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OPERATOR-VALUED FOURIER MULTIPLIERS ON PERIODIC BESOV SPACES AND APPLICATIONS

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Abstract Let $1 \leqslant p, q \leqslant \infty$, $s \in \mathbb{R}$ and let X be a Banach space. We show that the analogue of Marcinkiewicz's Fourier multiplier theorem on $L^p(\mathbb{T})$ holds for the Besov space $B^s_{p,q}(\mathbb{T}; X)$ if and only if $1 < p < \infty$ and X is a UMD-space. Introducing stronger conditions we obtain a periodic Fourier multiplier theorem which is valid without restriction on the indices or the space (which is analogous to Amann's result (*Math. Nachr.* **186** (1997), 5–56) on the real line). It is used to characterize maximal regularity of periodic Cauchy problems.

Keywords: Fourier multipliers; Besov spaces; periodic solutions; Cauchy problem; maximal regularity

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1. Introduction

In a series of recent publications, operator-valued Fourier multiplier theorems on diverse vector-valued function spaces have been studied (see, for example, [**1**, **3**–**5**, **10**, **11**, **13**, **14**, **19**, **22**–**24**]). They are needed to establish existence and uniqueness as well as regularity for differential equations in Banach spaces, and thus also for partial differential equations.

Besov spaces form one class of function spaces which are of special interest. They can be defined via dyadic decomposition and form scales $B^s_{p,q}$ carrying three indices $s \in \mathbb{R}$, $1 \leqslant p, q \leqslant \infty$. The relatively complicated definition is justified by very useful applications to differential equations (see, for example, [**2**] for a concrete and important model). Note also that the space $B^s_{\infty,\infty}$ is nothing else but the familiar space of all Hölder continuous functions of index s if $s \in (0, 1)$.

It was Amann [**1**] (see also [**22**]) who discovered another favourable property of vector-valued Besov spaces on the real line: a certain form of the (most efficient) Mikhlin's multiplier theorem does hold for arbitrary Banach spaces (see [**13**] for refinements). This is a dramatic contrast to the L^p-scale, where the corresponding theorem merely holds for Hilbert spaces even if $p = 2$ (see [**4**] for details). Whereas Amann and Girardi and Weis consider Besov spaces on $\mathbb{R}$, we are here interested in the periodic case. There are several reasons for this investigation. First of all, the periodic Besov spaces are much easier to define and to handle. Moreover, they are natural for periodic problems. Frequently, more general problems can also be reduced to this case (as in the case of classical L^p-regularity (see [**3**, Corollary 5.2])). Whereas scalar periodic Besov spaces have been studied by Triebel [**20**] and Schmeisser and Triebel [**18**], we believe that this paper represents the first work on vector-valued periodic Besov spaces, and it seems that Fourier multipliers on periodic Besov spaces have not previously appeared in the literature even in the scalar case.

There are several possibilities concerning the conditions to impose on a sequence in the attempt to establish a periodic Fourier multiplier theorem. It is interesting that success depends on the choice of the index p and on the Banach space X. We explain this in more detail. In view of the classical result, we say that a sequence $(M_k)_{k\in\mathbb{Z}}$ of operators on a Banach space X satisfies *the variational Marcinkiewicz condition* if

$$\sup_{k\in\mathbb{Z}}\|M_k\| + \sup_{j\geqslant 0}\sum_{2^j\leqslant|k|\leqslant 2^{j+1}}\|M_{k+1}-M_k\| < \infty. \tag{1.1}$$

If $X=\mathbb{C}$, then Marcinkiewicz's classical theorem says that such a sequence is an $L^p(\mathbb{T},\mathbb{C})$-Fourier multiplier for $1<p<\infty$. Our first main result states that each sequence satisfying (1.1) is a $B^s_{p,q}(\mathbb{T},X)$-Fourier multiplier if and only if $1<p<\infty$ and X is a UMD-space (where $1\leqslant q\leqslant\infty$ and $s\in\mathbb{R}$ are arbitrary). In particular, even in the scalar case there are sequences satisfying (1.1) which are not Fourier multipliers on the periodic Hölder spaces.

The following stronger condition was introduced in [**3**] in the L^p-context:

$$\sup_{k\in\mathbb{Z}}(\|M_k\| + \|k(M_{k+1}-M_k)\|) < \infty. \tag{1.2}$$

Here one may consider $M_{k+1}-M_k$ as the first derivative of the sequence (and condition (1.2) becomes analogous to Mikhlin's condition on the line). Taking the second derivative leads to the condition

$$\sup_{k\in\mathbb{Z}}\|k^2(M_{k+2}-2M_{k+1}+M_k)\| < \infty. \tag{1.3}$$

We call (1.2) *Marcinkiewicz's condition of order* 1. Moreover, we speak of *Marcinkiewicz's condition of order* 2 if (1.2) as well as (1.3) are satisfied. It is clear that condition (1.2) is stronger than the variational Marcinkiewicz condition.

In § 4 we show that the Marcinkiewicz condition of order 1 implies that $(M_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}(\mathbb{T};X)$-Fourier multiplier for every $s\in\mathbb{R}$ and every $1\leqslant p,q\leqslant\infty$ whenever X has a non-trivial Fourier type. This is analogous to the result of Girardi and Weis [**13**] on the real line. For arbitrary Banach spaces the Marcinkiewicz condition of order 2 is sufficient without restriction to the indices and the space. This is the periodic version of Amann's result [**1**] on the line.

Even though the Marcinkiewicz condition of order 2 is stronger than the variational Marcinkiewicz condition, in the context of resolvents, it is characteristic. In fact, given an operator A on a Banach space X such that $\mathrm{i}\mathbb{Z} \subset \rho(A)$, we show that $(k(\mathrm{i}k - A)^{-1})_{k\in\mathbb{Z}}$ is a $B^s_{p,q}(\mathbb{T}; X)$-Fourier multiplier if and only if the sequence is bounded. In view of the resolvent identity this is precisely the Marcinkiewicz condition of order 2. This result can be reformulated in terms of well-posedness of the periodic Cauchy problem:

$$P_{\mathrm{per}} \begin{cases} u'(t) = Au(t) + f(t), \\ u \in B^{1+s}_{p,q}(\mathbb{T}, X) \cap B^s_{p,q}(\mathbb{T}, D(A)), \end{cases}$$

where $f \in B^s_{p,q}(\mathbb{T}, X)$. The study of this and similar problems is another motivation of our investigation.

We are only interested in the study of $B^s_{p,q}(\mathbb{T}; X)$ for $1 \leqslant p, q \leqslant \infty$. It is possible to do this with all $-\infty < p, q \leqslant \infty$ and even for $\mathbb{T}^n$ in place of $\mathbb{T}$, but the notation would be more complicated. If S is a set and I is an index set, the notation $(s_i)_{i\in I} \subset S$ means that for each $i \in I$, we have $s_i \in S$.

2. Periodic Besov spaces

Let X be a Banach space and let $\mathbb{T} = [-\pi, \pi]$, where the points $-\pi$ and π are identified. Let $\mathcal{D}(\mathbb{T})$ be the space of all complex-valued infinitely differentiable functions on $\mathbb{T}$. The usual locally convex topology in $\mathcal{D}(\mathbb{T})$ is generated by the semi-norms $\|f\|_\alpha = \sup_{t\in\mathbb{T}} |f^{(\alpha)}(t)|$, where $\alpha \in \mathbb{N} \cup \{0\}$. We let $\mathcal{D}'(\mathbb{T}; X) := \mathcal{L}(\mathcal{D}(\mathbb{T}); X)$. In other words, $\mathcal{D}'(\mathbb{T}; X)$ is the set of all linear mappings T from $\mathcal{D}(\mathbb{T})$ to X such that

$$\|T(f)\|_X \leqslant C \sum_{\alpha\leqslant N} \|f\|_\alpha$$

for all $f \in \mathcal{D}(\mathbb{T})$ and for some $N \in \mathbb{N}$ and $C > 0$ independent of f. Elements in $\mathcal{D}'(\mathbb{T}; X)$ are called X-valued distributions on $\mathbb{T}$. We use the weak topology on $\mathcal{D}'(\mathbb{T}; X)$, i.e. a sequence T_k converges to T in $\mathcal{D}'(\mathbb{T}; X)$ if and only if $\lim_{k\to\infty} T_k(f) = T(f)$ for all $f \in \mathcal{D}(\mathbb{T})$.

Let $1 \leqslant p \leqslant \infty$. For $g \in L^p(\mathbb{T}; X)$ let

$$\|g\|_p = \left(\int_{-\pi}^{\pi} \|g(t)\|^p \frac{\mathrm{d}t}{2\pi}\right)^{1/p}, \quad \text{if } 1 \leqslant p < \infty,$$
$$\|g\|_\infty = \operatorname*{ess\,sup}_{t\in\mathbb{T}} \|g(t)\|.$$

Each element $g \in L^p(\mathbb{T}; X)$ can be interpreted in a natural way as an element of $\mathcal{D}'(\mathbb{T}; X)$, as in the scalar case. Let e_k be the function $e_k(t) = \mathrm{e}^{\mathrm{i}kt}$ for $k \in \mathbb{Z}$ and $t \in \mathbb{T}$. For $x \in X$, we denote by $e_k \otimes x$ the X-valued function $(e_k \otimes x)(t) = e_k(t)x$. We then have $e_k \otimes x \in \mathcal{D}'(\mathbb{T}; X)$.

Recall that a sequence $(a_k)_{k\in\mathbb{Z}} \subset X$ is of *at most polynomial growth* if there exist $c, N > 0$ such that for $k \in \mathbb{Z}$, we have $\|a_k\| \leqslant c(1 + |k|)^N$. We have the following characterization of $\mathcal{D}'(\mathbb{T}; X)$. The proof is similar to the scalar case [**12**, Chapter 12].

Proposition 2.1. *Let $(a_k)_{k\in\mathbb{Z}} \subset X$ be of at most polynomial growth. Then*

$$g(\phi) = \sum_{k=-\infty}^{+\infty} \hat{\phi}(k) a_k$$

converges for all $\phi \in \mathcal{D}(\mathbb{T})$ and defines a distribution $g \in \mathcal{D}'(\mathbb{T}, X)$. We write

$$g = \sum_{k=-\infty}^{\infty} e_k \otimes a_k.$$

Conversely, each distribution $g \in \mathcal{D}'(\mathbb{T}, X)$ is of this form with $a_k = g(e_k)$.

Let $g \in \mathcal{D}'(\mathbb{T}; X)$ and

$$g = \sum_{k\in\mathbb{Z}} e_k \otimes a_k$$

as in Proposition 2.1. Then we call $\hat{g}(k) := a_k$ the kth Fourier coefficient of g. We say that g is an X-valued *trigonometric polynomial* if there exist $N \in \mathbb{N}$ and $a_k \in X$ such that

$$g = \sum_{k=-N}^{N} e_k \otimes a_k.$$

Let $\mathcal{S}$ be the Schwartz space on $\mathbb{R}$ and let $\mathcal{S}'$ be the space of all tempered distributions on $\mathbb{R}$. Let $\varPhi(\mathbb{R})$ be the set of all systems $\phi = (\phi_j)_{j\geqslant 0} \subset \mathcal{S}$ satisfying

$$\begin{gathered} \operatorname{supp}(\phi_0) \subset [-2, 2], \\ \operatorname{supp}(\phi_j) \subset [-2^{j+1}, -2^{j-1}] \cup [2^{j-1}, 2^{j+1}], \quad j \geqslant 1, \\ \sum_{j\geqslant 0} \phi_j(t) = 1, \quad t \in \mathbb{R}, \end{gathered}$$

and for $\alpha \in \mathbb{N} \cup \{0\}$, there exists $C_\alpha > 0$ such that

$$\sup_{j\geqslant 0,\ x\in\mathbb{R}} 2^{\alpha j} \|\phi_j^{(\alpha)}(x)\| \leqslant C_\alpha. \tag{2.1}$$

The set $\varPhi(\mathbb{R})$ is not empty [**20**, Remark 1, p. 45] (see also Lemma 4.1 below).

Let $1 \leqslant p, q \leqslant \infty$, $s \in \mathbb{R}$ and $\phi = (\phi_j)_{j\geqslant 0} \in \varPhi(\mathbb{R})$. We define the X-valued *periodic Besov spaces* by

$$B_{p,q}^{s,\phi}(\mathbb{T}; X) := \biggl\{ f \in \mathcal{D}'(\mathbb{T}; X) : \|f\|_{B_{p,q}^{s,\phi}} = \biggl(\sum_{j\geqslant 0} 2^{sjq} \biggl\| \sum_{k\in\mathbb{Z}} e_k \otimes \phi_j(k) \hat{f}(k) \biggr\|_p^q \biggr)^{1/q} < \infty \biggr\},$$

with the usual modification when $q = \infty$.

Note that for $f \in \mathcal{D}'(\mathbb{T}; X)$ and $j \geqslant 0$,

$$\sum_{k\in\mathbb{Z}} e_k \otimes \phi_j(k) \hat{f}(k)$$

is a trigonometric polynomial by Proposition 2.1. So

$$\Big\| \sum_{k\in\mathbb{Z}} e_k \otimes \phi_j(k)\hat{f}(k)\Big\|_p < \infty \quad \text{for } 1 \leqslant p \leqslant \infty.$$

We can identify $B^{s,\phi}_{p,q}(\mathbb{T};X)$ with the space of all sequences $(a_k)_{k\in\mathbb{Z}}$ in X such that

$$\bigg(2^{sjq}\Big\| \sum_{k\in\mathbb{Z}} e_k \otimes \phi_j(k)a_k\Big\|_p\bigg)_{j\geqslant 0} \in \ell^q.$$

This shows that $\|\cdot\|_{B^{s,\phi}_{p,q}}$ is a norm and that $B^{s,\phi}_{p,q}(\mathbb{T},X)$ is complete.

We begin by proving a result on operator-valued Fourier multipliers on $L^p(\mathbb{T};X)$ which will enable us to show the equivalence of the norms $\|\cdot\|_{B^{s,\phi}_{p,q}}$ for different $\phi \in \Phi(\mathbb{R})$. Let X and Y be Banach spaces. We denote by $\mathcal{L}(X;Y)$ the set of all bounded linear operators from X to Y. When $X=Y$, we will simply denote it by $\mathcal{L}(X)$. For $f \in L^1(\mathbb{R};\mathcal{L}(X,Y))$, we denote by $\mathcal{F}f$, given by

$$(\mathcal{F}f)(t) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}st} f(s)\,\mathrm{d}s \quad (t\in\mathbb{R}),$$

the Fourier transform of f.

Proposition 2.2. *Let X and Y be Banach spaces. For every $M \in C_c(\mathbb{R};\mathcal{L}(X;Y)) \cap \mathcal{F}L^1(\mathbb{R};\mathcal{L}(X,Y))$ and $1\leqslant p\leqslant\infty$, we have*

$$\Big\| \sum_{k\in\mathbb{Z}} e_k \otimes M(k)\hat{f}(k)\Big\|_p \leqslant \frac{1}{\sqrt{2\pi}}\|\mathcal{F}^{-1}M\|_1\|f\|_p$$

whenever f is an X-valued trigonometric polynomial, where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform.

Proof. We have

$$\begin{aligned}\sum_{k\in\mathbb{Z}} e_k(t)M(k)\hat{f}(k) &= \sum_{k\in\mathbb{Z}} (\mathcal{F}\mathcal{F}^{-1}M)(k)\hat{f}(k)\mathrm{e}^{\mathrm{i}kt}\\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} (\mathcal{F}^{-1}M)(y)\bigg[\sum_{k\in\mathbb{Z}}\hat{f}(k)\mathrm{e}^{\mathrm{i}k(t-y)}\bigg]\,\mathrm{d}y\\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} (\mathcal{F}^{-1}M)(y)f(t-y)\,\mathrm{d}y\\ &= \frac{1}{\sqrt{2\pi}}(\mathcal{F}^{-1}M)*f(t).\end{aligned}$$

The claim now follows from Young's inequality. □

Notice that for $M\in L^1(\mathbb{R};\mathcal{L}(X;Y))$ and $a>0$, we have $\|\mathcal{F}^{-1}M\|_1 = \|\mathcal{F}^{-1}M(a\cdot)\|_1$. If $M\in W^2_1 = \{f\in L^2(\mathbb{R}) : f'\in L^2(\mathbb{R})\}$ with the natural norm $\|f\|_{W^2_1} = (\|f\|_2^2 + \|f'\|_2^2)^{1/2}$, then $\mathcal{F}^{-1}M \in L^1(\mathbb{R})$ and there exists a numerical constant $C>0$ such that

$$\|\mathcal{F}^{-1}M\|_1 \leqslant C\|M\|_{W^2_1}.$$

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Let $\phi, \varphi \in \Phi(\mathbb{R})$, $f \in \mathcal{D}'(\mathbb{T}; X)$, $1 \leqslant p \leqslant \infty$ and $j \geqslant 1$. We have $\operatorname{supp}(\phi_j(2^j \cdot)) \subset [-2, -\frac{1}{2}] \cup [\frac{1}{2}, 2]$. By (2.1), $\sup_{j \geqslant 1} \|\phi_j(2^j \cdot)\|_{W_1^2} < \infty$. Thus, by Proposition 2.2,

$$\begin{aligned}
\Big\| \sum_{k \in \mathbb{Z}} e_k \otimes \phi_j(k) \hat{f}(k) \Big\|_p &\leqslant \sum_{l=j-1,j,j+1} \Big\| \sum_{k \in \mathbb{Z}} e_k \otimes \phi_j(k) \varphi_l(k) \hat{f}(k) \Big\|_p \\
&\leqslant \frac{1}{\sqrt{2\pi}} \|\mathcal{F}^{-1} \phi_j\|_1 \sum_{l=j-1,j,j+1} \Big\| \sum_{k \in \mathbb{Z}} e_k \otimes \varphi_l(k) \hat{f}(k) \Big\|_p \\
&\leqslant \frac{1}{\sqrt{2\pi}} \|\mathcal{F}^{-1} \phi_j(2^j \cdot)\|_1 \sum_{l=j-1,j,j+1} \Big\| \sum_{k \in \mathbb{Z}} e_k \otimes \varphi_l(k) \hat{f}(k) \Big\|_p \\
&\leqslant \frac{C}{\sqrt{2\pi}} \|\phi_j(2^j \cdot)\|_{W_1^2} \sum_{l=j-1,j,j+1} \Big\| \sum_{k \in \mathbb{Z}} e_k \otimes \varphi_l(k) \hat{f}(k) \Big\|_p \\
&\leqslant C' \sum_{l=j-1,j,j+1} \Big\| \sum_{k \in \mathbb{Z}} e_k \otimes \varphi_l(k) \hat{f}(k) \Big\|_p
\end{aligned}$$

for $f \in \mathcal{D}'(\mathbb{T}; X)$, where C' is a constant independent of f and j.

By the definition of $\Phi(\mathbb{R})$, we have $\operatorname{supp}(\phi_0) \subset [-2, 2]$, so $\operatorname{supp}(\phi_0) \cap \operatorname{supp}(\phi_j) = \emptyset$ when $j \geqslant 2$. Hence there exists $C'' > 0$ such that

$$\Big\| \sum_{k \in \mathbb{Z}} e_k \otimes \phi_0(k) \hat{f}(k) \Big\|_p \leqslant C'' \bigg(\Big\| \sum_{k \in \mathbb{Z}} e_k \otimes \varphi_0(k) \hat{f}(k) \Big\|_p + \Big\| \sum_{k \in \mathbb{Z}} e_k \otimes \varphi_1(k) \hat{f}(k) \Big\|_p \bigg).$$

We deduce that there exists a constant $\alpha > 0$ independent of f such that

$$\|f\|_{B_{p,q}^{s,\phi}} \leqslant \alpha \|f\|_{B_{p,q}^{s,\varphi}} \tag{2.2}$$

for $f \in \mathcal{D}'(\mathbb{T}; X)$. Thus the space $B_{p,q}^s(\mathbb{T}; X)$ is independent of $\phi \in \Phi(\mathbb{R})$ and the norms $\|\cdot\|_{B_{p,q}^{s,\phi}}$ are equivalent. We will simply denote $B_{p,q}^{s,\phi}(\mathbb{T}; X)$ by $B_{p,q}^s(\mathbb{T}; X)$, and $\|\cdot\|_{B_{p,q}^{s,\phi}}$ by $\|\cdot\|_{B_{p,q}^s}$ for some $\phi \in \Phi(\mathbb{R})$.

The following theorem summarizes some useful properties of $B_{p,q}^s(\mathbb{T}; X)$. Here $\hookrightarrow$ denotes that the natural injection is a continuous linear operator.

Theorem 2.3.

(i) *$B_{p,q}^s(\mathbb{T}; X)$ is a Banach space. Furthermore, when $p, q < \infty$, the set of all X-valued trigonometric polynomials is dense in $B_{p,q}^s(\mathbb{T}; X)$.*

(ii) *If $s > 0$, then $B_{p,q}^s(\mathbb{T}; X) \hookrightarrow L^p(\mathbb{T}; X)$.*

(iii) *One has*

$$\begin{aligned}
&B_{p,q_0}^s(\mathbb{T}; X) \hookrightarrow B_{p,q_1}^s(\mathbb{T}; X), \quad q_0 \leqslant q_1,\ 1 \leqslant p \leqslant \infty, \\
&B_{p,q_0}^{s+\epsilon}(\mathbb{T}; X) \hookrightarrow B_{p,q_1}^s(\mathbb{T}; X), \quad \epsilon > 0,\ 1 \leqslant q_0, q_1, p \leqslant \infty.
\end{aligned}$$

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(iv) *Let* $1 \leqslant p, q_0, q_1, q \leqslant \infty$, $s_0, s_1 \in \mathbb{R}$, $s_0 \neq s_1$, $0 < \theta < 1$ *and* $s = \theta s_1 + (1-\theta)s_0$. *Then*

$$(B^{s_0}_{p,q_0}(\mathbb{T};X), B^{s_1}_{p,q_1}(\mathbb{T};X))_{\theta,q} = B^s_{p,q}(\mathbb{T};X).$$

(v) *The lifting property: let* $f \in \mathcal{D}'(\mathbb{T};X)$ *and* $\alpha \in \mathbb{R}$. *Then* $f \in B^s_{p,q}(\mathbb{T};X)$ *if and only if*

$$\sum_{k\neq 0} e_k \otimes (ik)^\alpha \hat{f}(k) \in B^{s-\alpha}_{p,q}(\mathbb{T};X).$$

(vi) *Let* $s > 0$. *Then* $f \in B^{1+s}_{p,q}(\mathbb{T};X)$ *if and only if* f *is differentiable a.e. and* $f' \in B^s_{p,q}(\mathbb{T};X)$.

Proof. Parts (i), (ii) and (iii) follow easily from the definition of periodic Besov spaces. Part (iv) follows by the same argument as in the scalar Besov spaces case [**20**, Theorem 2.4.2, p. 64]. We will only give the proof for (v) and (vi).

Let $f \in B^s_{p,q}(\mathbb{T};X)$ and let $\phi = (\phi_j)_{j\geqslant 0} \in \Phi(\mathbb{R})$. Let $\varphi \in \mathcal{S}$ be such that $\varphi(t) = 1$ for $1/2 \leqslant |t| \leqslant 2$ and $\operatorname{supp}(\varphi) \subset [\frac{1}{4}, 4] \cup [-4, -\frac{1}{4}]$. Then by Proposition 2.2 for $j \geqslant 1$:

$$\begin{aligned}
\Big\| \sum_{k\neq 0} e_k \otimes \phi_j(k)(\mathrm{i}k)^\alpha \hat{f}(k) \Big\|_p &= \Big\| \sum_{2^{j-1}\leqslant |k| \leqslant 2^{j+1}} e_k \otimes (\mathrm{i}k)^\alpha \phi_j(k)\varphi(2^{-j}k)\hat{f}(k) \Big\|_p \\
&\leqslant \|\mathcal{F}^{-1}(x \mapsto \varphi(2^{-j}x)(\mathrm{i}x)^\alpha)\|_1 \Big\| \sum_{k\in\mathbb{Z}} e_k \otimes \phi_j(k)\hat{f}(k) \Big\|_p \\
&= 2^{j\alpha} \|\mathcal{F}^{-1}(x \mapsto \varphi(2^{-j}x)(2^{-j}\mathrm{i}x)^\alpha)\|_1 \Big\| \sum_{k\in\mathbb{Z}} e_k \otimes \phi_j(k)\hat{f}(k) \Big\|_p \\
&\leqslant 2^{j\alpha} C \|(x \mapsto \varphi(x)(\mathrm{i}x)^\alpha)\|_{W^2_1} \Big\| \sum_{k\in\mathbb{Z}} e_k \otimes \phi_j(k)\hat{f}(k) \Big\|_p \\
&\leqslant C' 2^{j\alpha} \Big\| \sum_{k\in\mathbb{Z}} e_k \otimes \phi_j(k)\hat{f}(k) \Big\|_p,
\end{aligned}$$

where C' depends only on φ and α. This implies that

$$\sum_{k\neq 0} e_k \otimes k^\alpha \hat{f}(k) \in B^{s-\alpha}_{p,q}(\mathbb{T};X).$$

Since $\alpha \in \mathbb{R}$ is arbitrary, we have proved (v).

Now let $f \in B^{1+s}_{p,q}(\mathbb{T};X)$. Then (v) implies that

$$\sum_{k\neq 0} e_k \otimes k\hat{f}(k) \in B^s_{p,q}(\mathbb{T};X).$$

By (ii) the functions f and $\sum_{k\neq 0} e_k \otimes k\hat{f}(k)$ belong to $L^p(\mathbb{T};X)$. So by Lemma 2.1 of [**3**], f is differentiable a.e. and

$$f'(t) = \sum_{k\neq 0} e_k \otimes \mathrm{i}k\hat{f}(k).$$

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This shows one implication of (vi). Conversely, assume that $g \in B^s_{p,q}(\mathbb{T};X)$ for some $s>0$, g is differentiable a.e. and $g' \in B^s_{p,q}(\mathbb{T};X)$. Then

$$\sum_{k\neq 0} e_k \otimes \mathrm{i}k\hat{g}(k) \in B^s_{p,q}(\mathbb{T};X).$$

Now (v) implies that $g \in B^{1+s}_{p,q}(\mathbb{T};X)$. □

3. Sobolev–Lebesgue spaces

In this section we will see that the periodic Besov spaces $B^s_{p,q}(\mathbb{T};X)$ contain several classical periodic function spaces as special cases.

We let

$$C(\mathbb{T};X) := \{f : \mathbb{T} \to X : f \text{ is continuous}\},$$
$$\|f\|_{C(\mathbb{T};X)} := \sup_{t\in\mathbb{T}} \|f(t)\|.$$

For $m \in \mathbb{N}$, we denote by $C^m(\mathbb{T},X)$ the space of all m-times continuously differentiable functions f with the norm

$$\|f\|_{C^m(\mathbb{T};X)} := \sum_{\alpha\leqslant m} \|f^{(\alpha)}\|_{C(\mathbb{T};X)}.$$

For $s>0$, let $s = s_1 + s_2$, where $s_1 \in \mathbb{Z}$ and $0 \leqslant s_2 < 1$. Then, if $s \notin \mathbb{Z}$,

$$C^s(\mathbb{T};X) := \left\{ f \in C^{s_1}(\mathbb{T};X) : \sup_{x\neq y,\ x,y\in\mathbb{T}} \frac{\|f^{s_1}(x) - f^{s_1}(y)\|}{|x-y|^{s_2}} < \infty \right\},$$

$$\|f\|_{C^s(\mathbb{T};X)} := \|f\|_{C^{s_1}(\mathbb{T};X)} + \sup_{x\neq y,\ x,y\in\mathbb{T}} \frac{\|f^{s_1}(x) - f^{s_1}(y)\|}{|x-y|^{s_2}}.$$

Let $1 \leqslant p < \infty$, $m \in \mathbb{N}$. Then the periodic Sobolev spaces are defined by

$$W^p_m(\mathbb{T};X) := \{f \in L^p(\mathbb{T};X) : f^{(\alpha)} \in L^p(\mathbb{T};X) \text{ for all } \alpha \leqslant m\},$$
$$\|f\|_{W^p_m(\mathbb{T};X)} := \sum_{\alpha\leqslant m} \|f^{(\alpha)}\|_p.$$

Furthermore, we let $W^p_0(\mathbb{T};X) = L^p(\mathbb{T};X)$. Let $s = [s] + \{s\}$ with $[s] \in \mathbb{N} \cup \{0\}$, $0 < \{s\} \leqslant 1$. If f is defined on $\mathbb{T}$ and extended 2π-periodically on $\mathbb{R}$, then we define

$$\begin{aligned} (\Delta^1_h f)(x) &:= f(x+h) - f(x), && h,x \in \mathbb{T}, \\ (\Delta^2_h f)(x) &:= (\Delta^1_h f)(x+h) - (\Delta^1_h f)(x), && h,x \in \mathbb{T}. \end{aligned}$$

We let

$$\mathfrak{C}^s(\mathbb{T};X) := \left\{ f \in C^{[s]}(\mathbb{T};X) : \sup_{h\in\mathbb{T},\ h\neq 0} |h|^{-\{s\}} \|\Delta^2_h f^{([s])}\|_{C(\mathbb{T};X)} < \infty \right\},$$

$$\|f\|_{\mathcal{C}^s(\mathbb{T};X)} = \|f\|_{C^{[s]}} + \sup_{h\in\mathbb{T},\ h\neq 0} |h|^{-\{s\}} \|\Delta^2_h f^{([s])}\|_{C(\mathbb{T};X)}.$$

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Let $s > 0$, $1 \leqslant p, q < \infty$. Then

$$\Lambda^s_{p,q}(\mathbb{T}; X) := \left\{ f \in W^p_{[s]}(\mathbb{T}; X) : \int_{\mathbb{R}} |h|^{-\{s\}q} \|\Delta^2_h f^{([s])}\|^q_p \frac{\mathrm{d}h}{|h|} < \infty \right\},$$

$$\|f\|_{\Lambda^s_{p,q}} := \|f\|_{W^p_{[s]}(\mathbb{T};X)} + \left(\int_{\mathbb{R}} |h|^{-\{s\}q} \|\Delta^2_h f^{([s])}\|^q_p \frac{\mathrm{d}h}{|h|} \right)^{1/q},$$

$$\Lambda^s_{p,\infty}(\mathbb{T}; X) := \left\{ f \in W^p_{[s]}(\mathbb{T}; X) : \sup_{h \in \mathbb{T},\ h \neq 0} |h|^{-\{s\}} \|\Delta^2_h f^{([s])}\|_p < \infty \right\},$$

$$\|f\|_{\Lambda^s_{p,\infty}} := \|f\|_{W^p_{[s]}} + \sup_{h \in \mathbb{T},\ h \neq 0} |h|^{-\{s\}} \|\Delta^2_h f^{([s])}\|_p.$$

Theorem 3.1.

(i) *If $s > 0$, then $B^s_{\infty,\infty}(\mathbb{T}; X) = \mathfrak{C}^s(\mathbb{T}; X)$.*

(ii) *If $s > 0$ and $s \notin \mathbb{N}$, then $B^s_{\infty,\infty}(\mathbb{T}; X) = C^s(\mathbb{T}; X)$.*

(iii) *If $1 \leqslant p < \infty$, $1 \leqslant q \leqslant \infty$ and $s > 0$, then $B^s_{p,q}(\mathbb{T}; X) = \Lambda^s_{p,q}(\mathbb{T}; X)$.*

This is the periodic counterpart in the vector-valued case of [**20**, Theorem 2.5.7 p. 90]. The proof is similar. The main ingredients are (iv) of Theorem 2.3 and the continuous embeddings

$$B^0_{\infty,1}(\mathbb{T}; X) \hookrightarrow C(\mathbb{T}; X) \hookrightarrow B^0_{\infty,\infty}(\mathbb{T}; X), \tag{3.1}$$

$$B^m_{\infty,1}(\mathbb{T}; X) \hookrightarrow C^m(\mathbb{T}; X) \hookrightarrow B^m_{\infty,\infty}(\mathbb{T}; X) \quad (m \in \mathbb{N}), \tag{3.2}$$

$$B^0_{p,1}(\mathbb{T}; X) \hookrightarrow L^p(\mathbb{T}; X) \hookrightarrow B^0_{p,\infty}(\mathbb{T}; X) \qquad (1 \leqslant p < \infty), \tag{3.3}$$

$$B^m_{p,1}(\mathbb{T}; X) \hookrightarrow W^p_m(\mathbb{T}; X) \hookrightarrow B^m_{p,\infty}(\mathbb{T}; X) \quad (1 \leqslant p < \infty,\ m \in \mathbb{N}). \tag{3.4}$$

The embeddings (3.1) and (3.3) are easy to prove by using the definition of periodic Besov spaces and Proposition 2.2. The embeddings (3.2) and (3.4) follow from (3.1) and (3.3) and (v) of Theorem 2.3.

4. Fourier multipliers on $B^s_{p,q}(\mathbb{T}; X)$

In this section we will establish Fourier multiplier theorems on $B^s_{p,q}(\mathbb{T}; X)$. We will see that the most general Marcinkiewicz Theorem on periodic L^p-spaces [**25**, Theorem 4.14, p. 232] is no longer true on $B^s_{p,q}(\mathbb{T}; X)$ when $p = 1$ or ∞ even if $X = \mathbb{C}$. However, it remains true in the operator-valued case for $1 < p < \infty$ whenever the underlying Banach space is a UMD-space. We will also show that under some stronger conditions (see (4.3) and (4.4)), the Marcinkiewicz Theorem on periodic Besov spaces remains true in the operator-valued case for all $1 \leqslant p, q \leqslant \infty$, $s \in \mathbb{R}$ and for all Banach spaces.

We will use the following lemma.

Lemma 4.1. *There exists a system $\phi = (\phi_j)_{j \geqslant 0} \in \Phi(\mathbb{R})$ such that $\phi_j \geqslant 0$ for $j \geqslant 0$ and $\phi_j(t) = 1$ whenever $j \geqslant 3$ and $|t| \in [7 \cdot 2^{j-3}, 3 \cdot 2^{j-1}]$.*

Proof. Let $\phi_0 \in \mathcal{S}$ be such that $\mathrm{supp}(\phi_0) = [-2, 2]$. Let $\phi_1 \in \mathcal{S}$ be such that $\phi_1 \geqslant 0$, $\mathrm{supp}(\phi_1) \subset [\frac{3}{2}, 7/2] \cup [-\frac{7}{2}, -\frac{3}{2}]$ and $\phi_1(t) = 1$ for $t \in [2, 3] \cup [-3, -2]$.

For $j \geqslant 2$, let $\phi_j = \phi_1(2^{-j+1}\cdot)$. Then for all $t \in \mathbb{R}$, we have $\sum_{j\geqslant 0} \phi_j(t) \neq 0$. Let $\varphi_j = \phi_j / \sum_{k\geqslant 0} \phi_k$. Then it is easy to verify that $\varphi = (\varphi_j)_{j\geqslant 0} \in \Phi(\mathbb{R})$ and $\phi_j(t) = 1$ for $|t| \in [7 \cdot 2^{j-3}, 3 \cdot 2^{j-1}]$ whenever $j \geqslant 3$. □

Let X and Y be Banach spaces and let $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X; Y)$. We will say that $(M_k)_{k\in\mathbb{Z}}$ is a *Fourier multiplier* from $B^s_{p,q}(\mathbb{T}; X)$ to $B^s_{p,q}(\mathbb{T}; Y)$ if for each $f \in B^s_{p,q}(\mathbb{T}; X)$, there exists $g \in B^s_{p,q}(\mathbb{T}, Y)$ such that $\hat{g}(k) = M_k \hat{f}(k)$ for $k \in \mathbb{Z}$. In this case, it follows from the Closed Graph Theorem that there exists $C > 0$ such that for $f \in B^s_{p,q}(\mathbb{T}; X)$ we have

$$\Big\| \sum_{k\in\mathbb{Z}} e_k \otimes M_k \hat{f}(k) \Big\|_{B^s_{p,q}} \leqslant C \|f\|_{B^s_{p,q}}.$$

The classical Marcinkiewicz Theorem on Fourier multipliers asserts that when a sequence $(m_k)_{k\in\mathbb{Z}} \subset \mathbb{C}$ is bounded and

$$\sup_{j\geqslant 0} \sum_{2^j \leqslant |k| \leqslant 2^{j+1}} |m_k - m_{k-1}| < \infty,$$

then $(m_k)_{k\in\mathbb{Z}}$ is a Fourier multiplier on $L^p(\mathbb{T}; \mathbb{C})$ for $1 < p < \infty$ [**25**, Theorem 4.14, p. 232]. We will consider operator-valued sequences $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X; Y)$ satisfying a similar estimate, i.e.

$$\sup_{j\geqslant 0} \sum_{2^j \leqslant |k| \leqslant 2^{j+1}} \|M_k - M_{k-1}\| < \infty, \qquad \sup_{k\in\mathbb{Z}} \|M_k\| < \infty. \tag{4.1}$$

We call condition (4.1) the *variational Marcinkiewicz condition.*

Theorem 4.2. *Let X be a Banach space and let $s \in \mathbb{R}$. Then each sequence $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X)$ satisfying (4.1) is a Fourier multiplier on $B^s_{p,q}(\mathbb{T}; X)$ if and only if $1 < p < \infty$ and X is a UMD-space.*

Proof. Assume that each sequence $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X)$ satisfying (4.1) is a Fourier multiplier on $B^s_{p,q}(\mathbb{T}; X)$. Let $\phi = (\phi_j)_{j\geqslant 0} \in \Phi(\mathbb{R})$ satisfying the conclusion of Lemma 4.1. Let $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X)$ be defined by $M_k = I$ when $|k| \in [7\cdot 2^{j-3}, 2^j]$ for some $j \geqslant 3$, $M_k = 0$ otherwise. Then $(M_k)_{k\in\mathbb{Z}}$ satisfies the condition (4.1). By our assumption, $(M_k)_{k\in\mathbb{Z}}$ is a Fourier multiplier on $B^s_{p,q}(\mathbb{T}; X)$. Then there exists $C > 0$ such that

$$\Big\| \sum_{k\in\mathbb{Z}} e_k \otimes M_k \hat{f}(k) \Big\|_{B^s_{p,q}} \leqslant C \|f\|_{B^s_{p,q}} \tag{4.2}$$

whenever $f \in B^s_{p,q}(\mathbb{T}; X)$. Let $j \geqslant 3$ be fixed and let f be an X-valued trigonometric polynomial such that $\hat{f}(k) = 0$ for $k \notin [7 \cdot 2^{j-3}, 3 \cdot 2^{j-1}]$. By Lemma 4.1 $\phi_j(t) = 1$ for

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$t \in [7\cdot 2^{j-3}, 3\cdot 2^{j-1}]$. We deduce that $\phi_k(t) = 0$ for $t \in [7\cdot 2^{j-3}, 3\cdot 2^{j-1}]$ and $k \neq j$. Hence

$$\begin{aligned}\Big\| \sum_{k\in\mathbb{Z}} e_k \otimes M_k \hat{f}(k) \Big\|^q_{B^s_{p,q}} &= 2^{jsq} \Big\| \sum_{k\in\mathbb{Z}} e_k \otimes \phi_j(k) M_k \hat{f}(k) \Big\|^q_p \\ &= 2^{jsq} \Big\| \sum_{k=7\cdot 2^{j-3}}^{2^j} e_k \otimes \hat{f}(k) \Big\|^q_p\end{aligned}$$

and

$$\|f\|^q_{B^s_{p,q}} = 2^{jsq} \Big\| \sum_{k=7\cdot 2^{j-3}}^{3\cdot 2^{j-1}} e_k \otimes \hat{f}(k) \Big\|^q_p.$$

It follows from this and (4.2) that

$$\Big\| \sum_{k=7\cdot 2^{j-3}-2^j}^{0} e_k \otimes x_k \Big\|_p \leqslant C \Big\| \sum_{k=7\cdot 2^{j-3}-2^j}^{2^{j-1}} e_k \otimes x_k \Big\|_p$$

for $j \geqslant 3$ and $x_k \in X$. Thus the Riesz projection is bounded on $L^p(\mathbb{T}; X)$. We deduce that $1 < p < \infty$ and X is a UMD-space [**7**, **9**] (see also [**21**]).

Conversely, assume that $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, $s \in \mathbb{R}$, X is a UMD-space and $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X)$ satisfies (4.1). We will show that $(M_k)_{k\in\mathbb{Z}}$ is a Fourier multiplier on $B^s_{p,q}(\mathbb{T}; X)$. Let $\phi = (\phi_j)_{j\geqslant 0} \in \Phi(\mathbb{R})$, $j \geqslant 1$, and $f \in B^s_{p,q}(\mathbb{T}; X)$. We have

$$\begin{aligned}&\Big\| \sum_{k\in\mathbb{Z}} e_k \otimes \phi_j(k) M_k \hat{f}(k) \Big\|_p \\ &\qquad\leqslant \Big\| \sum_{k=2^{j-1}}^{2^{j+1}} e_k \otimes \phi_j(k) M_k \hat{f}(k) \Big\|_p + \Big\| \sum_{k=-2^{j+1}}^{-2^{j-1}} e_k \otimes \phi_j(k) M_k \hat{f}(k) \Big\|_p.\end{aligned}$$

Let

$$S_n = \sum_{k=2^{j-1}}^{n} e_k \otimes \phi_j(k) \hat{f}(k) \quad \text{for } 2^{j-1} \leqslant n \leqslant 2^{j+1}.$$

Since X is a UMD-space and $1 < p < \infty$, the Riesz projection is bounded on $L^p(\mathbb{T}; X)$ [**7**, **9**] (see also [**21**]). Thus

$$\begin{aligned}\Big\| \sum_{k=2^{j-1}}^{2^{j+1}} e_k \otimes \phi_j(k) M_k \hat{f}(k) \Big\|_p &= \Big\| \sum_{k=2^{j-1}}^{2^{j+1}-1} (M_k - M_{k+1}) S_k + M_{2^{j+1}} S_{2^{j+1}} \Big\|_p \\ &\leqslant \sum_{k=2^{j-1}}^{2^{j+1}-1} \|M_k - M_{k+1}\| \, \|S_k\|_p + \|M_{2^{j+1}}\| \, \|S_{2^{j+1}}\|_p \\ &\leqslant C \sup_{2^{j-1}\leqslant n\leqslant 2^{j+1}} \Big\| \sum_{k=2^{j-1}}^{n} e_k \otimes \phi_j(k) \hat{f}(k) \Big\|_p \\ &\leqslant C' \Big\| \sum_{k\in\mathbb{Z}} e_k \otimes \phi_j(k) \hat{f}(k) \Big\|_p,\end{aligned}$$

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where C, C' are constants depending only on $(M_k)_{k\in\mathbb{Z}}$ and the norm of the Riesz projection on $L^p(\mathbb{T};X)$. Similarly we show that

$$\Big\|\sum_{k=-2^{j+1}}^{-2^{j-1}} e_k\otimes\phi_j(k)M_k\hat f(k)\Big\|_p \leqslant C'\Big\|\sum_{k\in\mathbb{Z}} e_k\otimes\phi_j(k)\hat f(k)\Big\|_p.$$

Hence

$$\Big\|\sum_{k\in\mathbb{Z}} e_k\otimes\phi_j(k)M_k\hat f(k)\Big\|_p \leqslant 2C'\Big\|\sum_{k\in\mathbb{Z}} e_k\otimes\phi_j(k)\hat f(k)\Big\|_p.$$

On the other hand, it is obvious that there exists $C''>0$ depending only on $(M_k)_{k\in\mathbb{Z}}$ and $(\phi_j)_{j\geqslant 0}$ such that

$$\Big\|\sum_{k\in\mathbb{Z}} e_k\otimes\phi_0(k)M_k\hat f(k)\Big\|_p \leqslant C''\Big\|\sum_{k\in\mathbb{Z}} e_k\otimes\phi_0(k)\hat f(k)\Big\|_p.$$

We deduce that there exists a constant $C'''>0$ such that

$$\Big\|\sum_{k\in\mathbb{Z}} e_k\otimes M_k\hat f(k)\Big\|_{B^s_{p,q}} \leqslant C'''\|f\|_{B^s_{p,q}},$$

i.e. $(M_k)_{k\in\mathbb{Z}}$ is a Fourier multiplier on $B^s_{p,q}(\mathbb{T};X)$. □

The case of Hölder continuous functions is of particular interest. We reformulate in particular the 'negative' assertion of Theorem 4.2 which even holds in the scalar case.

Corollary 4.3. *Let $0<\alpha<1$. Then there exists a sequence $(M_k)_{k\in\mathbb{Z}}$ in $\mathbb{C}$ satisfying the variational Marcinkiewicz condition (4.1), which is not a Fourier multiplier from $C^\alpha_{2\pi}$ into $C^\alpha_{2\pi}$, where $C^\alpha_{2\pi}$ denotes the space of all 2π-periodic and s-Hölder continuous functions on $\mathbb{R}$.*

By the first part of the proof of Theorem 4.2, the scalar sequence $(m_k)_{k\in\mathbb{Z}}$ given by

$$m_k=\begin{cases}1, & |k|\in\bigcup_{j\geqslant 3}[\tfrac{7}{8}2^j,2^j],\\ 0, & \text{otherwise,}\end{cases}$$

satisfies the variational Marcinkiewicz condition (4.1), but it is not a Fourier multiplier from $C^\alpha_{2\pi}$ into $C^\alpha_{2\pi}$. Things are different if we consider an assumption that is slightly stronger than (4.1). The following condition on sequences $(M_k)_{k\in\mathbb{Z}}\subset\mathcal{L}(X;Y)$ was introduced in [**3**] to study Fourier multipliers in the L^p-context:

$$\sup_{k\in\mathbb{Z}}\|M_k\|<\infty,\qquad \sup_{k\in\mathbb{Z}}\|k(M_{k+1}-M_k)\|<\infty, \tag{4.3}$$

$$\sup_{k\in\mathbb{Z}}\|k^2(M_{k+1}-2M_k+M_{k-1})\|<\infty. \tag{4.4}$$

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It is easy to verify that condition (4.3) is stronger than condition (4.1).

Recall that a Banach space X has *Fourier type* $r \in [1,2]$ if there exists $C_r > 0$ such that

$$\|\mathcal{F}f\|_{r'} \leqslant C_r \|f\|_r, \quad f \in L^r(\mathbb{R}; X), \tag{4.5}$$

where $(1/r) + (1/r') = 1$ [**17**], the smallest constant C_r in (4.5) is called the Fourier type constant of X. The trivial estimate $\|\mathcal{F}f\|_\infty \leqslant (1/\sqrt{2\pi})\|f\|_1$ shows that each Banach space has Fourier type 1. A Banach space has Fourier type 2 if and only if it is isomorphic to a Hilbert space [**15**] (see also [**16**, pp. 73, 74]). A space $L^q(\Omega, \Sigma, \mu)$ has Fourier type $r = \min\{q, q'\}$ [**17**]. Each closed subspace and each quotient space of a Banach space X has the same Fourier type as X. Bourgain has shown that each B-convex Banach space (thus, in particular, each uniformly convex Banach space) has some non trivial Fourier type $r > 1$ [**6**, **8**].

We will use the following lemma. Its proof is similar to that of Theorem 4.3 of [**13**] and two ingredients are essential. The first one is the continuous injection $W_1^p(\mathbb{R}; X) \hookrightarrow B_{p,1}^s(\mathbb{R}; X)$ for $p > 1$, $0 < s < 1$, where

$$W_1^p(\mathbb{R}; X) := \{f \in L^p(\mathbb{R}; X) : f' \in L^p(\mathbb{R}; X)\}$$

equipped with the norm $\|f\|_{W_1^p} = \|f\|_p + \|f'\|_p$, and $B_{p,q}^s(\mathbb{R}; X)$ is the X-valued Besov space on $\mathbb{R}$ [**1**]. The second one is that when X has Fourier type $r \in (1,2]$, then the inverse Fourier transform defines a bounded linear operator $\mathcal{F}^{-1} : B_{r,1}^{1/r}(\mathbb{R}; X) \to L^1(\mathbb{R}; X)$ [**13**].

Lemma 4.4. *Let X and Y be Banach spaces having Fourier type $r \in (1,2]$, $1 \leqslant p \leqslant \infty$ and let $M \in C_c(\mathbb{R}; \mathcal{L}(X,Y)) \cap \mathcal{F}L^1(\mathbb{R}; \mathcal{L}(X,Y))$. Then*

$$\Big\| \sum_{k \in \mathbb{Z}} e_k \otimes M(k)\hat{f}(k) \Big\|_p \leqslant C_p \eta_r(M) \|f\|_p$$

whenever $f \in L^p(\mathbb{T}; X)$ is a trigonometric polynomial, where C_p is a constant only depending on p and the Fourier type constants of X and Y, and $\eta_r(M) := \inf\{\|M(a\cdot)\|_{W_1^r} : a > 0\}$.

Now we can prove the following general multiplier theorem.

Theorem 4.5.

(i) *Let X and Y be arbitrary Banach spaces and let $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X; Y)$ be a sequence satisfying conditions (4.3) and (4.4). Then for $1 \leqslant p, q \leqslant \infty$, $s \in \mathbb{R}$, $(M_k)_{k\in\mathbb{Z}}$ is a Fourier multiplier from $B_{p,q}^s(\mathbb{T}; X)$ to $B_{p,q}^s(\mathbb{T}; Y)$.*

(ii) *Let X and Y be Banach spaces having Fourier type $r \in (1,2]$ and let $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X; Y)$ be a sequence satisfying condition (4.3). Then for $1 \leqslant p, q \leqslant \infty$, $s \in \mathbb{R}$, $(M_k)_{k\in\mathbb{Z}}$ is a Fourier multiplier from $B_{p,q}^s(\mathbb{T}; X)$ to $B_{p,q}^s(\mathbb{T}; Y)$.*

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Proof. Let $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X;Y)$ be fixed and let $\phi = (\phi_j)_{j\geqslant 0} \in \Phi(\mathbb{R})$. In order to show that $(M_k)_{k\in\mathbb{Z}}$ is a Fourier multiplier from $B^s_{p,q}(\mathbb{T};X)$ to $B^s_{p,q}(\mathbb{T};Y)$, it will suffice to show that there exists $C>0$ such that for $j\geqslant 0$ and $f\in B^s_{p,q}(\mathbb{T};X)$,

$$\Big\|\sum_{k\in\mathbb{Z}} e_k\otimes\phi_j(k)M_k\hat{f}(k)\Big\|_p \leqslant C\Big\|\sum_{k\in\mathbb{Z}} e_k\otimes\phi_j(k)\hat{f}(k)\Big\|_p.$$

Let $j\geqslant 1$ be fixed. Define $\tilde{M}_j$ as the piecewise affine function in $\mathbb{C}_c(\mathbb{R};\mathcal{L}(X;Y))$ by $\tilde{M}_j(x)=0$ if $|x|\geqslant 2^{j+2}$ or $|x|\leqslant 2^{j-2}$, $\tilde{M}_j(k)=M_k$ for some $k\in\mathbb{Z}$ satisfying $2^{j-1}\leqslant |k|\leqslant 2^{j+1}$ and which is affine on $[k,k+1]$ for all $k\in\mathbb{Z}$.

By Proposition 2.2,

$$\begin{aligned}\Big\|\sum_{k\in\mathbb{Z}} e_k\otimes\phi_j(k)M_k\hat{f}(k)\Big\|_p &= \Big\|\sum_{2^{j-1}\leqslant|k|\leqslant 2^{j+1}} e_k\otimes\phi_j(k)\tilde{M}_j(k)\hat{f}(k)\Big\|_p \\ &\leqslant \frac{1}{\sqrt{2\pi}}\|\mathcal{F}^{-1}\tilde{M}_j\|_1\Big\|\sum_{k\in\mathbb{Z}} e_k\otimes\phi_j(k)\hat{f}(k)\Big\|_p.\end{aligned}$$

To prove (i) it will suffice to show that $\sup_{j\geqslant 1}\|\mathcal{F}^{-1}\tilde{M}_j\|_1<\infty$. Let $N_j=\tilde{M}_j(2^j\cdot)$. Notice that

$$\operatorname{supp}(N_j)\subset\{x:\tfrac14\leqslant|x|\leqslant 4\}\quad\text{and}\quad \sup_{x\in\mathbb{R}}\|N_j(x)\|\leqslant\sup_{k\in\mathbb{Z}}\|M_k\|.$$

Therefore,

$$\sup_{j\geqslant 1,\ x\in\mathbb{R}}\|\mathcal{F}^{-1}N_j(x)\|\leqslant 8\sup_{k\in\mathbb{Z}}\|M_k\|. \tag{4.6}$$

On the other hand, for $x\neq 0$,

$$\begin{aligned}\sqrt{2\pi}(\mathcal{F}^{-1}N_j)(x) &= \int_{\mathbb{R}}\tilde{M}_j(2^jy)\mathrm{e}^{\mathrm{i}xy}\,\mathrm{d}y \\ &= 2^{-j}\int_{2^{j-2}\leqslant|y|\leqslant 2^{j+2}}\tilde{M}_j(y)\mathrm{e}^{\mathrm{i}2^{-j}xy}\,\mathrm{d}y.\end{aligned}$$

Integrating by parts twice we obtain

$$\begin{aligned}\sqrt{2\pi}(\mathcal{F}^{-1}N_j)(x) &= M_{-2^{j+1}}(\mathrm{e}^{-\mathrm{i}2x}-\mathrm{e}^{-\mathrm{i}4x})x^{-2}/2 - M_{2^{j+1}}(\mathrm{e}^{\mathrm{i}4x}-\mathrm{e}^{\mathrm{i}2x})x^{-2}/2 \\ &\quad - 4M_{-2^{j-1}}(\mathrm{e}^{-\mathrm{i}x/4}-\mathrm{e}^{-\mathrm{i}x/2})x^{-2} + 4M_{2^{j-1}}(\mathrm{e}^{\mathrm{i}x/2}-\mathrm{e}^{\mathrm{i}x/4})x^{-2} \\ &\quad + \sum_{k=2^{j-1}}^{2^{j+1}-1}(M_{k+1}-M_k)(\mathrm{e}^{\mathrm{i}2^{-j}(k+1)x}-\mathrm{e}^{\mathrm{i}2^{j}kx})2^jx^{-2} \\ &\quad + \sum_{k=-2^{j+1}}^{-2^{j-1}-1}(M_{k+1}-M_k)(\mathrm{e}^{\mathrm{i}2^{-j}(k+1)x}-\mathrm{e}^{\mathrm{i}2^{j}kx})2^jx^{-2}.\end{aligned}$$

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The first four terms are bounded by $12\sup_{k\in\mathbb{Z}}\|M_k\|/x^2$. For the fifth term,

$$\begin{aligned}
&\bigg\|2^j x^{-2}\sum_{k=2^{j-1}}^{2^{j+1}-1}(M_{k+1}-M_k)(\mathrm{e}^{\mathrm{i}2^{-j}(k+1)x}-\mathrm{e}^{\mathrm{i}2^j kx})\bigg\| \\
&\quad=\bigg\|2^j x^{-2}\sum_{k=2^{j-1}}^{2^{j+1}-2}(M_{k+2}-2M_{k+1}+M_k)\mathrm{e}^{\mathrm{i}2^j(k+1)x} \\
&\qquad\qquad -2^j x^{-2}(M_{2^{j-1}+1}-M_{2^{j-1}})\mathrm{e}^{\mathrm{i}x/2}+2^j x^{-2}(M_{2^{j+1}}-M_{2^{j+1}-1})\mathrm{e}^{\mathrm{i}2x}\bigg\| \\
&\quad\leqslant\Big(16\sup_{k\in\mathbb{Z}}(k^2\|M_{k+2}-2M_{k+1}+M_k\|)+4\sup_{k\in\mathbb{Z}}\|M_k\|\Big)x^{-2}.
\end{aligned}$$

The same argument gives a similar estimate for the sixth term. We deduce that

$$\sup_{j\geqslant 1,\ x\in\mathbb{R}}\|x^2\mathcal{F}^{-1}N_j(x)\|<\infty. \tag{4.7}$$

This estimate together with (4.6) shows that

$$\sup_{j\geqslant 1}\|\mathcal{F}^{-1}\tilde{M}_j\|_1=\sup_{j\geqslant 1}\|\mathcal{F}^{-1}N_j\|_1<\infty.$$

We have proved (i).

Now assume that X and Y have Fourier type $r\in(1,2]$ and let $(M_k)_{k\in\mathbb{Z}}\subset\mathcal{L}(X,Y)$ satisfy (4.3). Define for $j\geqslant 1$ the same function

$$\tilde{M}_j\in C_c(\mathbb{R};\mathcal{L}(X,Y))\cap\mathcal{F}L^1(\mathbb{R};\mathcal{L}(X,Y))$$

as above. Applying Lemma 4.4 to $\tilde{M}_j$ we have

$$\begin{aligned}
\bigg\|\sum_{k\in\mathbb{Z}}e_k\otimes\phi_j(k)M_k\hat{f}(k)\bigg\|_p&=\bigg\|\sum_{2^{j-1}\leqslant|k|\leqslant 2^{j+1}}e_k\otimes\phi_j(k)\tilde{M}_j(k)\hat{f}(k)\bigg\|_p \\
&\leqslant C_p\eta_r(\tilde{M}_j)\bigg\|\sum_{k\in\mathbb{Z}}e_k\otimes\phi_j(k)\hat{f}(k)\bigg\|_p \\
&\leqslant C_p\|\tilde{M}_j(2^j\cdot)\|_{W_1^r}\bigg\|\sum_{k\in\mathbb{Z}}e_k\otimes\phi_j(k)\hat{f}(k)\bigg\|_p.
\end{aligned}$$

If we set $N_j=\tilde{M}_j(2^j\cdot)$, then

$$\begin{gathered}
\operatorname{supp}(N_j)\subset\{x:\tfrac14\leqslant|x|\leqslant 4\}, \\
\sup_{x\in\mathbb{R}}\|N_j(x)\|\leqslant\sup_{k\in\mathbb{Z}}\|M_k\|, \\
\sup_{x\in\mathbb{R}}\|N_j'(x)\|\leqslant 2\sup_{k\in\mathbb{Z}}\|k(M_{k+1}-M_k)\|+8\sup_{k\in\mathbb{Z}}\|M_k\|.
\end{gathered}$$

We deduce that $\sup_{j\geqslant 1}\|\tilde{M}_j(2^j\cdot)\|_{W_1^r}<\infty$. This proves (ii) and finishes the proof. $\square$

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Remark 4.6.

(i) Let X be an arbitrary Banach space and let $1 \leqslant p, q \leqslant \infty$, $s \in \mathbb{R}$. Let $M_k = I$ for $k \geqslant 0$ and $M_k = 0$ for $k < 0$. Then $(M_k)_{k\in\mathbb{Z}}$ defines a Fourier multiplier on $B^s_{p,q}(\mathbb{T}; X)$ by Theorem 4.5. The associated operator is called the *Riesz projection*. Similarly, letting $M_k = \mathrm{i} \cdot (\operatorname{sgn} k)I$ defines a Fourier multiplier on $B^s_{p,q}(\mathbb{T}; X)$. The associated operator is called the *Hilbert transform*.

(ii) For vector-valued Besov spaces on the real line, the corresponding result of the first part of Theorem 4.5 has been established by Amann [**1**] (for arbitrary Banach spaces) under a suitable condition on the growth at infinity of the first and the second derivatives of the multipliers. The non-periodic counterpart of the second part of Theorem 4.5 has been established recently by Girardi and Weis [**13**] (for Banach spaces having a non-trivial Fourier type) under a suitable condition on the growth of the first derivative of the multipliers. Note that in Theorem 4.5 the quantities $M_{k+1} - M_k$ and $M_{k+1} - 2M_k + M_{k-1}$ also somehow represent the first and the second derivative of the sequence $(M_k)_{k\in\mathbb{Z}}$.

(iii) In the case $p = q = \infty$, $s \in (0, 1)$, the Besov space $B^s_{p,q}(\mathbb{T}, X)$ is nothing else but the space of all periodic functions which are Hölder continuous of index s. For this case, Theorem 4.5 (ii) has been proved in [**5**] by a direct estimate which does not use dyadic decomposition or interpolation.

5. Applications

The Marcinkiewicz-type theorems established in the previous section enable us to study maximal regularity in vector-valued periodic Besov spaces for evolution equations with periodic boundary conditions.

Let A be a closed operator on X. We consider the periodic problem

$$P_{\text{per}} \begin{cases} u'(t) = Au(t) + f(t), & t \in [-\pi, \pi], \\ u(-\pi) = u(\pi), \end{cases}$$

where $f \in B^s_{p,q}(\mathbb{T}; X)$ for some $1 \leqslant p, q \leqslant \infty$ and $s > 0$. We say that the problem P_{per} has $B^s_{p,q}$-*maximal regularity* if for each $f \in B^s_{p,q}(\mathbb{T}; X)$, there exists a unique $u \in B^{1+s}_{p,q}(\mathbb{T}; X)$ (by (vi) of Theorem 2.3, u is differentiable a.e. and $u' \in B^s_{p,q}(\mathbb{T}; X)$), such that $u(t) \in D(A)$ and $u'(t) = Au(t) + f(t)$ for almost all $t \in [-\pi, \pi]$.

By the Closed Graph Theorem, it is easy to see that when the problem P_{per} has $B^s_{p,q}$-maximal regularity, then there exists $C > 0$ independent of f and u such that

$$\|u\|_{B^{1+s}_{p,q}} + \|Au\|_{B^s_{p,q}} \leqslant C\|f\|_{B^s_{p,q}}. \tag{5.1}$$

We will prove the following result.

Theorem 5.1. *Let A be a closed operator on X. Then the following assertions are equivalent.*

(i) $\mathrm{i}\mathbb{Z} \subset \rho(A)$ *and* $\sup_{k\in\mathbb{Z}} \|k(\mathrm{i}k - A)^{-1}\| < \infty$.

(ii) *The problem* P_{per} *has* $B^s_{p,q}$*-maximal regularity for some (equivalently, for all)* $s > 0$, $1 \leqslant p, q \leqslant \infty$.

Proof. (ii) $\Rightarrow$ (i). Assume that the problem P_{per} has $B^s_{p,q}$-maximal regularity for some $s > 0$, $1 \leqslant p, q \leqslant \infty$ and let $x \in X$ be fixed. Let $k \in \mathbb{Z}$ and let $f = e_k \otimes x$. It is obvious that $f \in B^s_{p,q}(\mathbb{T}; X)$. Hence there exists $u \in B^{1+s}_{p,q}(\mathbb{T}; X)$ such that $u(t) \in D(A)$ and $u'(t) = Au(t) + f(t)$ for almost all $t \in [-\pi, \pi]$ by assumption. By Lemma 3.1 of [**3**], this implies that $\hat{u}(k) \in D(A)$ and $\mathrm{i}k\hat{u}(k) = A\hat{u}(k) + x$. Thus $(\mathrm{i}k - A)\hat{u}(k) = x$. This shows that $\mathrm{i}k - A$ is surjective. If $(\mathrm{i}k - A)x = 0$, then $u = e_k \otimes x \in B^{1+s}_{p,q}(\mathbb{T}; X) \cap B^s_{p,q}(\mathbb{T}; D(A))$ defines a solution of $u' = Au$, $u(-\pi) = u(\pi)$. Hence $u = 0$ by the uniqueness of the solution of P_{per}. We have shown that $\mathrm{i}k - A$ is bijective. Since A is closed we deduce that $\mathrm{i}k \in \rho(A)$.

We consider $f = e_k \otimes x$ for some $k \in \mathbb{Z}$ and $x \in X$, the solution u is given by $u = (\mathrm{i}k - A)^{-1} e_k \otimes x$. The estimate (5.1) implies that

$$\|k(\mathrm{i}k - A)^{-1}x\| \leqslant C\|x\|.$$

Hence $\sup_{k\in\mathbb{Z}} \|k(\mathrm{i}k - A)^{-1}\| < \infty$. This proves (ii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii). Assume that $\mathrm{i}\mathbb{Z} \subset \rho(A)$ and $\sup_{k\in\mathbb{Z}} \|k(\mathrm{i}k - A)^{-1}\| < \infty$. Let $M_k = \mathrm{i}k(\mathrm{i}k - A)^{-1}$, $N_k = (\mathrm{i}k - A)^{-1}$. Then $(M_k)_{k\in\mathbb{Z}}$ and $(N_k)_{k\in\mathbb{Z}}$ satisfy conditions (4.3) and (4.4). By Theorem 4.5, $(M_k)_{k\in\mathbb{Z}}$ and $(N_k)_{k\in\mathbb{Z}}$ are Fourier multipliers on $B^s_{p,q}(\mathbb{T}; X)$. For each $f \in B^s_{p,q}(\mathbb{T}; X)$, there exist $u, v \in B^s_{p,q}(\mathbb{T}; X)$ such that for $k \in \mathbb{Z}$, we have $\hat{u}(k) = \mathrm{i}k(\mathrm{i}k - A)^{-1}\hat{f}(k)$ and $\hat{v}(k) = (\mathrm{i}k - A)^{-1}\hat{f}(k)$. By Lemma 2.1 of [**3**], v is differentiable a.e. and $v' = u$. By (vi) of Theorem 2.3, this implies that $v \in B^{1+s}_{p,q}(\mathbb{T}; X)$. $(\mathrm{i}k - A)\hat{v}(k) = \hat{f}(k)$ together with Lemma 3.1 of [**3**] implies that $v(t) \in D(A)$ and $v'(t) = Av(t) + f(t)$ for almost all $t \in [-\pi, \pi]$. This proves (i) $\Rightarrow$ (ii). □

Remark 5.2. The case $p = q = \infty$ and $0 < s < 1$ in which $B^s_{\infty,\infty}(\mathbb{T}, X) = C^\alpha(\mathbb{T}, X)$ has been considered in [**5**].

Theorem 4.5 may also be applied to the second order problem with periodic boundary conditions and gives a necessary and sufficient condition for such a problem to have $B^s_{p,q}$-maximal regularity. The proof of the following result is similar to that of Theorem 6.1 in [**3**]. We omit the details.

Theorem 5.3. *Let* A *be a closed operator on* X *and let* $1 \leqslant p, q \leqslant \infty$, $s > 0$. *The following assertions are equivalent.*

(i) *For all* $f \in B^s_{p,q}(\mathbb{T}; X)$, *there exists a unique*

$$u \in B^s_{p,q}(\mathbb{T}; D(A)) \cap B^{2+s}_{p,q}(\mathbb{T}; X)$$

such that

$$u'' + Au = f \text{ a.e.}$$

(ii) *One has* $k^2 \in \varrho(A)$ *for all* $k \in \mathbb{Z}$ *and* $\{k^2 R(k^2, A) : k \in \mathbb{Z}\}$ *is bounded.*

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