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COMPUTING WITH NILPOTENT ORBITS IN SIMPLE LIE ALGEBRAS OF EXCEPTIONAL TYPE

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Abstract

Let G be a simple algebraic group over an algebraically closed field with Lie algebra $\mathfrak{g}$. Then the orbits of nilpotent elements of $\mathfrak{g}$ under the adjoint action of G have been classified. We describe a simple algorithm for finding a representative of a nilpotent orbit. We use this to compute lists of representatives of these orbits for the Lie algebras of exceptional type. Then we give two applications. The first one concerns settling a conjecture by Elashvili on the index of centralizers of nilpotent orbits, for the case where the Lie algebra is of exceptional type. The second deals with minimal dimensions of centralizers in centralizers.

1. *Introduction*

Let G be a simple algebraic group over an algebraically closed field of characteristic 0. Let $\mathfrak{g}$ denote its Lie algebra. Then G acts on $\mathfrak{g}$ via the adjoint representation. It is a natural question what the G-orbits in $\mathfrak{g}$ are. Recall that an element $e \in \mathfrak{g}$ is said to be nilpotent if the map $\mathrm{ad}e : \mathfrak{g} \to \mathfrak{g}$ is nilpotent. Now the G-orbits of nilpotent elements in $\mathfrak{g}$ are called nilpotent orbits. These have drawn a lot of attention in the past decades. On some occasions it turns out that using conceptual arguments to prove their properties is a lot harder for the exceptional types than it is for the classical types. However, for the former an approach based on a case by case analysis is possible. It is the objective of this paper to describe how this can be carried out using computer calculations.

The nilpotent orbits in $\mathfrak{g}$ are classified in terms of so-called weighted Dynkin diagrams. The first problem that we consider is to find a nilpotent element in $\mathfrak{g}$ given the corresponding weighted Dynkin diagram. We describe a straightforward algorithm for this (Section 3). The algorithm is used to compute lists of explicit representatives of the nilpotent orbits in the Lie algebras of exceptional type. They are listed in Appendix A.

We use these lists to prove Elashvili's conjecture for the exceptional types by computer calculations. This conjecture concerns the index of centralizers of nilpotent elements. The concept of index is defined as follows. Let K be a finite-dimensional Lie algebra, and let K^* denote the dual space. For $f \in K^*$ set $K^f = \{x \in K \mid f([x,y]) = 0 \text{ for all } y \in K\}$. Then the index of K is defined as the number

$$\mathrm{ind}(K) = \inf_{f \in K^*} \dim K^f.$$

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For semisimple Lie algebras in characteristic zero it is known that the index is equal to the rank ([4], Proposition 1.11.12).

By $C_{\mathfrak{g}}(x)$ we denote the centralizer of of $x \in \mathfrak{g}$.

CONJECTURE 1 (Elashvili). *Let $\mathfrak{g}$ be a semisimple Lie algebra over an algebraically closed field of characteristic* 0. *Let $x \in \mathfrak{g}$. Then* $\operatorname{ind}(C_{\mathfrak{g}}(x))$ *is equal to the rank of* $\mathfrak{g}$.

This conjecture has recently received renewed attention, cf. [5], [6], [10]. Its proof immediately reduces to the case where $\mathfrak{g}$ is simple, and x nilpotent (cf. [6], §3). Also an inequality of Vinberg states that $\operatorname{ind}(C_{\mathfrak{g}}(x))$ is at least the rank of $\mathfrak{g}$ (see [6], 1.6, 1.7). The conjecture has been proved for $\mathfrak{g}$ of classical type in [11] (see also the discussion in [5]). In Section 4 we report on computer calculations that settle the conjecture for the exceptional types.

In [9] the question is considered whether for a given nilpotent $e \in \mathfrak{g}$ there exists $x \in C_{\mathfrak{g}}(e)$ such that the dimension of $C_{\mathfrak{g}}(e, x)$ equals the rank of $\mathfrak{g}$. There an example is given where such an x does not exist, for the case where $\mathfrak{g}$ is of type F_4. In Section 5 we approach this question using our lists of representatives of nilpotent orbits. This way we are able to give a complete list of all e for which such an x does not exist, in all exceptional types. For the Lie algebra of type E_8 this solves an open problem from [9]. For type G_2 this corrects a statement in [9].

The paper ends with two appendices. The first contains the lists of representatives of nilpotent orbits. The second (Appendix B) has lists of positive roots as they appear in the computer algebra system GAP. They have been added to help reading the tables of Appendix A.

All algorithms described in this paper have been implemented in the language of the computer algebra system GAP4. The implementations are available from

http://www.lms.ac.uk/jcm/11/lms2007-059/appendix-a

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2. *Preliminaries on nilpotent orbits*

In this section we give a short overview of the theory behind the classification of nilpotent orbits. For more detailed accounts we refer to [2], [3].

Let $e \in \mathfrak{g}$ be a nilpotent element. Then by the Jacobson–Morozov theorem e lies in a subalgebra of $\mathfrak{g}$ that is isomorphic to $\mathfrak{sl}_2$. In other words, there are elements $f, h \in \mathfrak{g}$ with $[e, f] = h$, $[h, f] = -2f$, $[h, e] = 2e$. In this case we say that (f, h, e) is an $\mathfrak{sl}_2$-triple.

Now let (f, h, e) be an $\mathfrak{sl}_2$-triple. Then by the representation theory of $\mathfrak{sl}_2$ we get a direct sum decomposition $\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}(k)$, where $\mathfrak{g}(k) = \{x \in \mathfrak{g} \mid [h, x] = kx\}$. Fix a Cartan subalgebra H of $\mathfrak{g}$ with $h \in H$. Let Φ be the corresponding root system of $\mathfrak{g}$. For $\alpha \in \Phi$ we let x_α be a corresponding root vector. For each α there is a $k \in \mathbb{Z}$ with $x_\alpha \in \mathfrak{g}(k)$. We write $\eta(\alpha) = k$. It can be shown that there exists a basis of simple roots $\Delta \subset \Phi$ such that $\eta(\alpha) \geqslant 0$ for all $\alpha \in \Delta$. Furthermore, for such a Δ

we have $\eta(\alpha) \in \{0, 1, 2\}$ for all $\alpha \in \Delta$. Write $\Delta = \{\alpha_1, \ldots, \alpha_l\}$. Then the Dynkin diagram of Φ has l nodes, the ith node corresponding to α_i. Now to each node we add the label $\eta(\alpha_i)$; the result is called the weighted Dynkin diagram. It is denoted $\Delta(e)$, and it depends only on e, and not on the choice of $\mathfrak{sl}_2$-triple containing e.

Let e, e' be two nilpotent elements in $\mathfrak{g}$. It can be shown that e, e' lie in the same G-orbit if and only if $\Delta(e) = \Delta(e')$. So the weighted Dynkin diagram of e uniquely identifies the nilpotent orbit Ge. The weighted Dynkin diagrams corresponding to nilpotent orbits have been classified. For the exceptional types there are explicit lists. For the classical types there is a classification in terms of partitions. In particular, the nilpotent orbits in $\mathfrak{g}$ have been classified.

Let $e \in \mathfrak{g}$ be a representative of a nilpotent orbit. We may assume that e is a linear combination of root vectors, corresponding to positive roots. Let $\beta_1, \ldots, \beta_r$ be the positive roots involved in this linear combination. Let x_{β_i} (respectively y_{β_i}) be the root vector corresponding to β_i (respectively $-\beta_i$). Let $\mathfrak{l} \subset \mathfrak{g}$ be the subalgebra generated by H along with the x_{β_i} and y_{β_i}. Then $\mathfrak{l}$ is reductive, and $e \in \mathfrak{l}$. Let (f, h, e) be an $\mathfrak{sl}_2$-triple containing e, contained in $\mathfrak{l}$. Then $\mathfrak{l}$ decomposes with respect to the action of ad$h$ as $\mathfrak{l} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{l}(k)$. Let $\mathfrak{p} = \bigoplus_{k \geqslant 0} \mathfrak{l}(k)$, which is a subalgebra of $\mathfrak{l}$. Now it can be shown that the nilpotent orbit containing e is uniquely determined by the pair $(\mathfrak{l}, \mathfrak{p})$ (cf. [3], Chapter 8). Corresponding to this the nilpotent orbit has the label $X_n(a_i)$, where X_n is the type of the semisimple part of $\mathfrak{l}$, and i is the number of simple roots in the semisimple part of $\mathfrak{p}$. If the latter algebra is solvable, then we omit the a_i. Furthermore, if the roots of $\mathfrak{l}$ are short (seen as roots of $\mathfrak{g}$), then a tilde is put over the X_n. On some occasions, two different orbits can have the same label. Then a $'$ is added to one of them, whereas the other gets $''$. We note that, although the pair $(\mathfrak{l}, \mathfrak{p})$ uniquely determines the nilpotent orbit, it is also true that the same nilpotent orbit can have more than one (non-isomorphic) such pair. So the same nilpotent orbit can have more than one label.

The nilpotent element e from above also has a Dynkin diagram, which is simply the Dynkin diagram of the roots β_i. This diagram has r nodes, and node i is connected to node j by $\langle \beta_i, \beta_j^\vee \rangle \langle \beta_j, \beta_i^\vee \rangle = 0, 1, 2, 3$ lines. Furthermore, if these scalar products are positive, then the lines are dotted. This only occurs when $\mathfrak{p}$ is not solvable.

3. *Finding representatives of nilpotent orbits*

In this section we consider the problem of finding a nilpotent element in $\mathfrak{g}$ corresponding to a given weighted Dynkin diagram D. We write D_i for the label at node i. Let H be a fixed Cartan subalgebra of $\mathfrak{g}$.

Let $e \in \mathfrak{g}$ be a nilpotent element such that $\Delta(e) = D$. Then there is an $\mathfrak{sl}_2$-triple (f, h, e), containing e. Since we can conjugate any Cartan subalgebra of $\mathfrak{g}$ to H by an element of G, we may assume that $h \in H$. As in the previous section we write $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ for a basis of simple roots. By choosing a Chevalley basis in $\mathfrak{g}$ we get basis elements $h_1, \ldots, h_l$ of H, and root vectors x_{α_i} with $[h_j, x_{\alpha_i}] = \langle \alpha_i, \alpha_j^\vee \rangle x_{\alpha_i}$.

Each $h \in H$ yields a decomposition $\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}(k)$, and a weighted Dynkin diagram, as described in the previous section. This weighted Dynkin diagram is equal to D if and only if $[h, x_{\alpha_i}] = D_i x_{\alpha_i}$ for $1 \leqslant i \leqslant l$. But this happens if and only if $\sum_{j=1}^{l} \langle \alpha_i, \alpha_j^\vee \rangle a_j = D_i$, where the a_j are such that $h = \sum_j a_j h_j$. Let $C = (\langle \alpha_i, \alpha_j^\vee \rangle)_{1 \leqslant i,j \leqslant l}$ be the Cartan matrix of Φ. It follows that h yields the weighted

Dynkin diagram D if and only if $C(a_1, \dots, a_l)^t = (D_1, \dots, D_l)$. Hence there is a unique such h, and we can compute it by solving a system of linear equations. However, not every weighted Dynkin diagram corresponds to a nilpotent orbit. In other words, not every weighted Dynkin diagram yields a h that lies in an $\mathfrak{sl}_2$-triple. The next two lemmas lead to a probabilistic algorithm to decide whether this is the case or not.

LEMMA 1. *Let $h \in H$. Then h belongs to an $\mathfrak{sl}_2$-triple if and only if there is an $x \in \mathfrak{g}(2)$ such that $h \in [x, \mathfrak{g}(-2)]$.*

Proof. The condition is clearly necessary. If $h \in [x, \mathfrak{g}(-2)]$ then there is a $y \in \mathfrak{g}(-2)$ with $[x, y] = h$. Then (y, h, x) is an $\mathfrak{sl}_2$-triple. ☐

LEMMA 2. *Let $h \in H$ be contained in an $\mathfrak{sl}_2$-triple (y, h, x). Let E be the set of $x' \in \mathfrak{g}(2)$ such that $h \in [x', \mathfrak{g}(-2)]$. Then E is Zariski dense in $\mathfrak{g}(2)$.*

Proof. (cf. [2], Proposition 5.6.2). Let $G_h = \{g \in G \mid \mathrm{Ad}(g)(h) = h\}$ be the stabilizer of h in G. Then G_h is an algebraic subgroup of G. Now $\mathrm{Lie}(G_h) = \{u \in \mathfrak{g} \mid \mathrm{ad}(u)(h) = 0\}$. This is the centralizer of h in $\mathfrak{g}$. Hence $\mathrm{Lie}(G_h) = \mathfrak{g}(0)$. For $u \in \mathfrak{g}(2)$ and $g \in G_h$ we have $[h, \mathrm{Ad}(g)(u)] = \mathrm{Ad}(g)[\mathrm{Ad}(g^{-1})(h), u] = \mathrm{Ad}(g)[h, u] = 2\mathrm{Ad}(g)(u)$. Hence $\mathrm{Ad}(g)$ stabilizes $\mathfrak{g}(2)$. Let $\varphi : G_h \to \mathfrak{g}(2)$ be the morphism defined by $\varphi(g) = \mathrm{Ad}(g)(x)$. Then the image of φ is the G_h-orbit of x in $\mathfrak{g}(2)$. The differential of φ is $d\varphi : \mathfrak{g}(0) \to \mathfrak{g}(2)$, $d\varphi(u) = [u, x]$. But this is surjective because $[\mathfrak{g}(0), x] = \mathfrak{g}(2)$ (this follows from the representation theory of $\mathfrak{sl}_2$). So φ is a dominant morphism. Hence $\varphi(G_h)$ is a dense subset of $\mathfrak{g}(2)$. Furthermore $\varphi(G_h) \subset E$. ☐

Based on this we have a probabilistic algorithm for finding a representative of a nilpotent orbit, given a weighted Dynkin diagram. First we determine the unique $h \in H$ corresponding to the diagram. Then we select a random $x \in \mathfrak{g}(2)$, in the following way. Let $x_1, \dots, x_s$ be a basis of $\mathfrak{g}(2)$. Let Ω be a finite subset of $\mathbb{Q}$ and select $\mu_1, \dots, \mu_s$ randomly, uniformly and independently from Ω. Then set $x = \sum_i \mu_i x_i$. By the previous lemma the probability that $h \in [x, \mathfrak{g}(-2)]$ is high (and can be made arbitrarily close to 1 by enlarging Ω). If it happens to be the case that $h \notin [x, \mathfrak{g}(-2)]$ then we select another x and continue. This algorithm will terminate in very few steps.

The x found by the algorithm above will have "ugly" coefficients with respect to a Chevalley basis. We can obtain an element with "nice" coefficients in the following way. We write x with respect to a Chevalley basis of $\mathfrak{g}$. We fix every coefficient but the first. For the first coefficient we try the values $0, 1, 2, \dots$. The lemma ensures that we will quickly find an x' which is a representative of the same nilpotent orbit, with the first coefficient a nice integer. We continue this way until all coefficients are nice integers.

The above results also provide a probabilistic algorithm for testing whether a given weighted Dynkin diagram corresponds to a nilpotent orbit. We basically try the same algorithm a few times, and if it does not come up with an x then the weighted Dynkin diagram does not correspond to a nilpotent orbit with high probability. In principle we can make this absolutely sure by using Gröbner bases. This works as follows. Let $x_1, \dots, x_s$ and $y_1, \dots, y_s$ be bases of respectively $\mathfrak{g}(2)$ and $\mathfrak{g}(-2)$. Let $a_1, \dots, a_s, b_1, \dots, b_s$ be indeterminates. Let $u_1, \dots, u_r$ be a basis of $\mathfrak{g}(0)$, and write $[x_i, y_j] = \sum_k \gamma_{ij}^k u_k$, and $h = \sum_k \alpha_k u_k$. Then there is an $x \in \mathfrak{g}(2)$ with $h \in [x, \mathfrak{g}(-2)]$ if and only if the system of polynomial equations

$$\sum_{i=1}^{s} \sum_{j=1}^{s} \gamma_{ij}^k a_i b_j - \alpha_k = 0 \quad \text{for } 1 \leqslant k \leqslant r$$

has a solution. Now this system has a solution over $\mathbb{C}$ if and only if the reduced Gröbner basis of the ideal generated by the left hand sides of these equations is not $\{1\}$.

Remark. In [7] Popov has given an algorithm for determining the strata of the nullcone of a linear representation of a reductive algebraic group. This also yields an algorithm for classifying nilpotent orbits in reductive Lie algebras, and for finding representatives of them.

4. *Calculating the index*

In this section we describe a simple algorithm that for a Lie algebra gives an upper bound for its index. If the Lie algebra is defined over a sufficiently large field (e.g., of characteristic 0), then the probability that this upper bound is equal to the index can be made arbitrarily high. We use the same notation as in Section 1.

Let K be a finite-dimensional Lie algebra with basis $\{x_1, \ldots, x_n\}$. Let c_{ij}^k be the structure constants of K, i.e., $[x_i, x_j] = \sum_{k=1}^n c_{ij}^k x_k$. Let $\{\psi_1, \ldots, \psi_n\}$ be the dual basis of K^*, i.e., $\psi_i(x_j) = \delta_{ij}$. Let $f = \sum_i T_i \psi_i$ be an element of the dual space K^*. Let $x = \sum_i \alpha_i x_i \in K$. Then $x \in K^f$ if and only if $f([x, x_j]) = 0$ for $1 \leqslant j \leqslant n$. Now this is equivalent to

$$\sum_{i=1}^{n} (\sum_{k=1}^{n} c_{ij}^k T_k) \alpha_i = 0 \text{ for } j = 1, \ldots, n.$$

Define the $n \times n$-matrix A by $A(i,j) = \sum_{k=1}^n c_{ij}^k T_k$. Then $\dim K^f = n - \operatorname{rank}(A)$. So the dimension of K^f is minimal if and only if the rank of A is maximal. Now the rank of A is not maximal if and only if certain polynomial expressions in the T_k (i.e., determinants of certain minors of A) vanish. Therefore, if the T_k are chosen randomly and uniformly from a sufficiently large set, then with high probability the rank of A will be maximal.

Here we consider the case where $K = C_{\mathfrak{g}}(e)$, where e is a nilpotent element of the simple Lie algebra $\mathfrak{g}$. Then by Vinberg's inequality we have that $\operatorname{ind}(K)$ is at least the rank of $\mathfrak{g}$. So if we find an f such that $\dim(K^f) = \operatorname{rank}(\mathfrak{g})$, then we have proved that $\operatorname{ind}(K) = \operatorname{rank}(\mathfrak{g})$. Moreover, the above discussion shows that we will quickly find such an f (if it exists) by randomly choosing the T_k.

With the help of an implementation of this algorithm in GAP4, we have checked Elashvili's conjecture for the exceptional types (which, except G_2, are the remaining open cases). As a result we can conclude that Elashvili's conjecture holds for all simple Lie algebras.

5. *Centralizers in centralizers*

Let $e \in \mathfrak{g}$ be a nilpotent element. Let $C_e = C_{\mathfrak{g}}(e)$ be the centralizer of e in $\mathfrak{g}$. Let $x \in C_e$ and consider the centralizer $C_{e,x}$ of x in C_e (i.e., $C_{e,x}$ is the set of all elements of $\mathfrak{g}$ commuting with both e and x). From [8] it follows that $C_{e,x}$ contains a commutative subalgebra of dimension equal to $\mathrm{rank}(\mathfrak{g})$. Hence the dimension of $C_{e,x}$ is at least the rank of $\mathfrak{g}$. In [9] the following question is considered: given e does there exist $x \in C_e$ such that the dimension of $C_{e,x}$ equals the rank of $\mathfrak{g}$? The main result of that paper is a counter example to the question for the case where $\mathfrak{g}$ is of type F_4.

With the lists of representatives of the nilpotent orbits we can easily tackle this question in all simple Lie algebras of exceptional type. Let $e \in \mathfrak{g}$ be a nilpotent element, and let $x_1, \dots, x_m$ be a basis of C_e. Set $x = T_1x_1 + \cdots + T_mx_m$. Then the centralizer of x in C_e is equal to the kernel of $\mathrm{ad}x$ (restricted to C_e). So the dimension of $C_{e,x}$ is minimal if the rank of the matrix $\mathrm{ad}x$ is maximal. Now the entries of this matrix are linear polynomials in the T_i. It follows that for a random choice of the T_i, with very high probability, the rank of $\mathrm{ad}x$ is maximal. So this gives a probabilistic algorithm for determining the minimal dimension of $C_{e,x}$ (recall that we are varying x, and keeping e fixed). Once the minimal dimension is found with this algorithm we can prove it rigorously as follows. Let x be an element such that $\dim C_{e,x}$ is (hypothetically) minimal, as produced by the algorithm. If $\dim C_{e,x} = \mathrm{rank}(\mathfrak{g})$ then we have proved that the minimal dimension of a $C_{e,x}$ is $\mathrm{rank}(\mathfrak{g})$, as it cannot be smaller. Secondly, if the dimension that we find happens to be bigger, then we compute the rank of the matrix $\mathrm{ad}x$, where $x = T_1x_1 + \cdots + T_mx_m$ and we let the T_i be generators of a rational function field. The rank of that matrix will equal the maximal rank of any $\mathrm{ad}x$ for $x \in C_e$.

Using this algorithm we arrive at the following result.

Proposition 1. *Let $\mathfrak{g}$ be a simple Lie algebra of exceptional type, and $e \in \mathfrak{g}$ nilpotent. Then the minimal dimension of a $C_{e,x}$ is equal to* $\mathrm{rank}(\mathfrak{g})$, *except in three cases, which are listed in the following table:*

type of $\mathfrak{g}$	*label of e*	*dimension of minimal $C_{e,x}$*
G_2	$A_1 + \widetilde{A}_1$	3
F_4	$\widetilde{A}_2 + A_2$	6
E_8	$A_5 + A_2 + A_1$	12

In all three cases it turns out that a minimal $C_{e,x}$ is abelian. Furthermore, in each case it is possible to choose the element $x \in C_e$ such that it is homogeneous of degree -1 with respect to the grading of $\mathfrak{g}$ defined by the $\mathfrak{sl}_2$-triple containing e.

In relation to [9] we remark the following. In [9] it is wrongly stated that in G_2 all minimal $C_{e,x}$ have dimension equal to $\mathrm{rank}(\mathfrak{g})$. The result for F_4 is the same as in [9]. Finally, the problem for E_8 is left open in [9].

Also, as a straightforward corollary of the proposition, it follows that in the exceptional types a minimal $C_{e,x}$ is always abelian.

Appendix A. *Representatives of nilpotent orbits*

In the tables below we list the nilpotent orbits in the Lie algebras of exceptional type. For each orbit we have given a label, the weighted Dynkin diagram, and the Dynkin diagram of a representative. We remark the following. If more than one label was possible, we have chosen the simplest one that we could find. This means that we have preferred a label of the form X_n over a label of the form $X_n(a_i)$. Furthermore, we have preferred labels such that the Dynkin diagram of a corresponding representative has as few lines as possible. In the Dynkin diagram a black node means that the corresponding root is long. Finally, the labels corresponding to each node refer to the basis elements of the simple Lie algebras as present in GAP4. In Appendix B we list the positive roots of each root system of exceptional type, in the order in which they are used by GAP4. Now, if in the tables in this section a Dynkin diagram of a representative has labels $i_1, \dots, i_k$, then the corresponding representative is the sum of the root vectors corresponding to the i_jth positive root for $1 \leqslant j \leqslant k$.

Table 2: Nilpotent orbits in the Lie algebra of type G_2.

label	diagram ● ≡ ○	representative
A_1	1 0	● 6
$\widetilde{A}_1$	0 1	○ 4
$A_1 + \widetilde{A}_1$	2 0	● 2 ○ 4
G_2	2 2	○ 1 ≡ ● 2

Table 3: Nilpotent orbits in the Lie algebra of type F_4.

label	diagram ● — ● ═ ○ — ○	representative
A_1	1 0 0 0	● 24
$\widetilde{A}_1$	0 0 0 1	○ 21
$A_1 + \widetilde{A}_1$	0 1 0 0	○ 17 ● 22
A_2	2 0 0 0	● 16 — ● 18
$\widetilde{A}_2$	0 0 0 2	○ 11 — ○ 12
$\widetilde{A}_1 + A_2$	0 0 1 0	○ 14 ● 15 — ● 16
B_2	2 0 0 1	○ 9 ═ ● 15
$\widetilde{A}_2 + A_1$	0 1 0 1	○ 8 — ○ 14 ● 16
$B_2 + A_1$	1 0 1 0	● 10 ═ ○ 11 ● 15
$\widetilde{A}_2 + A_2$	0 2 0 0	○ 8 — ○ 9 ● 10 — ● 18

Table 3 (continued). *Nilpotent orbits in type F_4*

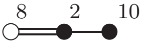
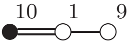
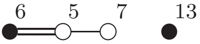
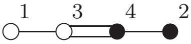
B_3	2 2 0 0	
C_3	1 0 1 2	
C_3+A_1	0 2 0 2	
B_4	2 2 0 2	
F_4	2 2 2 2	

Table 4: Nilpotent orbits in the Lie algebra of type E_6.

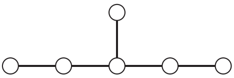
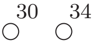
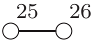
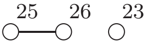
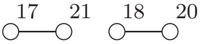
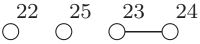
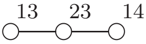
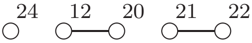
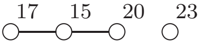
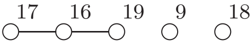
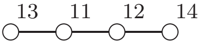
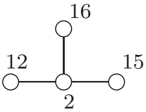
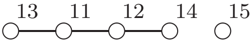
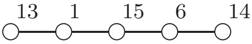
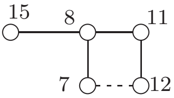
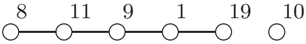
label	diagram 	representative
A_1	1 0 0 0 0 0	
$2A_1$	0 1 0 0 0 1	
$3A_1$	0 0 0 1 0 0	
A_2	2 0 0 0 0 0	
A_2+A_1	1 1 0 0 0 1	
$2A_2$	0 2 0 0 0 2	
$2A_1+A_2$	0 0 1 0 1 0	
A_3	2 1 0 0 0 1	
A_1+2A_2	0 1 0 1 0 1	
A_3+A_1	1 0 1 0 1 0	
A_3+2A_1	0 0 0 2 0 0	
A_4	2 2 0 0 0 2	
D_4	2 0 0 2 0 0	
A_4+A_1	1 1 1 0 1 1	
A_5	1 2 1 0 1 2	
$D_5(a_1)$	2 1 1 0 1 1	
A_5+A_1	0 2 0 2 0 2	

Table 4 (continued). *Nilpotent orbits in type E_6*

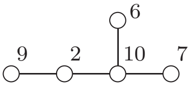
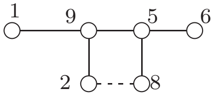
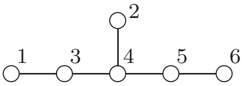
D_5	2 0 $\overset{2}{2}$ 0 2	
$E_6(a_1)$	2 2 $\overset{2}{0}$ 2 2	
E_6	2 2 $\overset{2}{2}$ 2 2	

Table 5: Nilpotent orbits in the Lie algebra of type E_7.

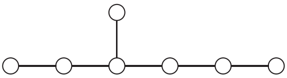
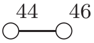
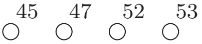
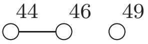
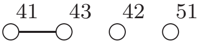
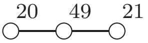
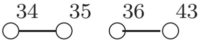
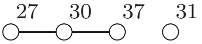
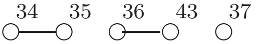
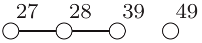
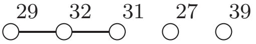
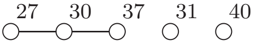
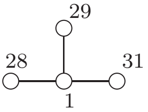
label	diagram 	representative
A_1	1 0 $\overset{0}{0}$ 0 0 0	
$2A_1$	0 0 $\overset{0}{0}$ 0 1 0	
$(3A_1)''$	0 0 $\overset{0}{0}$ 0 0 2	
$(3A_1)'$	0 1 $\overset{0}{0}$ 0 0 0	
A_2	2 0 $\overset{0}{0}$ 0 0 0	
$4A_1$	0 0 $\overset{1}{0}$ 0 0 1	
$A_2 + A_1$	1 0 $\overset{0}{0}$ 0 1 0	
$A_2 + 2A_1$	0 0 $\overset{0}{1}$ 0 0 0	
A_3	2 0 $\overset{0}{0}$ 0 1 0	
$2A_2$	0 0 $\overset{0}{0}$ 0 2 0	
$A_2 + 3A_1$	0 0 $\overset{2}{0}$ 0 0 0	
$(A_3 + A_1)''$	2 0 $\overset{0}{0}$ 0 0 2	
$2A_2 + A_1$	0 1 $\overset{0}{0}$ 0 1 0	
$(A_3 + A_1)'$	1 0 $\overset{0}{1}$ 0 0 0	
$(A_3 + 2A_1)'$	0 2 $\overset{0}{0}$ 0 0 0	
$(A_3 + 2A_1)''$	1 0 $\overset{0}{0}$ 1 0 1	
D_4	2 2 $\overset{0}{0}$ 0 0 0	

Table 5 (continued). *Nilpotent orbits in E_7.*

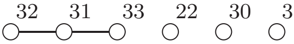
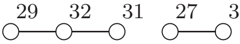
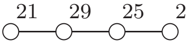
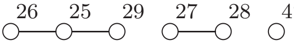
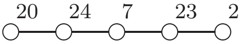
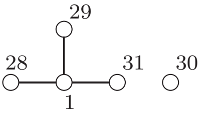
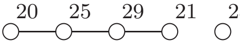
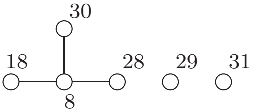
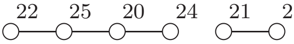
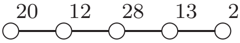
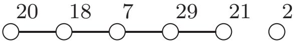
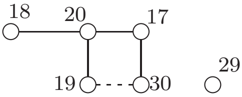
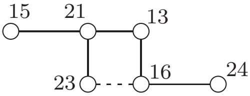
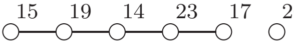
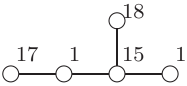
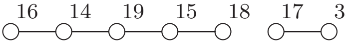
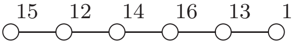
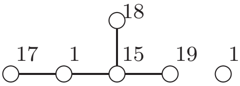
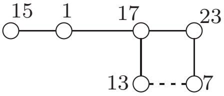
$A_3 + 3A_1$	$0\ \ 1\ \ \overset{1}{0}\ \ 0\ \ 0\ \ 1$	
$A_3 + A_2$	$0\ \ 0\ \ \overset{0}{1}\ \ 0\ \ 1\ \ 0$	
A_4	$2\ \ 0\ \ \overset{0}{0}\ \ 0\ \ 2\ \ 0$	
$A_3 + A_2 + A_1$	$0\ \ 0\ \ \overset{0}{0}\ \ 2\ \ 0\ \ 0$	
$(A_5)''$	$2\ \ 0\ \ \overset{0}{0}\ \ 0\ \ 2\ \ 2$	
$D_4 + A_1$	$2\ \ 1\ \ \overset{1}{0}\ \ 0\ \ 0\ \ 1$	
$A_4 + A_1$	$1\ \ 0\ \ \overset{0}{1}\ \ 0\ \ 1\ \ 0$	
$D_4 + 2A_1$	$2\ \ 0\ \ \overset{0}{1}\ \ 0\ \ 1\ \ 0$	
$A_4 + A_2$	$0\ \ 0\ \ \overset{0}{2}\ \ 0\ \ 0\ \ 0$	
$(A_5)'$	$1\ \ 0\ \ \overset{0}{1}\ \ 0\ \ 2\ \ 0$	
$(A_5 + A_1)''$	$1\ \ 0\ \ \overset{0}{1}\ \ 0\ \ 1\ \ 2$	
$D_5(a_1) + A_1$	$2\ \ 0\ \ \overset{0}{0}\ \ 2\ \ 0\ \ 0$	
$D_6(a_2)$	$0\ \ 1\ \ \overset{1}{0}\ \ 1\ \ 0\ \ 2$	
$(A_5 + A_1)'$	$0\ \ 2\ \ \overset{0}{0}\ \ 0\ \ 2\ \ 0$	
D_5	$2\ \ 2\ \ \overset{0}{0}\ \ 0\ \ 2\ \ 0$	
$A_5 + A_2$	$0\ \ 0\ \ \overset{0}{2}\ \ 0\ \ 0\ \ 2$	
A_6	$0\ \ 0\ \ \overset{0}{2}\ \ 0\ \ 2\ \ 0$	
$D_5 + A_1$	$2\ \ 1\ \ \overset{1}{0}\ \ 1\ \ 0\ \ 2$	
$D_6(a_1)$	$2\ \ 1\ \ \overset{1}{0}\ \ 1\ \ 0\ \ 2$	

Table 5 (continued). *Nilpotent orbits in E_7.*

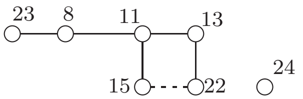
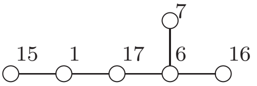
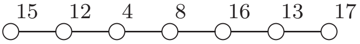
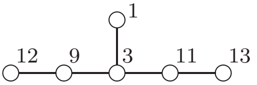
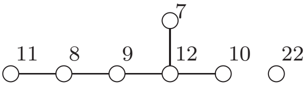
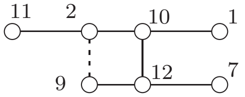
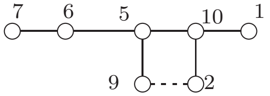
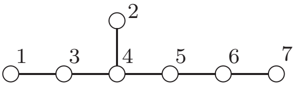
$D_6(a_1)+A_1$	$\begin{smallmatrix}&&0&&&\\2&0&2&0&0&2\end{smallmatrix}$	
D_6	$\begin{smallmatrix}&&1&&&\\2&1&0&1&2&2\end{smallmatrix}$	
A_7	$\begin{smallmatrix}&&0&&&\\2&0&2&0&2&0\end{smallmatrix}$	
E_6	$\begin{smallmatrix}&&0&&&\\2&2&2&0&2&0\end{smallmatrix}$	
D_6+A_1	$\begin{smallmatrix}&&0&&&\\2&0&2&0&2&2\end{smallmatrix}$	
$E_7(a_2)$	$\begin{smallmatrix}&&2&&&\\2&2&0&2&0&2\end{smallmatrix}$	
$E_7(a_1)$	$\begin{smallmatrix}&&2&&&\\2&2&0&2&2&2\end{smallmatrix}$	
E_7	$\begin{smallmatrix}&&2&&&\\2&2&2&2&2&2\end{smallmatrix}$	

Table 6: Nilpotent orbits in the Lie algebra of type E_8.

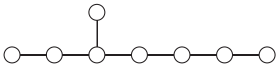
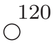
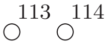
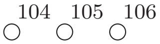
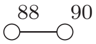
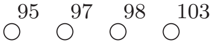
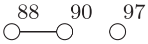
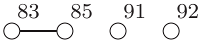
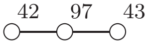
label	diagram	representative
		
A_1	$\begin{smallmatrix}&&0&&&&\\0&0&0&0&0&0&1\end{smallmatrix}$	
$2A_1$	$\begin{smallmatrix}&&0&&&&\\1&0&0&0&0&0&0\end{smallmatrix}$	
$3A_1$	$\begin{smallmatrix}&&0&&&&\\0&0&0&0&0&1&0\end{smallmatrix}$	
A_2	$\begin{smallmatrix}&&0&&&&\\0&0&0&0&0&0&2\end{smallmatrix}$	
$4A_1$	$\begin{smallmatrix}&&1&&&&\\0&0&0&0&0&0&0\end{smallmatrix}$	
A_2+A_1	$\begin{smallmatrix}&&0&&&&\\1&0&0&0&0&0&1\end{smallmatrix}$	
A_2+2A_1	$\begin{smallmatrix}&&0&&&&\\0&0&0&0&1&0&0\end{smallmatrix}$	
A_3	$\begin{smallmatrix}&&0&&&&\\1&0&0&0&0&0&2\end{smallmatrix}$	
A_2+3A_1	$\begin{smallmatrix}&&0&&&&\\0&1&0&0&0&0&0\end{smallmatrix}$	

Table 6 (continued). *Nilpotent orbits in E_8.*

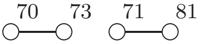
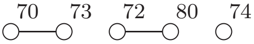
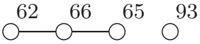
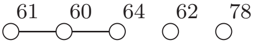
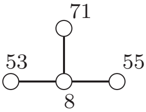
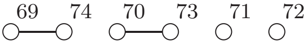
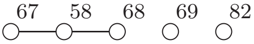
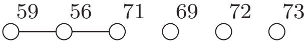
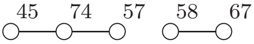
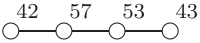
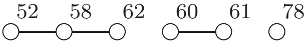
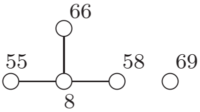
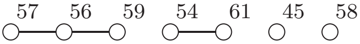
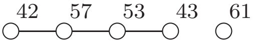
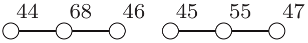
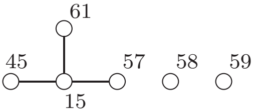
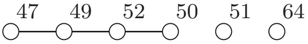
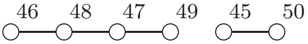
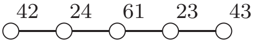
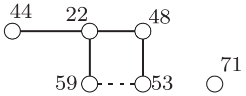
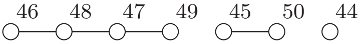
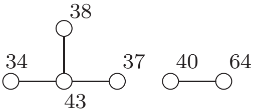
$2A_2$	$2\ 0\ \overset{0}{0}\ 0\ 0\ 0\ 0$	
$2A_2 + A_1$	$1\ 0\ \overset{0}{0}\ 0\ 0\ 1\ 0$	
$A_3 + A_1$	$0\ 0\ \overset{0}{0}\ 0\ 1\ 0\ 1$	
$(A_3 + 2A_1)'$	$0\ 0\ \overset{0}{0}\ 0\ 0\ 2\ 0$	
D_4	$0\ 0\ \overset{0}{0}\ 0\ 0\ 2\ 2$	
$2A_2 + 2A_1$	$0\ 0\ \overset{0}{0}\ 1\ 0\ 0\ 0$	
$(A_3 + 2A_1)''$	$0\ 1\ \overset{0}{0}\ 0\ 0\ 0\ 1$	
$A_3 + 3A_1$	$0\ 0\ \overset{1}{0}\ 0\ 0\ 1\ 0$	
$A_3 + A_2$	$1\ 0\ \overset{0}{0}\ 0\ 1\ 0\ 0$	
A_4	$2\ 0\ \overset{0}{0}\ 0\ 0\ 0\ 2$	
$A_3 + A_2 + A_1$	$0\ 0\ \overset{0}{1}\ 0\ 0\ 0\ 0$	
$D_4 + A_1$	$0\ 0\ \overset{1}{0}\ 0\ 0\ 1\ 2$	
$A_3 + A_2 + 2A_1$	$0\ 0\ \overset{2}{0}\ 0\ 0\ 0\ 0$	
$A_4 + A_1$	$1\ 0\ \overset{0}{0}\ 0\ 1\ 0\ 1$	
$2A_3$	$1\ 0\ \overset{0}{0}\ 1\ 0\ 0\ 0$	
$D_4 + 2A_1$	$1\ 0\ \overset{0}{0}\ 0\ 1\ 0\ 2$	
$A_4 + 2A_1$	$0\ 0\ \overset{0}{1}\ 0\ 0\ 0\ 1$	
$A_4 + A_2$	$0\ 0\ \overset{0}{0}\ 0\ 2\ 0\ 0$	
A_5	$2\ 0\ \overset{0}{0}\ 0\ 1\ 0\ 1$	
$D_5(a_1) + A_1$	$0\ 0\ \overset{0}{1}\ 0\ 0\ 0\ 2$	
$A_4 + A_2 + A_1$	$0\ 1\ \overset{0}{0}\ 0\ 1\ 0\ 0$	
$D_4 + A_2$	$0\ 0\ \overset{2}{0}\ 0\ 0\ 0\ 2$	

Table 6 (continued). *Nilpotent orbits in E_8.*

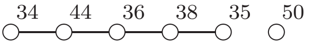
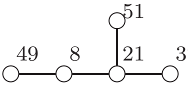
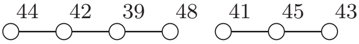
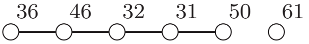
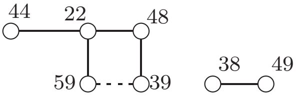
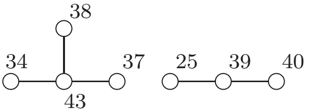
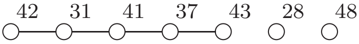
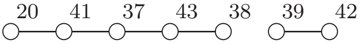
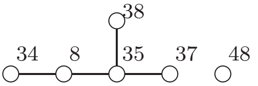
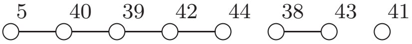
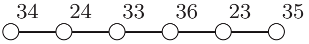
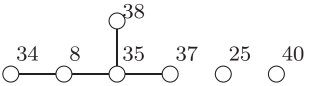
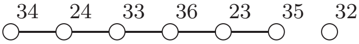
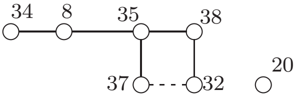
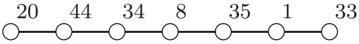
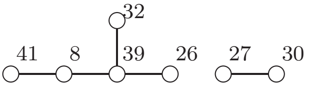
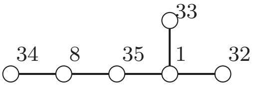
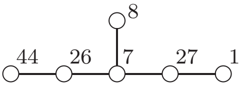
$(A_5+A_1)''$	$\begin{smallmatrix} & & 0 & & & & \\ 2 & 0 & 0 & 0 & 0 & 2 & 0 \end{smallmatrix}$	
D_5	$\begin{smallmatrix} & & 0 & & & & \\ 2 & 0 & 0 & 0 & 0 & 2 & 2 \end{smallmatrix}$	
A_4+A_3	$\begin{smallmatrix} & & 0 & & & & \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{smallmatrix}$	
$(A_5+A_1)'$	$\begin{smallmatrix} & & 0 & & & & \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{smallmatrix}$	
$D_5(a_1)+A_2$	$\begin{smallmatrix} & & 0 & & & & \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \end{smallmatrix}$	
D_4+A_3	$\begin{smallmatrix} & & 1 & & & & \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \end{smallmatrix}$	
A_5+2A_1	$\begin{smallmatrix} & & 0 & & & & \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 \end{smallmatrix}$	
A_5+A_2	$\begin{smallmatrix} & & 0 & & & & \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{smallmatrix}$	
D_5+A_1	$\begin{smallmatrix} & & 0 & & & & \\ 1 & 0 & 0 & 1 & 0 & 1 & 2 \end{smallmatrix}$	
$A_5+A_2+A_1$	$\begin{smallmatrix} & & 0 & & & & \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \end{smallmatrix}$	
A_6	$\begin{smallmatrix} & & 0 & & & & \\ 2 & 0 & 0 & 0 & 2 & 0 & 0 \end{smallmatrix}$	
D_5+2A_1	$\begin{smallmatrix} & & 1 & & & & \\ 0 & 1 & 0 & 0 & 0 & 1 & 2 \end{smallmatrix}$	
A_6+A_1	$\begin{smallmatrix} & & 0 & & & & \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{smallmatrix}$	
$D_6(a_1)+A_1$	$\begin{smallmatrix} & & 0 & & & & \\ 0 & 0 & 1 & 0 & 1 & 0 & 2 \end{smallmatrix}$	
$(A_7)''$	$\begin{smallmatrix} & & 0 & & & & \\ 2 & 0 & 0 & 0 & 2 & 0 & 2 \end{smallmatrix}$	
D_5+A_2	$\begin{smallmatrix} & & 0 & & & & \\ 0 & 0 & 0 & 2 & 0 & 0 & 2 \end{smallmatrix}$	
D_6	$\begin{smallmatrix} & & 1 & & & & \\ 2 & 1 & 0 & 0 & 0 & 1 & 2 \end{smallmatrix}$	
E_6	$\begin{smallmatrix} & & 0 & & & & \\ 2 & 0 & 0 & 0 & 2 & 2 & 2 \end{smallmatrix}$	

Table 6 (continued). *Nilpotent orbits in* E_8.

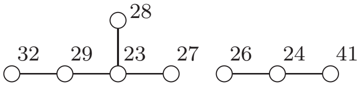
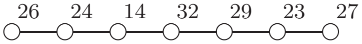
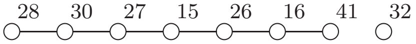
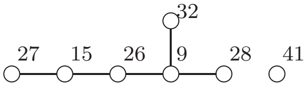
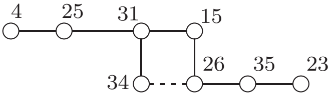
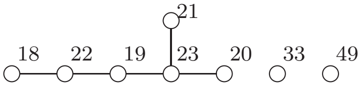
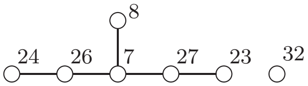
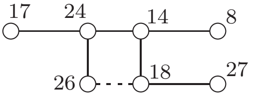
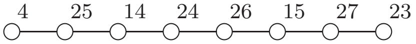
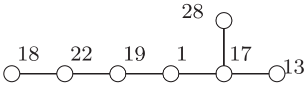
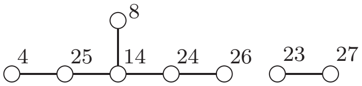
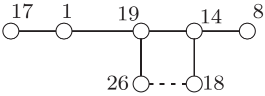
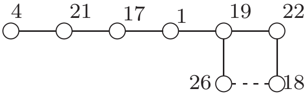
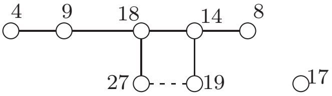
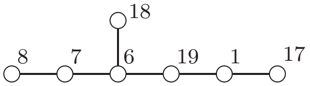
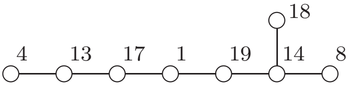
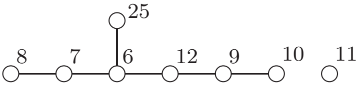
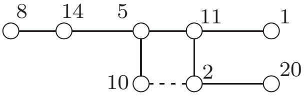
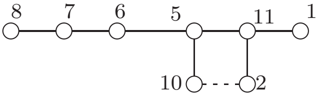
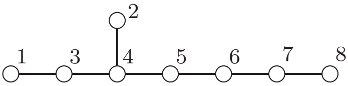
$D_5 + A_3$	$\begin{smallmatrix} & & 0 & & & & \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{smallmatrix}$	
$(A_7)'$	$\begin{smallmatrix} & & 0 & & & & \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 \end{smallmatrix}$	
$A_7 + A_1$	$\begin{smallmatrix} & & 0 & & & & \\ 1 & 0 & 1 & 0 & 1 & 0 & 2 \end{smallmatrix}$	
$D_6 + A_1$	$\begin{smallmatrix} & & 0 & & & & \\ 2 & 0 & 1 & 0 & 1 & 0 & 2 \end{smallmatrix}$	
$D_8(a_3)$	$\begin{smallmatrix} & & 0 & & & & \\ 0 & 0 & 2 & 0 & 0 & 0 & 2 \end{smallmatrix}$	
$D_6 + 2A_1$	$\begin{smallmatrix} & & 0 & & & & \\ 2 & 0 & 0 & 2 & 0 & 0 & 2 \end{smallmatrix}$	
$E_6 + A_1$	$\begin{smallmatrix} & & 0 & & & & \\ 1 & 0 & 1 & 0 & 1 & 2 & 2 \end{smallmatrix}$	
$E_7(a_2)$	$\begin{smallmatrix} & & 1 & & & & \\ 0 & 1 & 0 & 1 & 0 & 2 & 2 \end{smallmatrix}$	
A_8	$\begin{smallmatrix} & & 0 & & & & \\ 0 & 0 & 2 & 0 & 0 & 2 & 0 \end{smallmatrix}$	
D_7	$\begin{smallmatrix} & & 1 & & & & \\ 2 & 1 & 0 & 1 & 1 & 0 & 1 \end{smallmatrix}$	
$E_6 + A_2$	$\begin{smallmatrix} & & 0 & & & & \\ 0 & 0 & 2 & 0 & 0 & 2 & 2 \end{smallmatrix}$	
$E_7(a_1)$	$\begin{smallmatrix} & & 1 & & & & \\ 2 & 1 & 0 & 1 & 0 & 2 & 2 \end{smallmatrix}$	
$D_8(a_1)$	$\begin{smallmatrix} & & 0 & & & & \\ 2 & 0 & 2 & 0 & 0 & 2 & 0 \end{smallmatrix}$	
$E_7(a_1) + A_1$	$\begin{smallmatrix} & & 0 & & & & \\ 2 & 0 & 2 & 0 & 0 & 2 & 2 \end{smallmatrix}$	
E_7	$\begin{smallmatrix} & & 1 & & & & \\ 2 & 1 & 0 & 1 & 2 & 2 & 2 \end{smallmatrix}$	

Table 6 (continued). *Nilpotent orbits in E_8.*

D_8	$\begin{smallmatrix} & & 0 & & & & \\ 2 & 0 & 2 & 0 & 2 & 0 & 2 \end{smallmatrix}$	
$E_7 + A_1$	$\begin{smallmatrix} & & 0 & & & & \\ 2 & 0 & 2 & 0 & 2 & 2 & 2 \end{smallmatrix}$	
$E_8(a_2)$	$\begin{smallmatrix} & & 2 & & & & \\ 2 & 2 & 0 & 2 & 0 & 2 & 2 \end{smallmatrix}$	
$E_8(a_1)$	$\begin{smallmatrix} & & 2 & & & & \\ 2 & 2 & 0 & 2 & 2 & 2 & 2 \end{smallmatrix}$	
E_8	$\begin{smallmatrix} & & 2 & & & & \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 \end{smallmatrix}$	

Appendix B. *The exceptional root systems in* GAP

In this appendix we list the positive roots of the exceptional root systems, in the order in which they appear in GAP4. The tables have to be read from left to right, and from top to bottom. So the first root is the one top left, the second root is the second one on the first line, and the last root is the one bottom right. For each root its coefficients with respect to a basis of simple roots are given.

The roots for F_4 in Table 8 may seem slightly strange. This is due to the fact that in GAP4 the positive roots are ordered differently than usual. In this table the coefficients of each root with respect to the "usual" ordering of a basis of simple roots is given (i.e., as in [1]). However, the roots are listed in the same order as they are in GAP4.

Table 7: Positive roots in the root system of type G_2.

10	01	11	21	31	32

Table 8: Positive roots in the root system of type F_4.

0001	1000	0010	0100	0011	1100	0110	0111
1110	0120	1111	0121	1120	1121	0122	1220
1221	1122	1231	1222	1232	1242	1342	2342

Table 9: Positive roots in the root system of type E_6.

0 10000	1 00000	0 01000	0 00100	0 00010	0 00001	0 11000	1 00100	0 01100
0 00110	0 00011	0 11100	1 01100	1 00110	0 01110	0 00111	1 11100	0 11110
1 01110	1 00111	0 01111	1 11110	0 11111	1 01210	1 01111	1 11210	1 11111
1 01211	1 12210	1 11211	1 01221	1 12211	1 11221	1 12221	1 12321	2 12321

Table 10: Positive roots in the root system of type E_7.

0 100000	1 000000	0 010000	0 001000	0 000100	0 000010	0 000001
0 110000	1 001000	0 011000	0 001100	0 000110	0 000011	0 111000
1 011000	1 001100	0 011100	0 001110	0 000111	1 111000	0 111100
1 011100	1 001110	0 011110	0 001111	1 111100	0 111110	1 012100
1 011110	1 001111	0 011111	1 112100	1 111110	0 111111	1 012110
1 011111	1 122100	1 112110	1 111111	1 012210	1 012111	1 122110
1 112210	1 112111	1 012211	1 122210	1 122111	1 112211	1 012221
1 123210	1 122211	1 112221	2 123210	1 123211	1 122221	2 123211
1 123221	2 123221	1 123321	2 123321	2 124321	2 134321	2 234321

Table 11: Positive roots in the root system of type E_8.

0 1000000	1 0000000	0 0100000	0 0010000	0 0001000	0 0000100
0 0000010	0 0000001	0 1100000	1 0010000	0 0110000	0 0011000
0 0001100	0 0000110	0 0000011	0 1110000	1 0110000	1 0011000
0 0111000	0 0011100	0 0001110	0 0000111	1 1110000	0 1111000
1 0111000	1 0011100	0 0111100	0 0011110	0 0001111	1 1111000
0 1111100	1 0121000	1 0111100	1 0011110	0 0111110	0 0011111
1 1121000	1 1111100	0 1111110	1 0121100	1 0111110	1 0011111
0 0111111	1 1221000	1 1121100	1 1111110	0 1111111	1 0122100
1 0121110	1 0111111	1 1221100	1 1122100	1 1121110	1 1111111
1 0122110	1 0121111	1 1222100	1 1221110	1 1122110	1 1121111

Table 11 (continued). *Positive roots in* E_8

1 0122210	1 0122111	1 1232100	1 1222110	1 1221111	1 1122210
1 1122111	1 0122211	2 1232100	1 1232110	1 1222210	1 1222111
1 1122211	1 0122221	2 1232110	1 1232210	1 1232111	1 1222211
1 1122221	2 1232210	2 1232111	1 1233210	1 1232211	1 1222221
2 1233210	2 1232211	1 1233211	1 1232221	2 1243210	2 1233211
2 1232221	1 1233221	2 1343210	2 1243211	2 1233221	1 1233321
2 2343210	2 1343211	2 1243221	2 1233321	2 2343211	2 1343221
2 1243321	2 2343221	2 1343321	2 1244321	2 2343321	2 1344321
2 2344321	2 1354321	2 2354321	3 1354321	3 2354321	2 2454321
3 2454321	3 2464321	3 2465321	3 2465421	3 2465431	3 2465432

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