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GROUP GRADINGS OF $M_2(K)$

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We describe all group gradings of the matrix algebra $M_2(k)$, where k is an arbitrary field. We prove that any such grading reduces to a grading of type C_2, a grading of type $C_2 \times C_2$, or to a good grading. We give new simple proofs for the description of C_2-gradings and $C_2 \times C_2$-gradings on $M_2(k)$.

0. INTRODUCTION AND PRELIMINARY RESULTS

Let k be a field, G a group and A a k-algebra. We say that A is G-graded if $A = \bigoplus_{g \in G} A_g$, a direct sum of k-vector subspaces, such that $A_g A_h \subseteq A_{gh}$ for any $g, h \in G$. The following general problem was posed by E. Zelmanov (see [**8**]): find all G-gradings of the matrix algebra $M_n(k)$, where G is a group, k a field, and n a positive integer. The answer depends on the structure of G and k, so it is hard to expect the problem can be solved in the general. However, several results have been obtained in special cases. In [**7**], gradings on $A = M_n(k)$ for which every matrix unit e_{ij} (the matrix having 1 on the (i,j)-position, and zero elsewhere) is a homogeneous element (that is, it belongs to one of the subspaces A_g) have been studied. These gradings have been further investigated in [**6**], where they were called *good gradings*. Good gradings are fundamental in the study of all gradings, since as we shall see below, in certain cases any grading is isomorphic to a good grading. The description of all gradings of $M_2(k)$ by the cyclic group C_2 with two elements was done in [**6**], by using computational methods and the duality between group actions and group gradings. This was explained in [**4**] in terms of actions and coactions of Hopf algebras, the basic underlying idea being that a G-grading on an algebra A is precisely a structure of a kG-comodule algebra on A. This idea was very useful for studying gradings of matrix algebras by cyclic groups, see [**4**]. In particular all the isomorphism types of C_2-gradings on $M_2(k)$ were obtained in [**4**] in the case where $\mathrm{char}(k) \neq 2$. For $\mathrm{char}(k) = 2$, the classification has been completed in [**2**]. We note that in the case where k is algebraically closed, any C_m-grading on $M_n(k)$ is isomorphic to a good grading (see [**5, 10**]). This fact led to a different approach to the classification of all gradings of $M_n(k)$ over cyclic groups for an arbitrary field k, by using descent theory;

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see [5]. For non-cyclic groups, the description of gradings seems to be more difficult. One result in this direction is in [3], where gradings of $M_2(k)$ over the Klein group $C_2 \times C_2$ were classified for an arbitrary field k. The method also used the Hopf algebra approach and duality. A strong result was obtained in [1], where the gradings of $M_n(k)$ by an Abelian group were described for an algebraically closed field k, a special role being played again by the good gradings.

In this paper we give a complete answer to Zelmanov's problem in the case $n = 2$, by classifying all group gradings of $M_2(k)$, for any field k. We reobtain the results about C_2-gradings and $C_2 \times C_2$-gradings with a new, elementary technique, that does not make use of Hopf algebras or duality between gradings and actions. Our main result is the following.

THEOREM *Let G be a group with identity element 1, k be a field, and $A = M_2(k)$.*

(I) *If* $\mathrm{char}(k) \neq 2$, *then any G-grading of A is isomorphic to one of the following types.*

(1) *The trivial grading, that is, $A_1 = A$, $A_g = 0$ for any $g \neq 1$.*

(2) *A good grading of the form* $A_1 = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$, $A_g = \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}$, $A_h = 0$ *for* $h \in G - \{1, g\}$, *where $g \in G$ is an element of order 2.*

(3) *A grading of the form* $A_1 = \left\{ \begin{pmatrix} u & v \\ bv & u \end{pmatrix} \;\middle|\; u, v \in k \right\}$, $A_g = \left\{ \begin{pmatrix} u & v \\ -bv & -u \end{pmatrix} \;\middle|\; u, v \in k \right\}$, $A_h = 0$ *for* $h \in G - \{1, g\}$, *where $g \in G$ is an element of order 2, and $b \in k - k^2$.*

(4) *A grading of the form* $A_1 = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$, $A_g = \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix}$, $A_{g^{-1}} = \begin{pmatrix} 0 & 0 \\ k & 0 \end{pmatrix}$, $A_h = 0$ *for* $h \in G - \{1, g, g^{-1}\}$, *where $g \in G$ is an element of order greater than 2.*

(5) *A grading of the form* $A_1 = kI_2, A_g = kX, A_h = kY, A_{gh} = kXY$, $A_u = 0$ *for* $u \in G - \{1, g, h, gh\}$, *where $g, h \in G$ such that $\{1, g, h, gh\}$ is a subgroup of G isomorphic to $C_2 \times C_2$, and X, Y are invertible matrices such that $X^2, Y^2 \in kI_2$ and $XY = -YX$.*

(II) *If* $\mathrm{char}(k) = 2$, *then any grading is isomorphic to one of the gradings of type (1), (2), (4) in (I), or to a grading of the form.*

(3′) $A_1 = \left\{ \begin{pmatrix} x & x+y \\ b(x+y) & y \end{pmatrix} \;\middle|\; x, y \in k \right\}$, $A_g = \left\{ \begin{pmatrix} bx+y & x \\ y & bx+y \end{pmatrix} \;\middle|\; x, y \in k \right\}$, $A_h = 0$ *for* $h \in G - \{1, g\}$, *where $g \in G$ is an element of order 2, and* $b \in k - \{\alpha^2 + \alpha \mid \alpha \in k\}$.

Throughout the paper we denote $A = M_2(k)$. If $A = \bigoplus_{g\in G} A_g$ is a grading by the group G, we denote by $\operatorname{supp}(A) = \{g \in G \mid A_g \neq 0\}$. Note that $1 \in \operatorname{supp}(A)$, since $I_2 \in A_1$ for any grading. If H is a subgroup of G, we say that the grading is of type H, if $\operatorname{supp}(A) \subseteq H$. The theorem above says that any isomorphism type of group grading of $M_2(k)$ is either trivial, a good grading as in (c), or either of type C_2 or of type $C_2 \times C_2$ (when there exist subgroups of G isomorphic to C_2 or $C_2 \times C_2$). In Sections 1,2 and 3, respectively, we describe all gradings with $\operatorname{supp}(A)$ having 2, 3 and 4 elements, respectively. For basic facts about graded algebras we refer to [**9**].

We give now some easy results that will be used in the sequel.

Lemma 0.1. *Let X, Y be non-zero elements of $M_2(k)$ such that $X^2 \in kI_2$ and $XY = YX = 0$. Then $X^2 = 0$, and X, Y are linearly dependent.*

Proof: If $X^2 \in k^*I_2$, then X is invertible, so then $Y = 0$, a contradiction. Thus $X^2 = 0$. Since the conditions still hold if we replace X, Y by UXU^{-1}, UYU^{-1}, with U invertible, we may assume that X has the Jordan form. Since $X \neq 0$, we must have $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. An easy computation shows now that $XY = YX = 0$ implies that Y must be of the form $\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$, so $Y \in kX$. ∎

Lemma 0.2. *Let $A = \bigoplus_{g\in G} A_g$ be a grading of the matrix algebra. If $X \in A_g$ for some $g \in G$, then $X^2 \in A_1$. In particular if $g^2 \neq 1$ we have $X^2 = 0$.*

Proof: By the Cayley-Hamilton theorem we have that $X^2 \in A_1 + A_g$. On the other hand $X^2 \in A_{g^2}$, and then the result is clear. ∎

Corollary 0.3. *If $A = \bigoplus_{g\in G} A_g$ is a grading such that $A_1 = kI_2$, then for any $g, h \in G, g \neq h$ and any non-zero $X \in A_g$, $Y \in A_h$, we have either $XY \neq 0$ or $YX \neq 0$.*

Proof: We know by Lemma 0.2 that $X^2 \in kI_2$. If $XY = YX = 0$, then by Lemma 0.1 we see that X, Y are linearly dependent, a contradiction. ∎

Lemma 0.4. *Let $A = \bigoplus_{g\in G} A_g$ be a grading such that $A_1 \cap A_gA_h = 0$ for any $g, h \in \operatorname{supp}(A) - \{1\}$ (in particular this happens whenever $gh \neq 1$ for any $g, h \in \operatorname{supp}(A) - \{1\}$). Then the grading is trivial.*

Proof: Define $\varphi : A \to A_1$ by $\varphi(X) = X_1$, the homogeneous component of degree 1 of X. Then

$$\varphi(XY) = (XY)_1 = \sum_{g,h\in G, gh=1} X_gY_h = \sum_{g,h\in \operatorname{supp}(A), gh=1} X_gY_h = X_1Y_1 = \varphi(X)\varphi(Y)$$

so φ is an algebra morphism. Since $\varphi \neq 0$ and $\operatorname{Ker}(\varphi)$ is an ideal of $M_2(k)$, we get that $\operatorname{Ker}(\varphi) = 0$, and then $A = A_1$. ∎

1. Gradings of support 2

We first describe the gradings of the matrix algebra by a cyclic group with 2 elements.

Theorem 1.1. *Let k be a field and $C_2 = \{1, c\}$ be the cyclic group of order 2. Then the C_2-gradings of $A = M_2(k)$ are described as follows.*

(I) *If $\operatorname{char}(k) \neq 2$, then any grading is isomorphic to one of the following.*

(i) *The trivial grading $A_1 = M_2(k)$, $A_c = 0$;*

(ii) *The good grading $A_1 = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$, $A_c = \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}$*

(iii) *The grading $A_1 = \left\{ \begin{pmatrix} u & v \\ bv & u \end{pmatrix} \mid u, v \in k \right\}$, $A_c = \left\{ \begin{pmatrix} u & v \\ -bv & -u \end{pmatrix} \mid u, v \in k \right\}$ where $b \in k - k^2$.*

(II) *If $\operatorname{char}(k) = 2$, then any grading is isomorphic either to the trivial grading or to the good grading as in* (i), (ii) *in* (I), *or to a grading of the form*

(iii′) $A_1 = \left\{ \begin{pmatrix} x & x+y \\ b(x+y) & y \end{pmatrix} \mid x, y \in k \right\}$, $A_c = \left\{ \begin{pmatrix} bx+y & x \\ y & bx+y \end{pmatrix} \mid x, y \in k \right\}$ *for some $b \in k - \{\alpha^2 + \alpha \mid \alpha \in k\}$.*

Proof: Let k be of any characteristic and assume that the grading is not trivial. Then we have the following cases.

Case 1. $\dim A_1 = 3$ and $\dim A_c = 1$. Let $A_c = kX$, with $X^2 \in kI_2$. If $X^2 = 0$ we obtain a contradiction by Lemma 0.4. Thus $X^2 \neq 0$, and then X is invertible. Let $B \in A_1 - kI_2$. Then $BX \in A_c$, so $BX \in kX$, and then $B \in kI_2$, a contradiction. So this case is not possible.

Case 2. $\dim A_1 = 1$ and $\dim A_c = 3$. If $A_cA_c = 0$; then the grading is trivial by Lemma 0.4. Thus there exist $X, Y \in A_c$ with $XY \neq 0$. Then $XY \in k^*I_2$, so X and Y are invertible. Hence for $Z \in A_c - kX$ we have $X^2, XZ \in k^*I_2$, and since X is invertible this implies that $Z \in kX$, a contradiction. So this case is also not possible.

Case 3. $\dim A_1 = 2$ and $\dim A_c = 2$. Let $A_1 = kI_2 + kB$. By considering the grading (isomorphic to the initial one) $A = A'_1 \oplus A'_c$, where $A'_1 = UA_1U^{-1}$, $A'_c = UA_cU^{-1}$, for a certain invertible matrix U, we may assume that B has the Jordan form.

If B is diagonal, then it is easy to see that $A_1 = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$. Then if $X = \begin{pmatrix} x & y \\ z & t \end{pmatrix} \in A_c$, then $\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} = e_{11}Xe_{11} \in A_c$, so $x = 0$, and similarly $t = 0$. Thus $A_c \subseteq \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}$, and since $A = A_1 + A_c$ we must have $A_c = \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}$. Thus we obtain a grading of type (ii).

If the minimal polynomial of B is $t^2 - b$ for some $b \notin k^2$, then $B = \begin{pmatrix} 0 & 1 \\ b & 0 \end{pmatrix}$. We obtain that $A_1 = \left\{ \begin{pmatrix} u & v \\ bv & u \end{pmatrix} \;\middle|\; u, v \in k \right\}$. We see that if $X \in A$ and $X^2 \in A_1$, then either $X = \begin{pmatrix} x & y \\ by & x \end{pmatrix}$ or $X = \begin{pmatrix} x & y \\ z & -x \end{pmatrix}$, for some $x, y, z \in k$. Thus if we take $X \in A_c$, we must have $X = \begin{pmatrix} x & y \\ z & -x \end{pmatrix}$. Then $BX = \begin{pmatrix} z & -x \\ bx & by \end{pmatrix} \in A_c$, so $z = -by$. This shows that $A_c \subseteq H = \left\{ \begin{pmatrix} u & v \\ -bv & -u \end{pmatrix} \;\middle|\; u, v \in k \right\}$. If $\mathrm{char}(k) = 2$ we have $H = A_1$, a contradiction. If $\mathrm{char}(k) \neq 2$, we have $A = A_1 \oplus A_c \subseteq A_1 \oplus H$, implying that $A_c = H$, so the grading is of type (iii).

If the minimal polynomial of B is $(t - \lambda)^2$ for some $\lambda \in k$, replace B by $B - \lambda I_2 \in A_1$, and find $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then with the same computations as above (for $b \notin k^2$) we obtain $A_c \subseteq \left\{ \begin{pmatrix} u & v \\ 0 & -u \end{pmatrix} \;\middle|\; u, v \in k \right\}$. But then we have $A_1 + A_c \subseteq \begin{pmatrix} k & k \\ 0 & k \end{pmatrix} \neq A$, a contradiction.

It remains to consider the case where the minimal polynomial of B is irreducible of the form $t^2 - \alpha t - \beta$ with $\alpha \neq 0$. If $\mathrm{char}(k) \neq 2$, then replace B by $B - (\alpha/2) I_2 \in A_1$, and then reducing to the Jordan form we are again in the case $B = \begin{pmatrix} 0 & 1 \\ b & 0 \end{pmatrix}$, $b \notin k^2$, as above. If $\mathrm{char}(k) = 2$, then replace B by $(1/\alpha) B \in A_1$ (with minimal polynomial of the form $t^2 + t + b$), and then reduce to the Jordan form. Thus we can assume that $B = \begin{pmatrix} 0 & 1 \\ b & 1 \end{pmatrix}$ for some $b \notin \{\alpha^2 + \alpha \mid \alpha \in k\}$. In this case we see that

$$A_1 = \left\{ \begin{pmatrix} x & u \\ bu & x + u \end{pmatrix} \;\middle|\; x, u \in k \right\} = \left\{ \begin{pmatrix} x & x + y \\ b(x + y) & y \end{pmatrix} \;\middle|\; x, y \in k \right\}$$

Let $X \in A_c$. Since $X^2 \in A_1$ we must have as above $X = \begin{pmatrix} v & y \\ z & v \end{pmatrix}$ for some $z, y, v \in k$. Using the fact that $(BX)^2 \in A_1$ we obtain that $z = by + v$. Hence

$$A_c \subseteq H' = \left\{ \begin{pmatrix} v & y \\ by + v & v \end{pmatrix} \;\middle|\; v, y \in k \right\} = \left\{ \begin{pmatrix} bx + y & x \\ y & bx + y \end{pmatrix} \;\middle|\; x, y \in k \right\}$$

Now again $A = A_1 \oplus A_c \subseteq A_1 \oplus H$ implies that $A_c = H'$ and we obtain the grading of type (iii′). ∎

REMARKS 1.2.

(i) The gradings in (I)(iii) are classified by the factor group $k^*/(k^*)^2$. Indeed, it was proved in [**4, 5**] that two gradings like these, corresponding to $b_1, b_2 \in k - k^2$, are isomorphic if and only if $b_1/b_2 \in k^2$.

(ii) The gradings in (II)(iii′) are classified by the factor group $k/S(k)$, where $S(k) = \{\alpha^2 + \alpha \mid \alpha \in k\}$ is an additive subgroup of k. It was proved in [**2, 5**] that two gradings of this type, corresponding to $b_1, b_2 \in k - S(k)$ are isomorphic if and only if $b_1 - b_2 \in S(k)$.

The following result shows that any grading of support with 2 elements is essentially a grading by C_2.

PROPOSITION 1.3. *Let G be a group. Then any G-grading of $A = M_2(k)$ with $|\text{supp}(A)| = 2$ is of type C_2.*

PROOF: Let $\text{supp}(A) = \{1, g\}$. If the order of g is 2, then the grading is of type C_2. If the order of g is different from 2, then the grading is trivial by Lemma 0.4, and this ends the proof. ∎

2. GRADINGS OF SUPPORT 3

In this section we consider gradings of A by a group G, such that $\text{supp}(A) = \{1, g, h\}$ has 3 elements. We discuss separately the cases where $\dim A_1 = 2$ and $\dim A_1 = 1$.

PROPOSITION 2.1. *If $\dim A_1 = 2$, then $h = g^{-1}$ and the grading is isomorphic to the good grading given by*

$$A_1 = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}, \ A_g = \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix}, \ A_{g^{-1}} = \begin{pmatrix} 0 & 0 \\ k & 0 \end{pmatrix}, A_p = 0 \text{ for } p \notin \{1, g, g^{-1}\}$$

PROOF: Since $\dim A_1 = 2$, there exists $B \in A_1$ such that A_1 is spanned by I_2 and B. We have $\dim A_g = \dim A_h = 1$, and let $A_g = kX$, $A_h = kY$ for some $X, Y \in A$. Since $A_1 A_g \subseteq A_g$ and $A_1 A_h \subseteq A_h$, we obtain that there exist $\alpha, \beta \in k$ such that $BX = \alpha X$ and $BY = \beta Y$. Therefore the linear map $\varphi : A \to A$, $\varphi(Z) = BZ$ has the eigenvalues α, β with corresponding eigenvectors X, Y. It is easy to see that $P_\varphi = P_B^2$, where P_φ and P_B denote the characteristic polynomials of φ and B. Hence P_B is a product of two linear factors. As in Section 2, we can assume that B has the Jordan form by changing B, X, Y by $UBU^{-1}, UXU^{-1}, UYU^{-1}$.

Assume first that $B = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}$ is diagonalisable. Obviously $b_1 \neq b_2$, since $B \notin kI_2$. Then clearly $A_1 = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$, as the span of I_2 and B. Therefore we can replace B by any matrix in A_1 which is not a scalar multiple of I_2, and let us take $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. This matrix B has eigenvalues 0, with corresponding eigenvectors of the form $\begin{pmatrix} 0 & 0 \\ r & s \end{pmatrix}$ $r, s \in k$, and 1, with corresponding eigenvectors of the form $\begin{pmatrix} u & v \\ 0 & 0 \end{pmatrix}$, $u, v \in k$. Therefore we have $A_g = k \begin{pmatrix} u & 1 \\ 0 & 0 \end{pmatrix}$ and $A_h = \begin{pmatrix} 0 & 0 \\ 1 & s \end{pmatrix}$ for some $u, s \in k$ (or the other way around, but that will lead to an isomorphic grading). Now the facts that $A_g A_1 \subseteq A_g$ and $A_h A_1 \subseteq A_h$ imply that $u = 0$ and $s = 0$. We obtain that

$$A_1 = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}, \ A_g = \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix}, \ A_h = \begin{pmatrix} 0 & 0 \\ k & 0 \end{pmatrix}$$

Since $\deg(e_{12}) = g$, $\deg(e_{21}) = h$, and $0 \neq e_{12}e_{21} \in A_1$, we must have $h = g^{-1}$.

If B is not diagonalisable, then $\alpha = \beta = b$ for some $b \in k$, and $B = \begin{pmatrix} b & 1 \\ 0 & b \end{pmatrix}$. In this case $A_1 = \left\{ \begin{pmatrix} c & d \\ 0 & c \end{pmatrix} \mid c, d \in k \right\}$, and again, since we can replace B by any matrix in A_1 which is not a scalar multiple of I_2, we may consider that $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, with only 0 as an eigenvalue. The corresponding eigenvectors are of the form $\begin{pmatrix} m & n \\ 0 & 0 \end{pmatrix}$, with $m, n \in k$. Since $kB + kX + kY$ has dimension 3 and consists of eigenvectors, we obtain a contradiction. $\square$

REMARK 2.2. If $g \in G$ is an element of order greater than 2, and $A = B = M_2(k)$ are the G-graded algebras defined by

$$A_1 = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}, \ A_g = \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix}, \ A_{g^{-1}} = \begin{pmatrix} 0 & 0 \\ k & 0 \end{pmatrix}, A_p = 0 \text{ for } p \notin \{1, g, g^{-1}\}$$

$$B_1 = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}, \ B_g = \begin{pmatrix} 0 & 0 \\ k & 0 \end{pmatrix}, \ B_{g^{-1}} = \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix}, B_p = 0 \text{ for } p \notin \{1, g, g^{-1}\}$$

then A and B are isomorphic as graded algebras. Indeed, the map $\varphi : A \to B$, $\varphi(X) = UXU^{-1}$, where $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, is a graded isomorphism.

PROPOSITION 2.3. *There do not exist gradings of A with $|\operatorname{supp}(A)| = 3$ and $\dim A_1 = 1$.*

PROOF: Assume that such a grading exists, so say that $\dim A_g = 2$ and $\dim A_h = 1$. Let $X \in A_g$, $Y \in A_h$ be non-zero elements. Then by Corollary 0.3 we have that either

$XY \neq 0$ or $YX \neq 0$. Since $gh, hg \notin \{g, h\}$, in any case we must have $gh = 1$, and since $A_1 = kI_2$, this implies that X and Y are invertible. Then $A_g Y \subseteq A_1 = kI_2$, and since Y is invertible, this implies that $A_g \subseteq kY^{-1}$, a contradiction with $\dim A_g = 2$. $\square$

3. Gradings of support 4

We start by describing all gradings of $A = M_2(k)$ by the Klein group.

Theorem 3.1. *Let $G = C_2 \times C_2 = \{1, g, h, gh\}$ be the Klein group. Then:*

(I) *If $\operatorname{char}(k) \neq 2$, any G-grading of A is isomorphic to one of the following.*

  (i) *A grading of type C_2.*

  (ii) *A grading of the form $A_1 = kI_2, A_g = kX, A_h = kY, A_{gh} = kXY$, where X, Y are invertible matrices such that $X^2, Y^2 \in kI_2$ and $XY = -YX$.*

(II) *If $\operatorname{char}(k) = 2$, any G-grading of A is isomorphic to a grading of type C_2.*

Proof: Assume that the grading is not of type C_2. Then by using the results of the previous section, we see that the support must have 4 elements, so each homogeneous component has dimension 1, in particular $A_1 = kI_2$. Let $A_g = kX$ and $A_h = kY$. Then by Corollary 0.3 we have that either $XY \neq 0$ or $YX \neq 0$. Say that $XY \neq 0$. Then $A_{gh} = kXY$.

If $X^2 = 0$, then $X(XY) = 0$, and again by Corollary 0.3 we must have $(XY)X \neq 0$, so $XYX = \alpha Y$ for some non-zero scalar α. But then $0 = X(XY)X = \alpha XY$, a contradiction. Hence $X^2 \neq 0$, and similarly $Y^2 \neq 0$, so X, Y are invertible.

Now $XYX^{-1} \in A_h$, so $XYX^{-1} = \alpha Y$ for some scalar α. Since $X^2 \in kI_2$ this implies that $\alpha^2 Y = Y$. If $\operatorname{char}(k) \neq 2$, we have $\alpha \in \{1, -1\}$. If $\alpha = 1$, then $XY = YX$ and A is commutative, since it is generated as an algebra by X and Y. We conclude that $XY = -YX$ and the grading is of type (ii). If $\operatorname{char}(k) = 2$, then $\alpha = 1$, and as above we obtain that A is commutative, a contradiction. We conclude that the only possible gradings in characteristic 2 are of type C_2. $\square$

Remark 3.2. The isomorphism types of the $C_2 \times C_2$-gradings are described in [**3**].

The general situation of a grading with support having 4 elements is solved by the following.

Proposition 3.3. *Any group grading of A with $|\operatorname{supp}(A)| = 4$ is of type $C_2 \times C_2$.*

Proof: Let $A = \bigoplus_{g \in G} A_g$ with $S = \operatorname{supp}(A) = \{1, g, h, s\}$. We first note that $uv \neq 1$ for any $u, v \in S$, $u \neq v$. Indeed, if $uv = 1$ for some u, v like this, pick some non-zero $X \in A_u$, $Y \in A_v$. Then $X^2 \in kI_2$ by Lemma 0.2, and then by Corollary 0.3 either $XY \in k^* I_2$ or $YX \in k^* I_2$. In any case, X is invertible. But $u^2 \neq 1$ (since $uv = 1$), so $X^2 = 0$, a contradiction.

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Since the grading is not trivial, we see from Lemma 0.4 that there exists $u \in S - \{1\}$ with $A_u A_u \cap A_1 \neq 0$, in particular $u^2 = 1$. Say for instance that $u = g$, and let $A_g = kX$, with $X^2 \in k^* I_2$; in particular X is invertible. Then $A_g A_h = X A_h \neq 0$, so $gh \in S$, and the only possibility is that $gh = s$. Similarly $A_h A_g \neq 0$, and then $gh = hg = s$. Hence we have $hs = sh$. If $hs \notin S$, we would have $A_h A_s = A_s A_h = 0$, and then if we take non-zero $Y \in A_h$, $Z \in A_s$, we have $YZ = ZY = 0$, a contradiction by Corollary 0.3.

Therefore $h(hg) = (hg)h \in S$, and since this can not be h, hg or 1, we must have $h(hg) = g$, implying that $h^2 = 1$. This shows that S is a subgroup isomorphic to $C_2 \times C_2$, and the proof is finished. ∎

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