LIBRATION OF RETROGRADE SATELLITE ORBITS IN THE CIRCULAR PLANE RESTRICTED THREE-BODY PROBLEM

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INTRODUCTION

In the circular plane restricted three-body problem, we study the stable large retrograde non-periodic satellite orbits. We use rotating axes with the origin in the body around which turns the satellite, called its primary. We choose the initial conditions such as $Y_0=0$ and $U_0=0$, so that an orbit can be represented by a point in the (X_0,V_0) plane. In this plane, the set of stable orbits is represented by a limited region, which we call the stability zone. This zone is composed in general by a large continental region, approximately limited by Lagrange points, and a peninsula more or less elongated. Inside, takes place the characteristic of the single-periodic symmetrical family f which can be called the backbone of the zone (figure 1).

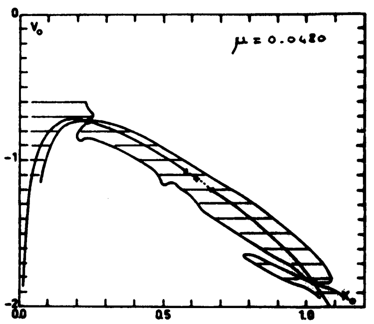

Figure 1. An example of the stability zone.

The numerical explorations have shown that the non-periodic orbits can be approximately decomposed into a fast "reference motion" and a slow libration of its centre around the primary of the satellite. Moreover, the amplitude of the libration, which is zero for the periodic

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orbits, increases when the initial conditions of the non-periobic orbits move off from the initial conditions of the periodic orbits.

HILL'S CASE

In Hill's case, the reference motion is elliptic, with a period of the order of 2π, and the centre of this ellipse librates on a very elongated oval with a period much larger than 2π. The analysis of this libration is developped in a paper published in 1976. In this paper, we establish the equations of motion for the coordinates of the centre of the ellipse and we found two integrals of motion: the first is the semi-major axis of the ellipse; the second is essentially Jacobi's integral, translated into the new coordinates. A numerical verification gives very good agreement for all these results.

GENERAL CASE

We turn now to the general case, i.e. $\mu \neq 0$. In this case, neither the reference motion, nor the trajectory of its centre cannot be described by simple curves (figure 2). The trajectory of the centre of the reference motion can be considered as a very narrow "bean", elongated along the circle of centre B_1 and radius 1.

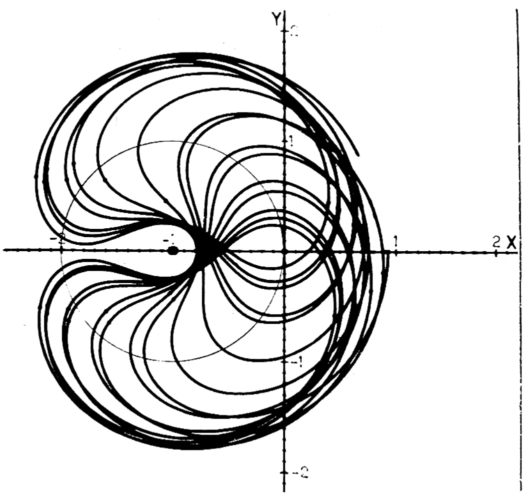

Figure 2. An example of libration.

To use the same analysis as in Hill's case, we must first "rectify the bean". This lead us to use the transformation defined by:

$$X = (\xi + 1)\cos\theta - 1 \quad \text{and} \quad Y = (\xi + 1)\sin\theta \, . \qquad (1)$$

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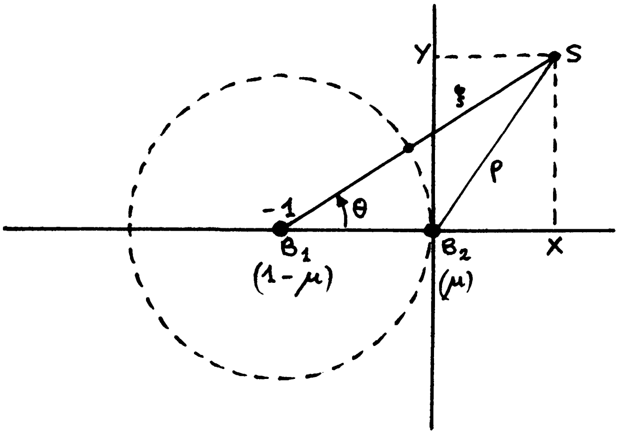

Figure 3. Representation of the coordinate systems (X,Y) and (ξ,θ).

Then, the classical equations of motion for the satellite become:

$$\left.\begin{aligned} &\dot{\xi} = \eta \;, \quad \dot{\theta} = \psi \;, \\ &\dot{\eta} = (\xi+1)(\psi+1)^2 - (\xi+1)^{-2} + \mu\, f(\xi,\theta) \;, \\ &\dot{\psi} = -2\eta\,(\psi+1)/(\xi+1) + \mu\, g(\xi,\theta) \,. \end{aligned}\right\} \quad (2)$$

In the limiting case $\mu=0$ (which corresponds to the two-body problem), the family f reduces (in fixed axes) to a family $\mathcal{E}$ of ellipses of focus B_1 and semi-major axis 1. In this case, the terms in μ vanish in equations (2) and the reduced equations have a well-known solution. Rather than the constants of integration, which are the elements of $\mathcal{E}$, the physically interesting quantities are the coordinates (x,y) of the centre of $\mathcal{E}$, the amplitude A in ξ and the phase difference φ between the motions of S and B_2, given by the equations:

$$\left.\begin{aligned} &x=a-1, \qquad y=(1-a^{3/2})\alpha+\omega+t_0, \\ &A=-ae, \qquad \varphi=t_0, \end{aligned}\right\} \quad (3)$$

where a, e, ω and t_0 are the elements of $\mathcal{E}$ and α is the eccentric anomaly of S on $\mathcal{E}$.

Then the solution of the reduced equations can be written:

$$\begin{aligned} &t=\sqrt{x+1}\,(A \sin\alpha+(x+1)\alpha)+\varphi, \\ &\xi=A \cos\alpha+x+1, \\ &\theta=-A\sqrt{x+1}\,\sin\alpha-\alpha+2 \,\mathrm{arc\,tg}\left(\sqrt{\frac{x+1-A}{x+1+A}}\,\mathrm{tg}\,\alpha/2\right)+Y. \end{aligned} \quad (4)$$

In the approximation $\mu=0$ (considering x, y, A and φ as constant), we differentiate equations (4) to obtain η and ψ :

$$\begin{aligned} &\eta=-A \sin\alpha/\sqrt{x+1}\,(A \cos\alpha+x+1), \\ &\psi=\sqrt{(x+1)^2-A^2}\,/\sqrt{x+1}\,(A \cos\alpha+x+1)^2-1. \end{aligned} \quad (5)$$

We return now to exact equations (2) and we effect a change of variable, replacing (ξ,θ,η,ψ) by (x,y,A,φ). Therefore, we differentiate equations (4) and (5), considering now x, y, A and φ as variables, to obtain the values of $(\dot{\xi},\dot{\theta},\dot{\eta},\dot{\psi})$; then, substituting in equations (2) and solving for $(\dot{x},\dot{y},\dot{A},\dot{\varphi})$, we obtain the differential equations for the new variables:

$$\left.\begin{aligned}&\dot{x}=\mu F(x,y,A,\varphi,\alpha),\\&\dot{y}=(1-(x+1)^{3/2})/\sqrt{x+1}(A\cos\alpha+x+1)+\mu\, G(x,y,A,\varphi,\alpha),\\&\dot{A}=\mu\, H(x,y,A,\varphi,\alpha),\qquad \dot{\varphi}=\mu\, K(x,y,A,\varphi,\alpha),\end{aligned}\right\}\qquad(6)$$

where F, G, H and K are rather complicated expressions, which integration does not seem to be feasible analytically in general.

Fortunately, we may put some approximations. The period T of the libration is much greater than the period 2π of the reference motion; therefore we can say that $\dot{x}$ is of the order of x/T and $\dot{y}$ of the order of y/T. On the other hand, the numerical results show that $x\ll 1$, while A and y are of the order of 1. And from the equations (6), we deduce that $\dot{x}$ is of the order of μ, so that finally μ must be much smaller than x. This means particularly that we can neglect in $\dot{y}$ the term in μ. Moreover, we can average the equations (6) over 2π with the assumption that x, y, A and φ stay constant over this period. Finally, we obtain the following equations:

$$\dot{x}=\mu\frac{\partial I}{\partial y},\qquad \dot{y}=-3x/2,\qquad \dot{A}=0,$$

$$\text{where}\quad I=\frac{1}{\pi}\int_0^{2\pi}\left((A\cos\alpha+1)/\rho-(A\cos\alpha+1)^2\cos\theta\right)d\alpha \qquad (7)$$

$$\text{with}\quad \rho^2=(A\cos\alpha+1)^2-2(A\cos\alpha+1)\cos\theta+1$$

$$\text{and}\quad \theta=-A\sin\alpha-\alpha+2\,\mathrm{arc\,tg}\left(\sqrt{\frac{1-A}{1+A}}\,\mathrm{tg}\,\alpha/2\right)+y.$$

The integration of the integral I does not seem to be feasible analytically in general; nevertheless, the numerical integration for a set of values of A and y is in progress.

As in Hill's case, we have two integrals of motion. The first is:

$$A=c^{st}. \qquad (8)$$

The other:

$$B(x,y,A)=3Ax^2/4+\mu AI=c^{st}, \qquad (9)$$

is essentially Jacobi's integral, translated into the new variables and averaged over 2π under the same assumptions which lead us to the final equations (7).

Now we have to compute numerically some examples of curves B=cst and verify all these results for some actual orbits.

REFERENCE

Benest,D.,1976,"Cel. Mech." 13, pp 203-215