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JAMES QUASI REFLEXIVE SPACE HAS THE FIXED POINT PROPERTY

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We prove that the classical sequence James space has the fixed point property. This gives an example of Banach space with a non-unconditional basis where the Maurey-Lin's method applies.

INTRODUCTION

Let K be a nonempty weakly compact convex subset of a Banach space X. We say that K has the fixed point property (f.p.p.) if every non-expansive mapping $T\colon K \to K$ (that is $\|T(x) - T(y)\| \leqslant \|x - y\|$ for any x, y in K) has a fixed point. We say that X has the fixed point property (f.p.p.) if every weakly convex compact subset of X has the f.p.p.

A theorem of Kirk [**9**] states that if K has normal structure, then it has the f.p.p. It was unknown whether the normal structure is essential. Karlovitz [**7**] answered the problem negatively.

Alspach [**1**] proved that L_1 fails the f.p.p., proving that weak compactness is not sufficient to have the f.p.p. The purpose of this paper is to give a proof that the classical James space [**5**] has the f.p.p., using the beautiful works of Maurey [**15**] and Lin [**12**].

Let me point out that in [**13**], Lin proved positive results concerning the f.p.p. in Banach spaces with unconditional basis. Our paper shows that the ideas arising from Lin's paper are applicable in some Banach spaces with a "good" Schauder basis.

For more detailed history of the f.p.p., we suggest the reader consults [**10**] and [**16**] and the references listed therein.

MAIN RESULT

First recall the definition of the James space J. This space consists of sequences $x = (x_n)$ for which $\mathrm{Lim}\,(x_n) = 0$, and $\|x\|_J < \infty$ where

$$\|x\|_J = \mathrm{Sup}\{[(x_{p_1} - x_{p_2})^2 + (x_{p_2} - x_{p_3})^2 + \ldots + (x_{p_{n-1}} - x_{p_n})^2 + (x_{p_n} - x_{p_1})^2]^{1/2}\}$$

and the supremum is taken over all positive integers n and all increasing sequences of positive integers $\{p_1, p_2, \ldots, p_n\}$.

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Remark. Sometimes the term $(x_{p_n} - x_{p_1})$ is dropped, and then we obtain a new space J_1 which is isomorphic to J. In [8] it is proved that any weakly compact convex subset of J_1 has the normal structure and therefore J_1 has the f.p.p.

The space J was used to disprove several long-standing conjectures [**14**,(I) p.25, 103, 132], [**14**,(II) p.36, 39], [**2**, **3**, **4**] and [**11**].

For the proof of our result, we need one technical lemma, which seems to be new.

LEMMA 1.

(1) *For integers $a \leqslant b$ we denote the interval of integers between a and b by F. Consider the natural projection P_F associated with the basis of J. Then:*

$$\|I - P_F\|^2 \leqslant 2$$

(2) *Let u and v be defined by:*

$$u = \sum_a^b \beta_i e_i \text{ and } v = \sum_c^d \alpha_i e_i \text{ with } a \leqslant b < c-1 \text{ and } c \leqslant d, \text{ then}$$

$$\|u + v\| \leqslant \sqrt{2}\,\|u - v\|$$

PROOF: Since the proof of (1) and (2) uses the same techniques, we give only the proof of (1):

Let x be in J with $\|x\| \leqslant 1$, we have

$$(I - P_F)(x) = x_F = \sum_{i<a} x_i e_i + \sum_{i>b} x_i e_i = \sum_i y_i e_i$$

Let (p_i) denote a strictly increasing finite sequence of integers. There are two cases:

First case. :

$$\{p_i\} \cap F = \emptyset \text{ then:}$$

$$\sum_1^n \left(y_{p_i} - y_{p_{i+1}}\right)^2 + \left(y_{p_n} - y_{p_1}\right)^2 \leqslant \|x\|^2 \leqslant 1$$

Second case. :

$$\{p_i\} \cap F \neq \emptyset \text{ then:}$$

$$\sum_1^{n-1} \left(y_{p_i} - y_{p_{i+1}}\right)^2 + \left(y_{p_n} - y_{p_1}\right)^2$$

$$= \sum_{i \leqslant j} \left(x_{p_i} - x_{p_{i+1}}\right)^2 + \sum_{i=k}^{i=n-1} \left(x_{p_i} - x_{p_{n+1}}\right)^2 + x_j^2 + x_k^2 + \left(x_{p_n} - x_{p_1}\right)^2$$

with $j \leqslant a \leqslant b \leqslant k$. But:

$$\sum_{i=1}^{i=j}(x_{p_i} - x_{p_{i+1}})^2 + \sum_{i=k}^{i=n-1}(x_{p_i} - x_{p_{i+1}})^2 + (x_{p_n} - x_{p_1})^2$$
$$\leqslant \sum_{i=1}^{i=j}(x_{p_i} - x_{p_{i+1}})^2 + \left(x_{p_j} - x_{p_k}\right)^2$$
$$+ \sum_{i=k}^{i=n-1}(x_{p_i} - x_{p_{i+1}})^2 + (x_{p_n} - x_{p_1})^2 \leqslant \|x\|^2 \leqslant 1$$

and $x_i^2 + x_k^2 \leqslant 1$ (because the sequence (x_n) is in c_o)

We deduce that:

$$\|x_f\|^2 \leqslant 1 + 1 = 2.$$

∎

Now we state the main theorem.

THEOREM. *Every weakly compact convex subset of J has the fixed point property.*

PROOF: Suppose that there exists a weakly compact, nonempty convex subset C of J and a non-expansive $T: C \to C$ without fixed point. By Zorn's Lemma C contains a nonempty closed convex subset K, T-invariant and minimal with respect to the inclusion. Our hypothesis on T implies that diam $K > 0$, without loss of generality we can assume that diam $K = 1$. It is easy to see that K contains a quasi-fixed sequence (x_n) (that is $\operatorname{Lim}\|x_n - T(x_n)\| = 0$). Using the fact that K is weakly compact, the sequence (x_n) has a subsequence which is weakly convergent. Since our problem is invariant by translation and by passing to a subsequence, we can assume that (x_n) converges weakly to 0.

The Karlovitz' Lemma [7] states that for any x in K we have:

(**) $$\operatorname{Lim}\|x_n - x\| = \operatorname{diam} K = 1.$$

Since (x_n) converges weakly to 0 and satisfies (**) then there exists a subsequence (x'_n) and a sequence of blocks (u_n) such that:

1) $\operatorname{Lim}\|x'_n - u_n\| = 0$,
2) $\operatorname{Lim}\left\|x'_{n+1} - x'_n\right\| = 1$,

where $u_n = \sum_{i=1_n}^{i=b_n} \beta_i^n e_i$ with $a_n < b_n = a_{n+1}$.

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Let P_n and Q_n denote the natural projections defined by:

$$P_n\left(\sum_i \beta_i e_i\right) = \sum_{i=a_n}^{i=b_n} \beta_i e_i \text{ and } Q_n\left(\sum_i \beta_i e_i\right) = \sum_{i>a_{n+1}} \beta_i e_i.$$

Then by the construction of (u_n) we have:

i) $\operatorname{Lim} \|x\prime'_n - P_n(x'_n)\| = 0$;
ii) $\operatorname{Lim} \left\|x'_{n+2} - Q_n\left(x'_{n+2}\right)\right\| = 0$ (because $Q_n(u_{n+2}) = u_{n+2}$);
iii) $\operatorname{Lim} \|P_n(x)\| = \operatorname{Lim} \|Q_n(x)\| = 0$ for every x in J.

Let $\mathcal{U}$ denote a non-trivial ultrafilter on $\mathbb{N}$. The ultraproduct space $\mathbf{J}$ of J is the quotient space of:

$$1_\infty(J) = \{(x_n); x_n \in J \text{ for all } n \in \mathbb{N} \text{ and } \|(x_n)\|_\infty = \sup \|x_n\| < \infty\}$$

by $\mathcal{N} = \{(x_n) \in 1_\infty(J) \ \operatorname{Lim}_{\mathcal{U}} \|x_n\| = 0\}$. We shall not distringuish between $(x_n) \in 1_\infty(J)$ and the coset $(x_n) + \mathcal{N} \in \mathbf{J}$. Clearly,

$$\|(x_n)\|_{\mathbf{J}} = \operatorname{Lim}_{\mathcal{U}} \|x_n\|_J.$$

It is also clear that J is isometric to a subspace of $\mathbf{J}$ by the mapping $x \to (x, x, \ldots)$. Hence, we may assume that J is a subspace of $\mathbf{J}$. We will write $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ for the general elements of $\mathbf{J}$ and x, y, z for the general element of J. In $\mathbf{J}$ we define:

$$\mathbf{K} = \{\mathbf{y} \in \mathbf{J}; \mathbf{y} = (y_n) \text{ with } y_n \in K\},$$

and

$$\mathbf{T}: \mathbf{K} \to \mathbf{K} \text{ with } \mathbf{T}(\mathbf{y}) = \mathbf{T}(y_n) = (T(y_n)).$$

Clearly $\mathbf{K}$ is a closed convex set with $\operatorname{diam}(\mathbf{K}) = \operatorname{diam}(K) = 1$, and $\mathbf{T}$ is a nonexpansive map on $\mathbf{K}$. Furthermore, $\mathbf{T}$ has fixed points in $\mathbf{K}$. Indeed, if (x_n) is quasi fixed sequence for T in K, then $\operatorname{Lim} \|x_n - T(x_n)\| = 0$ and hence:

$$\|\mathbf{T}(x_n) - (x_n)\|_{\mathbf{J}} = \operatorname{Lim}_{\mathcal{U}} \|x_n - T(x_n)\| = 0.$$

This means that $\mathbf{T}(x_n) = (x_n)$, that is (x_n) is a fixed point for $\mathbf{T}$ in $\mathbf{J}$. Also, if $\mathbf{T}(y_n) = (y_n)$, then some subsequence of (y_n) is a quasi fixed sequence for $\mathbf{T}$. The Karlovitz Lemma can be stated in the space $\mathbf{J}$ by the following:

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LEMMA 2. [12]: *Let* $(\mathbf{w}_n)$ *be a quasi-fixed sequence for* $\mathbf{T}$, *then:*

$$\operatorname{Lim} \|\mathbf{w}_n - x\| = \operatorname{diam}(K) = 1 \text{ for any } x \text{ in } K.$$

In other works, if $\mathbf{W}$ is any nonempty closed $\mathbf{T}$-invariant convex subset of $\mathbf{K}$, we have:

$$\operatorname*{Sup}_{w \in \mathbf{W}} \|\mathbf{w} - x\| = \operatorname{diam}(K) = 1 \text{ for any } x \text{ in } K.$$

Define $\mathbf{x}$ and $\mathbf{y}$ by: $\mathbf{x} = (x'_n)$ and $\mathbf{y} = (x'_{n+2})$ by ii) we have: $\mathbf{x} = (P_n(x'_n))$ and $\mathbf{y} = (Q_n(x'_{n+2}))$ and by 2) we have: $\|\mathbf{x} - \mathbf{y}\| = 1$.

From Lemma 1, we deduce that:

$$\|\mathbf{x} + \mathbf{y}\|^2 \leqslant 2\,\|\mathbf{x} - \mathbf{y}\|^2$$

Let $\mathbf{W} = \{\mathbf{w} \in \mathbf{K}, \exists x \in K s.t. \|\mathbf{w} - x\| \leqslant 2^{-1/2} \& \operatorname{Sup}(\|\mathbf{w} - \mathbf{x}\|, \|\mathbf{w} - \mathbf{y}\|) \leqslant 1/2\}$.

Everything was done to ensure that $\frac{\mathbf{x}+\mathbf{y}}{2}$ is in $\mathbf{W}$. Also it is easy to verify that $\mathbf{W}$ is a closed $\mathbf{T}$-invariant convex subset of $\mathbf{K}$. Consider the projections $\mathbf{P}$ and $\mathbf{Q}$ defined on $\mathbf{J}$ by:

$$\mathbf{P}(\mathbf{z}) = (P_n(z_n)) \text{ and } \mathbf{Q}(\mathbf{z}) = (Q_n(z_n)) \text{ where } \mathbf{z} = (z_n).$$

Since the basis of $\mathbf{J}$ is bimonotone, we have:

$$\begin{gathered}\|\mathbf{P}\| \leqslant \operatorname{Sup}\|\mathbf{P}_n\| \leqslant 1\\ \|\mathbf{Q}\| \leqslant \operatorname{Sup}\|\mathbf{Q}_n\| \leqslant 1,\\ \|\mathbf{P} + \mathbf{Q}\| \leqslant \operatorname{Sup}\|\mathbf{P}_n + \mathbf{Q}_n\| \leqslant 1,\\ \|\mathbf{I} - \mathbf{Q}\| \leqslant 1.\end{gathered}$$

Invoking Lemma 1, we have:

$$\|\mathbf{I} - \mathbf{P}\|^2 \leqslant 2.$$

Choose $\mathbf{w}$ in $\mathbf{W}$ and x in K such that $\|\mathbf{w} - x\| \leqslant 2^{-1/2}$.

One has:

$$(*) \qquad 2\mathbf{w} = (\mathbf{P} + \mathbf{Q})(\mathbf{w}) + (\mathbf{I} - \mathbf{P})(\mathbf{w}) + (\mathbf{I} - \mathbf{Q})(\mathbf{w}).$$

From the definitions of $\mathbf{P}$ and $\mathbf{Q}$, we can directly derive the following:

$$\mathbf{P}(x) = \mathbf{Q}(x) = 0,\ \mathbf{P}(\mathbf{x}) = \mathbf{x} \text{ and } \mathbf{Q}(\mathbf{y}) = \mathbf{y}.$$

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Using (*) we deduce that $2\mathbf{w} = (\mathbf{P}+\mathbf{Q})(\mathbf{w}-x)+(\mathbf{I}-\mathbf{P})(\mathbf{w}-\mathbf{x})+(\mathbf{I}-\mathbf{Q})(\mathbf{w}-\mathbf{y})$. And then we have

$$2\|\mathbf{w}\| \leqslant \|\mathbf{P}+\mathbf{Q}\|\,\|\mathbf{w}-x\| + \|\mathbf{I}-\mathbf{P}\|\,\|\mathbf{w}-\mathbf{x}\| + \|\mathbf{I}-\mathbf{Q}\|\,\|\mathbf{w}-\mathbf{y}\|.$$

And using all our previous inequalities, we obtain

$$2\|\mathbf{w}\| \leqslant 2^{-1/2} + 2^{-1/2} + 2^{-1}.$$

This implies

$$\operatorname*{Sup}_{\mathbf{w}} \|\mathbf{w}\| < 1,$$

which yields a contradiction to Lemma 2.

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