. The collineation then induces an automorphism of $\mathfrak{N}(G)$ corresponding to an autotopism of G which is extensible relative to Σ by virtue of the fact that adjoined lines are mapped into adjoined lines.

3. Homomorphisms. Let H be a normal subloop of G and let η be the natural homomorphism of G onto G/H. Then we may define a homomorphism of $\mathfrak{N}(G)$ onto $\mathfrak{N}(G/H)$ by a mapping of the same form as that used in Theorem 2 with $\alpha = \beta = \gamma = \eta$ and $\oplus$ taken as $+$, the operation in G/H. Furthermore, it follows easily from Theorem 3 that every homomorphism of a 3-net onto a 3-net can be viewed in this way for appropriate G and H.

Suppose σ is an adjoint of G with companion τ. A normal subloop H of G is a σ-*subloop* provided any one of the following implies the other two: (i) $x \equiv y \bmod H$, (ii) $x\sigma \equiv y\sigma \bmod H$, (iii) $x\tau \equiv y\tau \bmod H$. If Σ is a set of adjoints, then H is a normal Σ-subloop if H is a normal σ-subloop for each $\sigma \in \Sigma$. This definition states that σ^* defined by $(x + H)\sigma^* = x\sigma + H$ is a permutation of G/H. The same statement applies to τ^* and, moreover, $(x + H)(\epsilon^* + \tau^*) = (x + H) + (x + H)\tau^* = (x + H) + (x\tau + H) = (x + x\tau) + H = x\sigma + H = (x + H)\sigma^*$ for all x and thus σ^* is an adjoint of G/H. Furthermore, the point $(x, x\sigma)$ maps into $(x + H, (x + H)\sigma^*)$ so the adjoined line of $\mathfrak{N}(G)$ defined by σ maps onto the adjoined line of $\mathfrak{N}(G/H)$ defined by σ^*.

Suppose, conversely, that H is normal in G, that σ is an adjoint of G, and that the images $(x + H, x\sigma + H)$ form a line adjoined to $\mathfrak{N}(G/H)$. Then the mapping $x + H \to x\sigma + H$ is an adjoint of G/H. Hence, $x + H = y + H$ if and only if $x\sigma + H = y\sigma + H$. Also, if $x' + H$ is the unique solution of $(x + H) + (x' + H) = x\sigma + H$, then $x + H \to x' + H$ is a permutation of G/H so $x + H = y + H$ if and only if $x' + H = y' + H$. But $x\sigma + H = (x + x') + H = (x + x\tau) + H$ so $x' + H = x\tau + H$. Therefore, H is a normal σ-subloop of G.

4. Linear adjoints and extensibility. If $a \in G$, the permutation $\rho(a)$ is defined by $x\rho(a) = x + a$. A permutation of G is *linear* if it has the form $\sigma = \delta\rho(a)$ where δ is an automorphism of G. We say that σ is *strongly linear* if a is in the associator (see **2**) of G. An adjoint σ of G is linear (strongly linear) if *both* σ and its companion are linear (strongly linear) as permutations. If a linear adjoint σ has $a = 0$, σ is an *automorphism adjoint*.

LEMMA 1. (i) *If $\sigma = \delta\rho(a)$ is a linear adjoint with companion $\gamma\rho(b)$ then $a = b$. If σ is strongly linear, then δ is a strongly linear adjoint with companion γ.*

(ii) *If σ is any adjoint on G with companion τ and a is in the associator of G then $\sigma\rho(a)$ is an adjoint with companion $\tau\rho(a)$.*

Proof. (i) $0(\epsilon + \gamma\rho(b)) = 0\delta\rho(a)$ gives $a = b$. Under strong linearity, $x\delta + a = x + (x\gamma + a) = (x + x\gamma) + a$ for every $x \in G$ and thus $\delta = \epsilon + \gamma$.

(ii) $x(\epsilon + \tau\rho(a)) = x + (x\tau + a) = (x + x\tau) + a = x\sigma\rho(a)$.

A set Σ of strongly linear adjoints of G will be called *complete* if, for every automorphism δ such that $\delta\rho(a) \in \Sigma$, $\delta\rho(b) \in \Sigma$ for every b in the associator of G. Lemma 1 guarantees that $\delta\rho(b)$ is an adjoint.

Bruck (**2**, ch. II, § 4) has studied three types of autotopisms of a com-

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mutative loop G with the inverse property and has shown that these autotopisms generate the autotopism group of G. The following theorem shows that certain subclasses of types (1) and (2) and all autotopisms of type (3) are extensible relative to a complete set of strongly linear adjoints.

THEOREM 5. *Suppose G is a commutative loop with the inverse property and Σ is a complete set of strongly linear adjoints of G.*

(i) *If $U = (\alpha, \alpha, \alpha)$ where α is an automorphism of G, then U is extensible relative to Σ if and only if $\alpha^{-1}\Delta\alpha = \Delta$ where Δ is the set of automorphisms in Σ.*

(ii) *If $V = (\rho(a), \rho(a), \rho(a)^2)$ where a is in the associator of G, then V and all the autotopisms obtainable from V by Bruck's Lemma 4A are extensible relative to Σ.*

(iii) *If $W = (\epsilon, \rho(a), \rho(a))$ where ϵ is the identity and a is in the associator of G, then W and all the autotopisms obtainable from W by Bruck's Lemma 4A are extensible relative to Σ.*

Proof. (i) Suppose U is extensible. Let $\delta\rho(b) \in \Sigma$. Then, for all $x \in G$, $x\alpha^{-1}\delta\rho(b)\alpha = x\alpha^{-1}\delta\alpha + b\alpha = x\gamma + c$ for some $\gamma \in \Delta$, c an associator element. If $x = 0$, $b\alpha = c$ and $\alpha^{-1}\delta\alpha = \gamma$. Similarly $\alpha\Delta\alpha^{-1} \subseteq \Delta$.

Conversely, assume $\alpha^{-1}\Delta\alpha = \Delta$. Then, if $x \in G$, $x\alpha^{-1}\delta\rho(a)\alpha = x(\alpha^{-1}\delta\alpha)\rho(a\alpha)$ and $x\alpha\delta\rho(a)\alpha^{-1} = x(\alpha\delta\alpha^{-1})\rho(a\alpha^{-1})$. Since the associator is a characteristic subloop of G, U is extensible.

(ii) Consider $x\rho(a)^{-1}\delta\rho(b)\rho(a)^2 = [(x - a)\delta + b] + 2a = x\delta\rho(c)$ where c is in the associator. Similarly, $\rho(a)\delta\rho(b)\rho(a)^{-2} \in \Sigma$.

Extensibility for the autotopisms obtained by using Bruck's Lemma 4A may be verified with a certain amount of similar computation.

(iii) We compute $x\epsilon^{-1}\delta\rho(b)\rho(a) = (x\delta + b) + a = x\delta\rho(c)$ and $x\epsilon\delta\rho(b)\rho(a)^{-1} = (x\delta + b) - a = x\delta\rho(d)$ where c and d are in the associator.

Again, it is straightforward to verify that the "derived" autotopisms are extensible.

THEOREM 6. *Let G be an abelian group and let Σ be a complete set of (strongly) linear adjoints of G. If Δ is the set of automorphisms in Σ, let $\delta_1 - \delta_2$ be a permutation (and hence an automorphism) of G for every pair $\delta_1, \delta_2 \in \Delta$, $\delta_1 \neq \delta_2$. Then, the quasinet $\mathfrak{Q}(G, \Sigma)$ is a net each of whose parallel classes, besides those of $\mathfrak{N}(G)$, consists of the set of adjoined lines given by the adjoints $\delta\rho(c)$ where δ is fixed and c ranges over G. Moreover, the automorphisms of $\mathfrak{Q}(G, \Sigma)$ are exactly given by $(x, y) \to (x\alpha + r, y\alpha + s)$ where α is an automorphism of G in the centralizer of Δ and r, s are in G.*

Proof. The point $(a, b) \in \mathfrak{Q}(G, \Sigma)$ is on exactly the line determined by $\delta\rho(c)$, $c = b - a\delta$, in the class corresponding to δ. Any line from an adjoined class contains exactly one point in common with each line of $\mathfrak{N}(G)$ and thus we need only consider lines determined by $\delta_1\rho(a)$, $\delta_2\rho(b)$ where $\delta_1 \neq \delta_2$. It is easy to see that these lines share exactly the point (x, y) where $x = (b - a)(\delta_1 - \delta_2)^{-1}$, $y = x\delta_1\rho(a) = x\delta_2\rho(b)$. Hence $\mathfrak{Q}(G, \Sigma)$ is a net.

Every automorphism of the net $\mathfrak{Q}(G, \Sigma)$ induces an automorphism of $\mathfrak{N}(G)$ which, by virtue of Theorem 2, corresponds to an autotopism of G. By Bruck's Theorem 4D **(2**, ch. II, § 4**)**, this autotopism is of the form UVW where U, V, and W are of types (i), (ii), (iii) respectively, as described in Theorem 5. It is easy to check that V and W and hence VW correspond to "translations": $(x, y) \to (x + r, y + s)$. Thus, the autotopism UVW corresponds to $(x, y) \to (x\alpha + r, y\alpha + s)$ where α is an automorphism of G. Moreover, UVW is extensible in that it gives an automorphism of $\mathfrak{Q}(G, \Sigma)$. Furthermore, by Theorem 5 (ii), (iii), VW is extensible and thus $\rho(r)\Sigma\rho(s)^{-1} = \Sigma$. Therefore,

$$\alpha\Sigma\alpha^{-1} = \alpha[\rho(r)\ \Sigma\rho(s)^{-1}]\alpha^{-1} = [\alpha\rho(r)]\Sigma[\alpha\rho(s)]^{-1} = \Sigma,$$

showing that (α, α, α) is extensible and giving, by Theorem 5 (i), $\alpha\Delta\alpha^{-1} = \Delta$. But, even more, the translations, the automorphisms of $\mathfrak{Q}(G, \Sigma)$, and hence also the mappings $(x, y) \to (x\alpha, y\alpha)$ preserve parallel class. If we note the proof of Theorem 5 (i), we see that $\alpha^{-1}\delta\alpha = \delta$ for each $\delta \in \Delta$.

To prove the converse, suppose we have a mapping of $\mathfrak{Q}(G, \Sigma)$ as described in the statement of the theorem. The translations are automorphisms of $\mathfrak{N}(G)$ and the corresponding autotopism is a product of autotopisms of types (ii) and (iii):

$$(\rho(r), \rho(s - r), \rho(s)) = (\rho(r), \rho(r), \rho(r)^2)\ (\epsilon, \rho(s - 2r), \rho(s - 2r)).$$

Thus the translations correspond exactly to such products and, by Theorem 5 (ii), (iii), are extensible. In fact, they preserve parallel class. Also, by the proof of Theorem 5 (i), (α, α, α) is extensible and preserves parallel class in the extended net. Thus our mapping corresponds to the autotopism $(\alpha\rho(r), \alpha\rho(s - r), \alpha\rho(s))$ which is extensible and preserves parallel class and therefore gives an automorphism of $\mathfrak{Q}(G, \Sigma)$.

The translations $(x, y) \to (x + r, y + s)$ are transitive on the points of $\mathfrak{Q}(G, \Sigma)$ and we see that if $\mathfrak{Q}(G, \Sigma)$ is an affine plane, it is Veblen-Wedderburn **(8)**.

5. Linear adjoints and homomorphisms.

LEMMA 2. *Let G be a loop and $\sigma = \delta\rho(c)$ be a linear adjoint with companion $\gamma\rho(c)$. Then a normal subloop H of G is a σ-subloop if and only if $H\delta = H\gamma = H$.*

Proof. Assume $H\delta = H$. Then $x\delta + c = (y\delta + c) + h$ for $h \in H$ if and only if $x\delta + c = (y\delta + h') + c$ for $h' \in H$ if and only if $x\delta = y\delta + h'$. But $y\delta + h' = y\delta + h''\delta = (y + h'')\delta$ for $h'' \in H$. Thus $x\delta\rho(c) = y\delta\rho(c) + h$ if and only if $x = y + h''$. Similarly, $H\gamma = H$ implies $x \equiv y \bmod H$ if and only if $x\gamma\rho(c) \equiv y\gamma\rho(c) \bmod H$.

Now assume that H is a normal σ-subloop. Then $h \equiv 0 \bmod H$ if and only if $h\delta\rho(c) \equiv 0\delta\rho(c) \bmod H$ if and only if $h\delta + c \equiv c \bmod H$ if and only if $h\delta \equiv 0 \bmod H$; that is, $h \in H$ if and only if $h\delta \in H$. Similarly, $H\gamma = H$.

We remark that $H\delta \subseteq H$ if and only if $H\gamma \subseteq H$; for, if $h \in H$, $h\delta + c = h + (h\gamma + c) = (h' + h\gamma) + c$ for some $h' \in H$. Hence $h\delta = h' + h\gamma$ and thus

$h\delta \in H$ if and only if $h\gamma \in H$. Therefore, if G is an abelian group with descending chain condition or if G is a finite loop, we need only assume either $H\delta \subseteq H$ or $H\gamma \subseteq H$ in order to prove that a normal subloop H is a normal σ-subloop.

In any loop G, multiplication of mappings of G into G is left distributive over addition of mappings. If G has the inverse property, one readily verifies that the mappings of G into G form a loop (under addition) with the inverse property where $-\alpha$ is defined by $x(-\alpha) = -(x\alpha)$. Therefore, $-(\alpha + \beta) = (-\beta) + (-\alpha)$. Also, $-\alpha$ is one-one (onto) if and only if α is one-one (onto).

LEMMA 3. *Let G be a loop with the inverse property and let $\sigma_1 = \delta_1\rho(c_1)$ and $\sigma_2 = \delta_2\rho(c_2)$ be strongly linear adjoints on G where $\delta_1 - \delta_2$ is one-one on G. Then*

(i) *σ_1 and σ_2 are compatible,*

(ii) *if H is a normal $\{\sigma_1, \sigma_2\}$-subloop, the induced adjoints $\sigma^*{}_1$ and $\sigma^*{}_2$ are strongly linear on G/H. If $H \subseteq H(\delta_1 - \delta_2)$, then $\delta^*{}_1$ and $\delta^*{}_2$ are compatible.*

Proof. (i) Assume $x\delta_1 + c_1 = x\delta_2 + c_2$. Then $-(x\delta_2) + (x\delta_1 + c_1) = c_2$ and $-(x\delta_2) + x\delta_1 = x(-\delta_2 + \delta_1) = c_2 - c_1$. The solution is unique if $-\delta_2 + \delta_1$ is one-one. Since $\delta_1 - \delta_2$ is one-one so also is $-(\delta_1 - \delta_2) = \delta_2 - \delta_1$. Define the mapping θ on G by $x\theta = -x$. We see that θ is a permutation of G and that if α is an automorphism of G, then $\theta\alpha = -\alpha$. Therefore,

$$\theta(\delta_2 - \delta_1) = \theta\delta_2 + \theta(-\delta_1) = -\delta_2 + \theta(\theta\delta_1) = -\delta_2 + (\theta\theta)\delta_1 = -\delta_2 + \delta_1$$

is one-one. We note that if $\delta_1 - \delta_2$ is a permutation then $x\delta_1 + c_1 = x\delta_2 + c_2$ possesses a (unique) solution.

(ii) For $i = 1, 2$, $(x + H)\sigma^*{}_i = x\sigma_i + H = (x\delta_i + c_i) + \text{H} = (x\delta_i + H) + (c_i + H)$. Clearly, $c_i + H$ is in the associator of G/H and, by Lemma 2, $\delta^*{}_i : x + H \to x\delta_i + H$ is an automorphism of G/H. In the same way, the companion of $\sigma^*{}_i$ is a strongly linear permutation and thus $\sigma^*{}_i$ is a strongly linear adjoint.

Next, assume $(x + H)\delta^*{}_1 = (x + H)\delta^*{}_2$. Then, for some $h \in H$, $x\delta_1 = h + x\delta_2$ and thus $x\delta_1 - x\delta_2 = x(\delta_1 - \delta_2) = h = h'(\delta_1 - \delta_2)$ for some $h' \in H$. Since $\delta_1 - \delta_2$ is one-one, $x \in H$ and hence $0 + H$ is the unique solution of $X\delta^*{}_1 = X\delta^*{}_2$.

THEOREM 7. *Let G be an abelian group satisfying the descending chain condition. If Σ is a complete set of compatible linear adjoints and H is a Σ-subgroup, then the quasinets $\mathfrak{Q}(G, \Sigma)$* and *$\mathfrak{Q}(G/H, \Sigma^*)$ are nets in the sense described in Theorem 6.*

Proof. Let Δ be the set of automorphisms in Σ and suppose δ_1 and δ_2 are distinct elements of Δ. Now $\delta_1 - \delta_2$ is an endomorphism of G and $x(\delta_1 - \delta_2) = 0$ implies $x\delta_1 = x\delta_2$. Since δ_1 and δ_2 are compatible and since $0\delta_1 = 0\delta_2$ we see that $x = 0$; that is, the kernel of $\delta_1 - \delta_2$ is 0 and $\delta_1 - \delta_2$ is one-one and has a right inverse. By the descending chain condition, $G(\delta_1 - \delta_2)^{r+1} = G(\delta_1 - \delta_2)^r$ for some r and thus $G(\delta_1 - \delta_2) = G$ showing, by Theorem 6, that $\mathfrak{Q}(G, \Sigma)$ is a net.

The set Σ^* of induced adjoints of G/H is certainly complete. Let δ^*_1 and δ^*_2 be distinct elements of Δ^*. Then $\delta^*_1 - \delta^*_2$ is an endomorphism of G/H and $(x + H)(\delta^*_1 - \delta^*_2) = H$ implies $(x + H)\delta^*_1 - (x + H)\delta^*_2 = (x\delta_1 - x\delta_2) + H = x(\delta_1 - \delta_2) + H = H$. Thus $x(\delta_1 - \delta_2) \in H$. But $H(\delta_1 - \delta_2) \subseteq H$ because H is a Σ-subgroup and, by the descending chain condition, there is an integer s such that $H(\delta_1 - \delta_2)^{s+1} = H(\delta_1 - \delta_2)^s$. As above, $H(\delta_1 - \delta_2) = H$ giving $x \in H$ and showing that $\delta^*_1 - \delta^*_2$ is an isomorphism. Since G/H satisfies the descending chain condition, $\mathfrak{Q}(G/H, \Sigma^*)$ is a net as before.

If Σ is a set of strongly linear adjoints on a loop G, we have seen that the set Δ of automorphisms in Σ plays a special role in identifying normal Σ-subloops and in questions of compatibility and extensibility. We shall focus attention now on Δ and the normal Δ-subloops, realizing that, for every $\delta \in \Delta$ and every a in the associator of G, there is an adjoint $\delta\rho(a)$ whose normal subloops are exactly the normal δ-subloops.

LEMMA 4. *Let G be a finite loop of order $n > 1$ and suppose Δ is a set of automorphism adjoints such that every pair in Δ is compatible. Then the number of elements in Δ is not more than $n - 2$.*

Proof. The lines $x = 0$, $y = 0$, and $y = x$ of $\mathfrak{N}(G)$ have exactly the point $(0, 0)$ in common. Further, $(0, 0)$ is the point shared by each of these three lines and the adjoined line determined by $\delta \in \Delta$. Moreover, by compatibility, every pair of adjoined lines shares exactly this point. Let d be the number of elements in Δ. $\mathfrak{N}(G)$ has n^2 points and we count the points on the above $d + 3$ lines:

$$(d + 3)(n - 1) + 1 \leqslant n^2.$$

Rejecting $n = 1$, we have $d \leqslant n - 2$.

Definition. A *compatible triple* (G, Δ, H) consists of a loop G with the inverse property, a set Δ of automorphism adjoints on G and a normal Δ-subloop H of G such that, for every distinct pair δ_1 and δ_2 in Δ, $\delta_1 - \delta_2$ is one-one on G and is a permutation on H. The *degree* of Δ (deg Δ) is the number of elements in Δ. $(G : H)$ denotes the index of H in G.

THEOREM 8. *If (G, Δ, H) is a compatible triple where Δ has finite degree, then either $(G : H) = 1$ or $(G : H) \geqslant \deg \Delta + 2$.*

Proof. Suppose $(G : H) > 1$. By Lemma 3 (ii), G/H is a loop with automorphism adjoints Δ^* and every pair δ^*_1, δ^*_2 from Δ^* is compatible. Moreover, $\deg \Delta = \deg \Delta^*$ for suppose $(x + H)\delta^*_1 = (x + H)\delta^*_2$ for all x. Then, for each $x \in G$, there is an $h \in H$ such that $x\delta_1 = h + x\delta_2$ or $x(\delta_1 - \delta_2) = h$. But $\delta_1 - \delta_2$ is a permutation on H and hence $x \in H$, contradicting our assumption that $G \neq H$. If $(G : H)$ is finite, apply Lemma 4 to the loop G/H with adjoints Δ^* to obtain $\deg \Delta = \deg \Delta^* \leqslant (G : H) - 2$.

COROLLARY 1. *If the inverse property loop G contains a characteristic normal subloop H of index 2 then G has no automorphism adjoints.*

Proof. If δ is an automorphism adjoint, take $\Delta = \delta$ so deg $\Delta = 1$. (G, Δ, H) is a compatible triple and thus $2 \geqslant 1 + 2$, a contradiction.

Example. The symmetric group on n symbols has no automorphism adjoints.

COROLLARY 2. *Let G be a finite loop with the inverse property and let E be a characteristic normal subloop of G. Then, if (G, Δ, H) is a compatible triple for any H,* deg $\Delta \leqslant (G : E) - 2$.

Proof. (G, Δ, E) is a compatible triple because E is a Δ-subloop since it is characteristic and, by finiteness, $\delta_1 - \delta_2$ is a permutation on E for every distinct pair from Δ.

6. Irreducible sets of linear adjoints.

Definition. Let G be a loop with a set Σ of adjoints. We say that Σ is *irreducible* if G has no proper normal Σ-subloops.

THEOREM 9. *Let G be a loop with a set Σ of adjoints so that the quasinet $\mathfrak{Q}(G, \Sigma)$ is an affine plane. Then Σ is irreducible.*

Proof. Given $x \neq 0$, $y \neq 0$, $x \neq y$ in G, there is a unique line of the plane through $(0, 0)$ and (x, y). If the line is in $\mathfrak{N}(G)$, either $x = 0$, $y = 0$, or $y = x$ but these are excluded. Therefore, there exists $\sigma \in \Sigma$ such that $0\sigma = 0$ and $x\sigma = y$. Now suppose H is a proper normal Σ-subloop of G. Choose $x = h \neq 0$ in H and $y \notin H$. Then $h \equiv 0 \bmod H$ implies $h\sigma \equiv 0\sigma = 0 \bmod H$ and thus $h\sigma = y$ is in H, a contradiction.

LEMMA 5. *Let G be a finite loop with an irreducible set Σ of strongly linear adjoints. Let Δ be the set of automorphisms belonging to the adjoints of Σ and let Ω be the centralizer of Δ in the semigroup of endomorphisms of G. Then the non-zero elements of Ω are automorphisms. If G is an abelian group, the finiteness restriction can be dropped.*

Proof. Let K be the kernel of $\omega \in \Omega$. Then $k \in K$, $\delta \in \Delta$ implies $(k\delta)\omega = (k\omega)\delta = 0\delta = 0$ and thus $K\delta \subseteq K$. But ω commutes also with δ^{-1} giving $K\delta = K$. Now $\epsilon + \gamma = \delta$ and since ω is an endomorphism, multiplication by ω is distributive on both sides over mapping addition. Thus, $\omega + \gamma\omega = \delta\omega$ and $\omega + \omega\gamma = \omega\delta$ showing $\omega\gamma = \gamma\omega$. As before, $K\gamma = K$. By Lemma 2 and irreducibility, $K = G$ or $K = 0$ and hence ω is the zero endomorphism or is one-one on G. In the latter case, since G is finite, ω is an automorphism. If G is an infinite abelian group, $G\omega = (G\delta)\omega = (G\omega)\delta$ and $G\omega = (G\gamma)\omega = (G\omega)\gamma$ showing that $G\omega$ is a (normal) Σ-subloop. Therefore, either $G\omega = 0$, in which case $G = 0$, or $G\omega = G$. In either case ω is an automorphism of G.

THEOREM 10. *Let G be a finite loop or an abelian group with an irreducible complete set Σ of strongly linear adjoints whose subset of automorphisms is Δ. Let Ω be the centralizer of Δ in the endomorphisms of G. Then the autotopism*

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(ω, ω, ω) with $\omega \in \Omega$, ω not zero, is associated with an automorphism of $\mathfrak{N}(G)$ which leaves fixed every line in $\mathfrak{Q}(G, \Sigma)$ through the point $(0, 0)$. Conversely, every such automorphism of $\mathfrak{N}(G)$ is paired with an autotopism of the type described. Moreover, these autotopisms are extensible relative to Σ.

Proof. The automorphism associated with (ω, ω, ω) is $(x, y) \to (x\omega, y\omega)$. This obviously fixes the lines $x = 0$, $y = 0$, and $y = x$ of $\mathfrak{N}(G)$. If $\delta\rho(a) \in \Sigma$ determines an adjoined line through the origin, evidently $a = 0$ and $(x, x\delta) \to (x\omega, x\delta\omega) = (x\omega, (x\omega)\delta)$. Thus the line determined by δ maps into itself.

Conversely, let (α, γ, β) give an automorphism which fixes lines through $(0, 0)$. Then, in particular, $y = x$ is fixed and hence $(x, x) \to (x\alpha, x\beta)$ for all x implies $\alpha = \beta$. Therefore, $(x + y)\alpha = x\alpha + y\gamma$ for every x and y. Since $(0, 0)$ is fixed, $0\alpha = 0$ and we set $x = 0$ giving $y\alpha = y\gamma$. Therefore, the autotopism arises from an automorphism, $\alpha = \beta = \gamma$ of G. If $\delta \in \Delta$, the corresponding line is fixed and hence $(x, x\delta) \to (x\alpha, x\delta\alpha) = (x\alpha, (x\alpha)\delta)$ showing that α is in the centralizer of Δ.

Extensibility follows immediately and, in fact, if G and Σ satisfy the hypothesis of Theorem 6, α gives an automorphism of the quasinet $\mathfrak{Q}(G, \Sigma)$.

Suppose G is an abelian group and Σ, Δ, Ω are as in Theorem 10. Then Ω is a ring and, in fact, a division ring by Lemma 5. Therefore, we may regard G as a vector space over Ω and we note that the elements of Ω different from the zero and the identity are automorphism adjoints of G. If $\mathfrak{Q}(G, \Sigma)$ is a Veblen-Wedderburn plane as in Theorem 6, one may verify that Ω is André's "Kern" **(1)**.

7. A class of examples. The neofields of Paige **(7)** and their generalizations, the division neorings of Hughes **(5)**, provide a class of examples of loops with adjoints. If $(D, +, \cdot)$ is a division neoring **(**see **5** for definition**)**, take G to be $(D, +)$ and define the mapping R_a on G by $xR_a = xa$ where $a \in D$. Clearly, R_a is a permutation for every $a \neq 0$. Then, every R_a with $a \neq 0, a \neq 1$ is an automorphism adjoint on G because $(x + y)R_c = xR_c + yR_c$ for every $c \in D$ and $x(\epsilon + R_b) = x + xb = x(1 + b) = xR_a$ where $b \neq 0$ is the unique solution of $1 + b = a$. If E is any subdivision neoring for which $(E, +)$ is normal in $(D, +)$, then $(E, +)$ is a normal Δ-subloop where Δ consists of the mappings R_e, $e \in E$, $e \neq 0$, $e \neq 1$. Moreover, if $(D, +)$ has the inverse property, then $((D, +), \Delta, (E, +))$ is a compatible triple.

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