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HOLONOMY AND BASIC COHOMOLOGY OF FOLIATIONS

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Abstract

In this paper, we consider the relationship between the cohomologies of the basic differential forms and the transverse holonomy groupoid of a foliation. Applications to minimal models are given.

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Introduction

The minimal model of a foliation $\mathscr{F}$ on a manifold M is, in some sense, the algebraic analogue of the homotopy fibration $M \to M/\mathscr{F}$ onto the leaf space which is obtained by identifying each leaf of $\mathscr{F}$ to a point. More precisely, it is the minimal model of the inclusion of the sub-differential graded algebra of basic differential forms, $\Omega_B(\mathscr{F})$, into the de Rham algebra $\Omega_{DR}(M)$ of M [15]. In this set-up, it is of considerable interest to have a geometric interpretation for the fibre (also called the *relative minimal model* of the foliation) of the minimal model. For certain foliations, for example if M is compact and $\mathscr{F}$ is Riemannian, it is known that there is a locally trivial fibration $\tilde{L} \to M \to B\Gamma$, where $B\Gamma$ denotes the classifying space of the transverse holonomy groupoid of $\mathscr{F}$ and $\tilde{L}$ is the common holonomy covering of the leaves of $\mathscr{F}$. A study of the relationship between $\Omega_B(\mathscr{F})$ and $B\Gamma$ thus allows us to compare the fibre of the minimal model with $\tilde{L}$. This is the motivation for our work. It is also interesting to note that the relationship between $\Omega_B(\mathscr{F})$ and $B\Gamma$ ties the basic cohomology into considerations of characteristic classes and dual homotopy invariants. Our previous paper [16] fits into this scheme.

All our objects are in the C^∞ category. We refer to [9] for background to Sections 2,

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3 and 5; to [12] for background to Section 1; and to [15] for background to Section 6.

1. Basic cohomology

Let $\mathscr{F}$ denote a non-singular foliation of dimension p and codimension q on a connected manifold M without boundary. We recall that a differential form ω is said to be *basic* if

$$i(X)\omega = L_X\omega = 0$$

whenever X is a vector field tangent to the leaves of $\mathscr{F}$. Here, i and L denote the interior product and Lie derivative respectively. The set of all basic differential forms constitutes a sub-differential graded algebra, denoted by $\Omega_B(\mathscr{F})$, of the de Rham algebra $\Omega_{DR}(M)$ of M. The cohomology of $\Omega_B(\mathscr{F})$ is denoted by $H_B(\mathscr{F})$ and is called the basic cohomology of $\mathscr{F}$.

2. Cohomology of the transverse holonomy groupoid

The transverse holonomy semigroup of $\mathscr{F}$ is generated by the identification or transition maps on the non-empty intersections of local Frobenius coordinate charts, which can be considered as local diffeomorphisms of local transversals. The germs of these local diffeomorphisms generate a well-defined topological groupoid called the *transverse holonomy groupoid* of $\mathscr{F}$, denoted by Γ. The identities of Γ are given by the collection of local transversals, denoted by U. Two maps, $\alpha, \beta : \Gamma \to U$, are also defined and are called the source and target maps, respectively.

The following result is due to Winkelnkemper [17] :

THEOREM 2.1. *Γ admits a separable, locally q-dimensional Euclidean topology as well as a differentiable structure. If the holonomy of $\mathscr{F}$ is locally determined, such as for isometric or real analytic groupoids, then Γ is Hausdorff and is thus a q-dimensional paracompact differentiable manifold.*

A Γ-*sheaf* $\mathscr{A}$ is a sheaf on U which has a continuous Γ-action with respect to α and β, that is, for all $\xi \in \mathscr{A}_{\alpha(\gamma)}$, there is a continuous map defined by

$$(\xi, \gamma) \mapsto \xi \cdot \gamma \in \mathscr{A}_{\beta(\gamma)}.$$

The cohomology of Γ with coefficients in a Γ-sheaf $\mathscr{A}$ is defined to be the cohomology of the double complex $C^{r,s} = C^s(\Gamma; C^r(\mathscr{A}))$ with the differentials $\delta : C^{r,s} \to C^{r,s+1}$

defined by

$$(\delta f)(\gamma_0, \dots, \gamma_s) = \gamma_0 f(\gamma_1, \dots, \gamma_s) - \sum_{i=0}^{s-1}(-1)^i f(\gamma_0, \dots, \gamma_i\gamma_{i+1}, \dots, \gamma_s) + (-1)^s f(\gamma_0, \dots, \gamma_{s-1}),$$

and $\partial : C^{r,s} \to C^{r+1,s}$ where

$$0 \to \mathscr{A} \to C^0(\mathscr{A}) \xrightarrow{\partial} C^1(\mathscr{A}) \xrightarrow{\partial} C^2(\mathscr{A}) \cdots$$

is an injective resolution of $\mathscr{A}$ by flabby Γ-sheaves.

3. Resolution using differential forms

Let $\underline{\mathbb{R}}$ denote the constant trivial Γ-sheaf with stalk $\mathbb{R}$ and consider its resolution by the Γ-sheaves of germs of differential forms on U:

$$0 \to \underline{\mathbb{R}} \to \underline{\Omega}^0(U) \xrightarrow{d} \underline{\Omega}^1(U) \xrightarrow{d} \cdots$$

where d is induced by the exterior derivative. Let $C^s(\Gamma; \underline{\Omega}^r(U))$, $r, s \geq 0$, denote the double complex with the differentials d and δ which is given in Section 2.

THEOREM 3.1. *The double complex $C^s(\Gamma; \underline{\Omega}^r(U))$ gives rise to a convergent third quadrant spectral sequence with $E_2^{r,0} = H_B^r(\mathscr{F})$.*

PROOF. Using the standard filtration on a double complex, we get a spectral sequence with

$$E_1^{r,0} = H^0(\Gamma; \underline{\Omega}^r(U)) = \Omega^r(U)^\Gamma = \Omega_B^r(\mathscr{F})$$

since the differential forms on the transversals which are invariant under holonomy are exactly basic. It follows that $E_2^{r,0} = H_B(\mathscr{F})$.

If Γ is not Hausdorff, the resolution by differential forms is in general not injective. However, for an injective resolution $C\underline{\mathbb{R}}$ of $\underline{\mathbb{R}}$, there is a map $\underline{\Omega}(U) \to C\underline{\mathbb{R}}$ which induces a map

$$\mathscr{I} : C(\Gamma; \underline{\Omega}(U)) \to C(\Gamma; C\underline{\mathbb{R}}).$$

In the case that Γ is Hausdorff, and is thus a Hausdorff paracompact manifold, $\mathscr{I}$ induces an isomorphism in cohomology.

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4. Examples

In the following, we consider the computation of the cohomology of holonomy groupoids in several situations.

1. Since the holonomy of Riemannian foliations are local isometries, the map $\mathscr{I}$ induces an isomorphism in cohomology. Thus, the cohomology of the holonomy groupoid of a Riemannian foliation can be calculated using the resolution by differential forms.
2. Let $\mathscr{F}$ be a Riemannian foliation with all leaves compact. Then, the source and target maps of the holonomy groupoid give rise to locally trivial fibrations with fibre $\tilde{L}$, called the *universal leaf* of $\mathscr{F}$. Furthermore, $\tilde{L}$ is compact. It follows that in the double complex $C^s(\Gamma; \underline{\Omega}^r(U))$, one can define a chain homotopy for δ,

$$H \; : \; C^{s+1}(\Gamma; \underline{\Omega}^r(U)) \to C^s(\Gamma; \underline{\Omega}^r(U)),$$

by

$$(Hf)(\gamma_1, \dots, \gamma_s) = \int_{\tilde{L}} \gamma_0^{-1} f(\gamma_0, \dots, \gamma_s)\, d\gamma_0$$

where $d\gamma_0$ is the volume form on $\tilde{L}$ with $\int_{\tilde{L}} d\gamma_0 = 1$. Thus, the spectral seqence of the double complex collapses, that is, $E_2^{r,s} = 0$ for $s > 0$, which implies that $H(\Gamma; \mathbb{R}) = H_B(\mathscr{F})$.
3. Haefliger [10] showed that if $\mathscr{F}$ is a foliation on a manifold M such that the holonomy covering of the leaves are all contractible, then $H(\Gamma; \mathbb{R}) = H_{DR}(M)$. Thus, for example, the cohomology of the transverse holonomy groupoid of the Reeb foliation on S^3 is that of S^3.
4. Let $M = \tilde{B} \times_\phi G/H$, where $\tilde{B}$ is the universal covering of a manifold B, G is a Lie group, H is a closed subgroup of G, $\phi : \pi_1(B) \to G$ is a homomorphism, and $\pi_1(B)$ acts on $\tilde{B}$ by deck transformations and on G/H by ϕ. Let $\mathscr{F}$ be a flat transversely homogeneous foliation [1] on M, that is, $\mathscr{F}$ is induced by the product foliation with leaves $\tilde{B} \times \{x\}$ on $\tilde{B} \times G/H$. Then $\mathscr{F}$ is a foliation transverse to a fibration with fibre G/H. Now, suppose G is compact. Denote the image of ϕ by Γ, and the closure of Γ by $\bar{\Gamma}$. Then $\bar{\Gamma}$ is a compact Lie group. Denote the transverse holonomy groupoid of $\mathscr{F}$ by $\Gamma_{G/H}$. Then, since the holonomy of a transversly homogeneous foliation is analytic,

$$\begin{aligned} H(\Gamma_{G/H}; \mathbb{R}) &= H\left(C\left(\Gamma_{G/H}; \underline{\Omega}(G/H)\right)\right) = H\left(C\left(\Gamma^\delta; \Omega(G/H)\right)\right) \\ &= H\left(C\left(\Gamma^\delta; \Omega(G/H)^{\bar{\Gamma}}\right)\right) = H(\Gamma^\delta, \mathbb{R}) \otimes H_{DR}(G/H). \end{aligned}$$

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Here, Γ^δ denotes the group Γ with the discrete topology and $\Omega(G/H)^{\bar\Gamma}$ denotes the $\bar\Gamma$-invariant forms on G/H.

5. Let $\mathscr{F}$ be a Lie foliation on a compact connected manifold M which is modelled on the simply-connected Lie group G. Then $\mathscr{F}$ has a development map, that is, a locally trivial fibration $\tilde M \to G$, where $\tilde M$ denotes the universal covering of M, which induces the foliation $\mathscr{F}$ on M. Furthermore, the fibration is equivariant with respect to the holonomy representation $H : \pi_1(M) \to G$. Denote the image of H by Γ, and the transverse holonomy groupoid of $\mathscr{F}$ by Γ_G. Then, similar to the previous example, we have

$$H(\Gamma;\mathbb{R}) = H\left(C\left(\Gamma^\delta;\Omega(G)\right)\right).$$

5. The characteristic homomorphism

Let $\mathscr{F}$ be a foliation on a paracompact manifold M with transverse holonomy groupoid Γ. Then $\mathscr{F}$ is a Γ-structure in the sense of Haefliger, and hence given by a map $\gamma : M \to B\Gamma$, where $B\Gamma$ is the classifying space of Γ of Buffet-Lor [2]. The map induced in cohomology by γ is known as the *characteristic homomorphism* of $\mathscr{F}$. Using the double complex of Section 3, we have a map

$$\gamma^* : C(\Gamma;\underline{\Omega}(U)) \to C(\mathscr{U};\Omega(M))$$

where the double complex on the right is the Čech-de Rham complex of M with respect to a covering $\mathscr{U}$ of M, defined by

$$(\gamma^* f)\left(U_{i_0}\cap U_{i_1}\cap\cdots\cap U_{i_s}\right)(x) = \gamma^*_{i_0 i_0} f\left(\gamma_{i_0 i_1}(x),\ldots,\gamma_{i_{s-1}i_s}(x)\right)$$

if $x \in U_{i_0}\cap U_{i_1}\cap\cdots\cap U_{i_s}$, where $\gamma_{i_k i_{k+1}} : U_{i_k}\cap U_{i_{k+1}} \to \Gamma$ is the Γ-structure defining $\mathscr{F}$. Passing to cohomology, we have the following commutative diagram.

$$\begin{array}{ccc} H\left(C\left(\Gamma;\underline{\Omega}(U)\right)\right) & & \\ \downarrow & \overset{\gamma^*}{\searrow} & \\ & & H_{DR}(M) \\ \downarrow & \underset{\gamma^*}{\nearrow} & \\ H(\Gamma;\mathbb{R}) & & \end{array}$$

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6. The minimal model of a foliation

Let $\mathscr{F}$ be a foliation on a connected manifold M. Then there is a map of connected differential graded (DG) algebras

$$\iota : \Omega_B(\mathscr{F}) \hookrightarrow \Omega_{DR}(M).$$

The minimal model of ι is called the *minimal model of the foliation* $\mathscr{F}$. It consists of a DG algebra $(\mathscr{M}, d)$ which is minimal in the sense of Sullivan [7, 11], and a DG algebra map Φ such that the following diagram commutes:

$$\begin{array}{ccccc}
\Omega_B(\mathscr{F}) & \xrightarrow{\;\;i\;\;} & (\Omega_B(\mathscr{F})\tilde{\otimes}\mathscr{M},\ D) & \xrightarrow{\;\;j\;\;} & (\mathscr{M}, d) \\
 & \searrow^{\iota} & \sim \big\downarrow \Phi & & \\
 & & \Omega_{DR}(M) & &
\end{array}$$

where

(i) i is the inclusion in the first factor,
(ii) j is the projection onto the second factor,
(iii) Φ induces an isomorphism in cohomology, and
(iv) $\Omega_B(\mathscr{F})\tilde{\otimes}\mathscr{M}$ is the tensor product of graded algebras, and the differential D satisfies

$$D(1\tilde{\otimes}m) - 1\tilde{\otimes}dm \in \Omega_B^{>0}(\mathscr{F})\tilde{\otimes}\mathscr{M}.$$

The filtration

$$F^i(\Omega_B(\mathscr{F})\tilde{\otimes}\mathscr{M}) = \Omega_B^{\geq i}(\mathscr{F})\tilde{\otimes}\mathscr{M}$$

gives rise to a third quadrant spectral sequence converging to $H(M)$ with $E_2^{r,0} = H_B^r(\mathscr{F})$.

We would like to remark, as we have mentioned in the introduction, that the fibre $\mathscr{M}$ of the minimal model, also called the *relative minimal model* of $\mathscr{F}$, does not have an obvious geometric interpretation, as we illustrate by the following easy result that indeed it can have infinite cohomological dimension.

PROPOSITION 6.1. *Let M be a compact, connected, simply-connected manifold, and let $\mathscr{F}$ be a Riemannian foliation of codimension 3 on M. Then, either $H_{DR}^3(M) \neq 0$, or the fibre of the minimal model of $\mathscr{F}$ has infinite cohomological dimension.*

PROOF. For the spectral sequence introduced above, we have the isomorphism

$$H_B^1(\mathscr{F}) = E_2^{1,0} \cong E_\infty^{1,0} = \mathscr{G}^{1,0}H_{DR}^M.$$

Since M is simply-connected, it follows that $H_B^1(\mathscr{F}) = 0$. By the duality result of Kamber-Tondeur [13], $H_B(\mathscr{F})$ satisfies Poincaré duality. Hence, $H_B^0(\mathscr{F}) = H_B^3(\mathscr{F}) = \mathbb{R}$, $H_B^1(\mathscr{F}) = H_B^2(\mathscr{F}) = 0$.

Let $[\nu]$ be a basis for $H_B^3(\mathscr{F})$. Then, either (i) $\iota_*[\nu] \neq 0 \in H_{DR}^3(M)$, or (ii) $\iota_*[\nu] = 0$. In the case (ii), we observe that $[\nu]$ is a spherical class, that is, its image under the dual Hurewicz map $H_B(\mathscr{F}) \to \pi^*(\Omega_B(\mathscr{F}))$ is non-zero. It can be represented by a generator β in the minimal model of $\Omega_B(\mathscr{F})$. Thus,

$$\ker\left\{\Phi_* : H^3\left(\Omega_B(\mathscr{F})\tilde{\otimes}\mathscr{M}^{<2}\right) \to H_{DR}^3(M)\right\} \neq 0$$

and there exists a generator α of $\mathscr{M}$ of degree 2 such that $d\alpha = \beta$. From $d\alpha^k = k\alpha^{k-1}\beta$, $k = 1, 2, 3, \ldots$, we see that $d\alpha^k = 0$ and $\alpha^k \notin d\mathscr{M}$, that is,

$$0 \neq [\alpha^k] \in H^{2k}(\mathscr{M}), \qquad k = 1, 2, 3, \ldots.$$

EXAMPLE 6.1. It is well-known that a non-singular taut Riemannian flow $\mathscr{F}$ is given by a non-vanishing Killing vector field [13], and if the ambient manifold is connected and compact, there is a decomposition of the de Rham algebra of M, up to cohomology, as follows [6]:

$$\Omega_B(\mathscr{F}) \otimes \Lambda X \xrightarrow{\sim} \Omega_{DR}(M),$$

where ΛX denotes the exterior algebra with one generator of degree one with the trivial differential. Thus the minimal model of a non-singular taut Riemannian flow is described by

$$\begin{array}{ccccc} \Omega_B(\mathscr{F}) & \longrightarrow & \Omega_B(\mathscr{F})\tilde{\otimes}\Lambda X & \longrightarrow & \Lambda X \\ & \searrow & \Big\downarrow{\scriptstyle \sim\ \Phi} & & \\ & & \Omega_{DR}(M) & & \end{array}$$

and its relative minimal model is the algebra ΛX.

For the double complex $C(\Gamma; \underline{\Omega}(U))$, there is a DG algebra $\mathscr{C}$ and a map of complexes $\mathscr{C} \to C(\Gamma; \underline{\Omega}(U))$, where the complex on the right is equipped with the total differential $d + \delta$, which induces an isomorphism in cohomology [4]. Furthermore, this construction is natural in the category of complexes. In the following, we will abuse notation and refer to $\mathscr{C}$ by $C(\Gamma; \underline{\Omega}(U))$, and when we write the minimal model or rational homotopy of the Čech complex, we actually refer to that of the cohomologically equivalent DG algebra.

Let $\mathscr{F}$ be a foliation on a connected manifold M with transverse holonomy groupoid Γ. Then there is an injection of connected differential graded algebras

$$\Omega_B(\mathscr{F}) \hookrightarrow C(\Gamma; \underline{\Omega}(U)),$$

and we can consider its minimal model [11]:

$$\begin{array}{ccccc}
\Omega_B(\mathscr{F}) & \longrightarrow & \Omega_B(\mathscr{F})\tilde{\otimes}\Sigma & \longrightarrow & \Sigma \\
& \searrow & \sim \big\downarrow \Psi & & \\
& & C(\Gamma; \underline{\Omega}(U)) & &
\end{array}$$

where Σ is a minimal algebra in the sense of Sullivan.

For the remainder of this paper, we will need to make a technical assumption on our foliations as embodied in the following:

DEFINITION 6.1. In the notation established above, a foliation $\mathscr{F}$ on a connected manifold M is said to be of *finite type* if both the algebras $\mathscr{M}$ and Σ are of finite type, that is, finite-dimensional in each degree.

7. Comparison results

The aim of this section is to obtain a model for the universal leaf of a Riemannian foliation $\mathscr{F}$ on a compact connected manifold M through a comparison theorem. We note that the passage from the manifold M to the transverse holonomy groupoid Γ is by replacing the leaves by contractible spaces. Thus, it can be viewed as a fibration construction over the leaf space $M/\mathscr{F}$, and the fibre should be the model for the universal leaf. Following this line, we consider the 'fibrations' $M \to M/\mathscr{F}$ and $\Gamma \to M/\mathscr{F}$ and the models for their fibres, namely, $\mathscr{M}$ and Σ, respectively. $\mathscr{M}$ and Σ are then compared to obtain a model for the universal leaf.

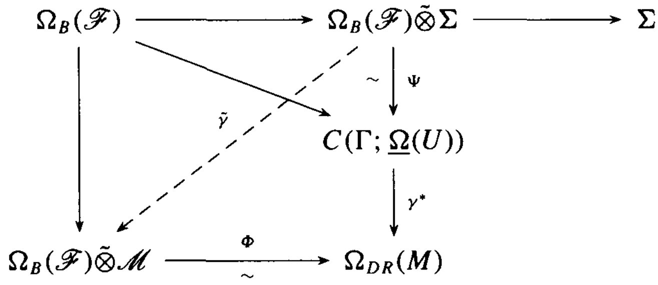
For our subsequent arguments involving the minimal model construction to be valid, we need to impose the technical condition that all foliations discussed are of finite type.

Analogous to the construction of the minimal model of the classifying map $\gamma : M \to B\Gamma$, we have:

THEOREM 7.1. *Let $\mathscr{F}$ be a foliation of finite type on a connected manifold M. Then there exists a commutative diagram of DG algebras as follows.*

$$\begin{array}{ccccc}
\Omega_B(\mathscr{F}) & \longrightarrow & \Omega_B(\mathscr{F})\tilde{\otimes}\Sigma & \longrightarrow & \Sigma \\
\Big\| & & \big\downarrow \tilde{\gamma} & & \\
\Omega_B(\mathscr{F}) & \longrightarrow & \Omega_B(\mathscr{F})\tilde{\otimes}\mathscr{M} & \longrightarrow & \mathscr{M}
\end{array}$$

PROOF. Consider the following diagram.

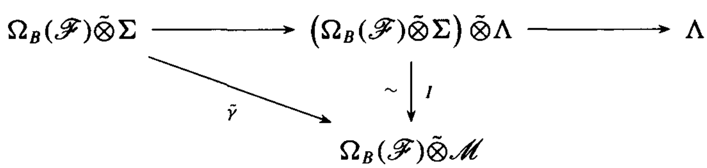


Since the map Φ induces an isomorphism in cohomology of DG algebras and $\Omega_B(\mathscr{F})\tilde{\otimes}\mathscr{M}$ is a free DG algebra over $\Omega_B(\mathscr{F})$, by obstruction theory [7, 11], there exists a map

$$\tilde{\gamma} : \Omega_B(\mathscr{F})\tilde{\otimes}\Sigma \to \Omega_B(\mathscr{F})\tilde{\otimes}\mathscr{M}$$

which makes the diagram commutative as claimed.

The minimal model of the map $\tilde{\gamma}$ can be constructed as follows.

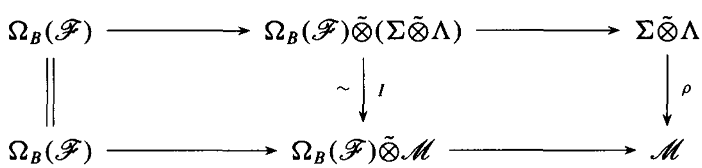


THEOREM 7.2. *Under the the conditions of Theorem* 7.1, *there is a DG algebra map*

$$\rho : \Sigma\tilde{\otimes}\Lambda \to \mathscr{M}$$

which induces an isomorphism in cohomology.

PROOF. Consider the following commutative diagram of DG algebras.



This is a morphism of KS-extension in the terminology of [11] where it is also proved that as I induces an isomorphism in cohomology, so does ρ ([11, Theorem 4.5]).

Let $\sigma : \mathscr{M}' \to \Sigma\tilde{\otimes}\Lambda$ be the minimal model of $\Sigma\tilde{\otimes}\Lambda$. By Theorem 7.2, by composing with ρ, $\mathscr{M}'$ is also the minimal model of $\mathscr{M}$. Since $\mathscr{M}$ is itself minimal, by the uniqueness of minimal models, we have:

COROLLARY 7.3. *The minimal model of* $\Sigma\tilde{\otimes}\Lambda$ *is given by*

$$\sigma : \mathscr{M} \to \Sigma\tilde{\otimes}\Lambda.$$

Thus, $\mathscr{M}$ *and* $\Sigma\tilde{\otimes}\Lambda$ *are isomorphic up to cohomology.*

In the next theorem, which is our main result, we identify the algebra Λ with the minimal model of the universal leaf under certain nilpotence assumptions. For this, we need the result of Winkelnkemper [17] (see also [10]) that for a Riemannian foliation $\mathscr{F}$ on a compact manifold M with transverse holonomy groupoid Γ, there is a locally trivial fibration $\tilde{L} \to M \to B\Gamma$ where $\tilde{L}$ is the common holonomy covering of the leaves of $\mathscr{F}$, that is, the universal leaf.

THEOREM 7.4. *Let* $\mathscr{F}$ *be a Riemannian foliation of finite type on a compact, connected manifold* M. *For the fibration* $\tilde{L} \to M \xrightarrow{\gamma} B\Gamma$, *suppose*

(i) $H(B\Gamma)$ *or* $H(\tilde{L})$ *has finite type,*
(ii) $\pi_1(B\Gamma) = \pi^1\big(C(\Gamma; \underline{\Omega}(U))\big)^*$ *is nilpotent and acts nilpotently on each* $H(\tilde{L})$.

Then, in the minimal model of the characteristic homomorphism $\tilde{\gamma}$, *the minimal algebra* Λ *is the minimal model of the universal leaf.*

PROOF. We note that the minimal model of the map

$$\gamma^* : C(\Gamma; \underline{\Omega}(U)) \to C(\mathscr{U}; \Omega(M))$$

gives rise to the following commutative diagram of DG algebras:

$$\begin{array}{ccccc}
\Omega_B(\mathscr{F})\tilde{\otimes}\Sigma & \longrightarrow & (\Omega_B(\mathscr{F})\tilde{\otimes}\Sigma)\tilde{\otimes}\Lambda & \longrightarrow & \Lambda \\
\big\downarrow{\scriptstyle\sim} & & \big\downarrow{\scriptstyle\sim} & & \\
C(\Gamma; \underline{\Omega}(U)) & \longrightarrow & C(\Gamma; \underline{\Omega}(U))\tilde{\otimes}\Lambda & \longrightarrow & \Lambda \\
 & {\scriptstyle\gamma^*}\searrow & \big\downarrow{\scriptstyle\sim} & & \\
 & & C(\mathscr{U}; \Omega(M)) & & \\
 & & \big\downarrow{\scriptstyle\sim} & & \\
 & & \Omega_{DR}(M) & &
\end{array}$$

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which is a minimal model of the fibration γ. By a result of Da Silveira [3] (see also [8, Theorem 6.4] when $B\Gamma$ is simply-connected), the algebra Λ is the model of the fibre, that is, $\tilde{L}$.

We remark that since a Riemannian foliation with all leaves compact is a generalized Seifert fibration with the universal leaf as fibre, and the leaf space is a Satake manifold whose cohomology is that of the algebra of basic forms [10], the relative minimal model $\mathscr{M}$ for $M \to M/\mathscr{F}$ is the minimal model of the universal leaf. On the other hand, the inclusion $\Omega_B(\mathscr{F}) \hookrightarrow C(\Gamma; \underline{\Omega}(U))$ induces an isomorphism in cohomology (cf. Example (2) in Section 4), hence the relative minimal model Σ of the fibration $B\Gamma \to M/\mathscr{F}$ is trivial. Thus, we have the following obvious result:

COROLLARY 7.5. *Let $\mathscr{F}$ be a Riemannian foliation of finite type on a compact connected manifold M with all leaves compact. Then the minimal algebra $\mathscr{M}$ is the minimal model of the universal leaf.*

Let M, N be connected manifolds, and let the fundamental group $\pi_1(M)$ act on the universal covering $\tilde{M}$ by deck transformations and on N by diffeomorphisms via $\phi : \pi_1(M) \to \mathrm{Diff}(N)$. Suppose there exists a submersion $\psi : \tilde{M} \to N$ which is equivariant with respect to the $\pi_1(M)$-actions. Then ψ induces a foliation $\mathscr{F}$ on M which is transversally modelled on N. Such a foliation is called *developable*, and ψ is called the *development map*. The image of ϕ, denoted by G, is called the *global holonomy group* of $\mathscr{F}$, and we denote the holonomy groupoid of $\mathscr{F}$ by Γ_G. Note that $\Omega_B(\mathscr{F})$ consists exactly of those differential forms on N which are invariant under G.

In the following, we will assume that the source map of the holonomy groupoid of $\mathscr{F}$ defines a locally trivial fibration

$$\tilde{L} \to \Gamma_G \xrightarrow{\alpha} N.$$

This class of foliations includes the Riemannian foliations on compact manifolds, foliations which are analytic and transverse to compact fibrations, and foliations for which the holonomy groups of closed leaves are infinite [10]. In this case, there is a locally trivial fibration

$$\tilde{L} \to E\Gamma_G \xrightarrow{\pi} B\Gamma_G$$

where $\tilde{L}$ denotes the common holonomy covering of the leaves of $\mathscr{F}$, and there is a homotopy equivalence $M \xrightarrow{\sim} E\Gamma_G$. This fibration is the universal principal Γ_G-bundle [2, 10] of $\mathscr{F}$.

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We recall that the Buffet-Lor [2] construction of $E\Gamma_G$ is a bundle $EG \to E\Gamma_G \xrightarrow{\tilde{\alpha}} N$, where $\tilde{\alpha}$ is induced by α, and hence is also a locally trivial fibration. Thus, we have a locally trivial fibration

$$BG \to B\Gamma_G \xrightarrow{\tilde{\alpha}} N/G$$

induced on $B\Gamma_G = E\Gamma_G/\Gamma_G$ with the quotient topology, where BG is the classifying space of the abstract group G. We remark that in this case, in the calculation of the cohomology of $B\Gamma_G$ using differential forms, the double complex $C(G; \Omega(N))$ is nothing else but the double complex of the fibration $\tilde{\alpha}$. In particular, the edge terms $E_2^{r,0}$ are the basic cohomology groups $H_B^r(\mathscr{F})$.

Now, assume that (i) $H(G)$ or $H_B(\mathscr{F})$ has finite type (this is satisfied for Riemannian foliations on compact manifolds [5, 14]), and (ii) $\pi_1(N/G)$ is nilpotent and acts nilpotently on $H(G)$. Then, by considering the minimal model of the fibration $\tilde{\alpha}$, we have the following DG algebra map which induces an isomorphism in cohomology

$$\zeta \; : \; \Omega_B(\mathscr{F}) \tilde{\otimes} \mathscr{M}_{BG} \xrightarrow{\sim} C(G; \Omega(N)).$$

We are now ready to formulate the following result, which gives the relative minimal model $\mathscr{M}$ of $\mathscr{F}$ in terms of the minimal models $\mathscr{M}_{BG}$ and $\mathscr{M}_{\tilde{L}}$ of the classifying space BG and the universal leaf $\tilde{L}$ respectively.

THEOREM 7.6. *Under the assumptions of the above discussion, there is a DG algebra map* $\mathscr{M} \xrightarrow{\sim} \mathscr{M}_{BG} \tilde{\otimes} \mathscr{M}_{\tilde{L}}$ *which induces an isomorphism in cohomology.*

PROOF. By using the homotopy equivalence $M \xrightarrow{\sim} E\Gamma_G$ and composing the minimal model of the fibration π with ζ, we obtain the following commutative diagram.

$$\begin{array}{ccccccc}
\Omega_B(\mathscr{F}) & \xrightarrow{\vartheta} & \Omega_B(\mathscr{F}) \tilde{\otimes} \mathscr{M}_{BG} & \longrightarrow & (\Omega_B(\mathscr{F}) \tilde{\otimes} \mathscr{M}_{BG}) \tilde{\otimes} \mathscr{M}_{\tilde{L}} & \longrightarrow & \mathscr{M}_{\tilde{L}} \\
 & & \zeta \downarrow \sim & & \sim \downarrow & & \\
 & & C(G; \Omega(N)) & \longrightarrow & \Omega(E\Gamma_G) & & \\
 & & & & \downarrow \sim & & \\
 & & & & \Omega_{DR}(M) & &
\end{array}$$

Comparing this with the relative minimal model $\mathscr{M}$ of $\mathscr{F}$, our result follows.

Corresponding to the DG algebra map $\pi^* \circ \zeta \circ \vartheta$, we consider the following diagram of fibrations.

$$
\begin{array}{ccccc}
 & & EG & \longrightarrow & BG \\
 & & \downarrow & & \downarrow \\
\tilde{L} & \longrightarrow & E\Gamma & \longrightarrow & B\Gamma \\
 & & \downarrow & & \downarrow \\
 & & N & \longrightarrow & N/G
\end{array}
$$

Our comparison result shows that the relative minimal model $\mathscr{M}$ of $\mathscr{F}$ corresponds to the Borel construction $EG \times_G \tilde{L}$ resulting from the the holonomy action on the universal leaf. We remark that in the case that the action of G on $\tilde{L}$ is free, that is, that the foliation is without holonomy, the minimal algebra $\mathscr{M}$ is in fact the minimal model of the leaves which are all diffeomorphic.

EXAMPLE 7.1. (i) Consider the foliation $\mathscr{F}$ of the torus T^2 by lines with a rational slope. Then all the leaves of $\mathscr{F}$ are compact (diffeomorphic to S^1) and $\mathscr{F}$ is without holonomy. By Theorem 7.6, the relative minimal model of $\mathscr{F}$ is the minimal model of S^1 which is the free DG algebra with one generator of degree one with the trivial differential.

(ii) Consider the foliation $\mathscr{F}$ of the torus T^2 by lines with an irrational slope. Then all the leaves of $\mathscr{F}$ are diffeomorphic to $\mathbb{R}^1$ and the transverse holonomy groupoid is the free cyclic group. By Theorems 7.2 and 7.4, the relative minimal model of $\mathscr{F}$ is the minimal model of the free cylic group which is the free DG algebra with one generator of degree one with the trivial differential.

We note that the relative minimal models of these examples have already been calculated in Example 6.1. We point out, however, that the two foliations considered have different minimal models as $\Omega_B(\mathscr{F})$ are different.

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