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Cusp Forms Like Δ

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Abstract. Let f be a square-free integer and denote by $\Gamma_0(f)^+$ the normalizer of $\Gamma_0(f)$ in $\mathrm{SL}(2,\mathbb{R})$. We find the analogues of the cusp form Δ for the groups $\Gamma_0(f)^+$.

Let G be a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$ acting on the upper half plane $\mathcal{H}$ by fractional linear transformations and let $\mathcal{H}^* = \mathcal{H} \cup \mathbb{Q} \cup \infty$. Suppose $G\backslash\mathcal{H}^*$ is compact, *i.e.*, G is a Fuchsian group of the first kind. For any meromorphic function h on $\mathcal{H}$ and $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$ define the slash operator by

$$h|[M]_k = (cz+d)^{-k}h(Mz).$$

The convention here for arguments and exponents, following Knopp [8], is that $z^r = |z|^r \exp(ir\arg(z))$, where $-\pi \leq \arg(z) < \pi$. (Note the non-standard choice of argument for the negative reals.) Recall that h is called an automorphic form for G of weight k and multiplier ν if in addition to being meromorphic on $\mathcal{H}$, it also satisfies the following two conditions:

(i) $h|[M]_k = \nu(M)h$ for all $M \in G$;
(ii) h is meromorphic at the cusps of G.

In (i) we require $|\nu(M)| = 1$ for all $M \in G$ and

$$\nu(M_3)(c_3z+d_3)^k = \nu(M_1)\nu(M_2)(c_1M_2z+d_1)^k(c_2z+d_2)^k$$

for all $M_1, M_2 \in G, M_3 = M_1M_2$. If k is integral this is just the condition that ν is a character of G. See for example Knopp [8, Chapter 2] and Shimura [13, Chapter 2]. If in addition h is holomorphic on $\mathcal{H}$ and vanishes at the cusps of G then it is called a cusp form.

Let

$$\eta = q^{1/24}\prod_{i\geq 1}(1-q^i), \quad q = \exp(2\pi iz).$$

Then with the above definitions η is a cusp form of weight 1/2 on $\mathrm{SL}(2,\mathbb{Z})$ for an appropriate multiplier system. Petersson [11], following Rademacher [12], gave an explicit formula for the multiplier system for η:

Theorem 1 *Let $a,b,c,d \in \mathbb{Z}$ with $ad-bc = 1$. Then the multiplier system ν for $\eta(z)$ is given by*

$$\nu\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} \left(\frac{d}{c}\right)^* \exp\left(\frac{\pi i}{12}[(a+d)c - bd(c^2-1) - 3c]\right) & \text{if } c \text{ is odd,} \\ \left(\frac{c}{d}\right)_* \exp\left(\frac{\pi i}{12}[(a+d)c - bd(c^2-1) + 3d - 3 - 3cd]\right) & \text{if } c \text{ is even,} \end{cases}$$

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where if $c \neq 0$ then

$$\left(\frac{c}{d}\right)^* = \left(\frac{c}{|d|}\right) \quad \textit{and} \quad \left(\frac{c}{d}\right)_* = \left(\frac{c}{|d|}\right)(-1)^{\frac{\mathrm{sign(c)}-1}{2}\frac{\mathrm{sign(d)}-1}{2}}$$

with $(\frac{d}{|c|})$ and $(\frac{c}{|d|})$ being the standard Jacobi symbols with $(\frac{c}{1}) = 1$. We also have $(\frac{0}{\pm 1})^* = (\frac{0}{1})_* = -(\frac{0}{-1})_* = 1$.

Note that this formula is for the non-standard choice of argument given above, as can be seen, for example, by considering the transformation $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. See the proof in [8, Chapter 4, Theorem 2] for details.

It follows that $\Delta = \eta^{24}$ is a weight 12 cusp form on $\mathrm{SL}(2,\mathbb{Z})$ with trivial multiplier system. The cusp form Δ has many remarkable properties and has been extensively studied. More generally there has been much study of automorphic forms that can be expressed as products of η functions. One particularly nice result is the following, which we shall use later:

Fix a positive integer N and define $h(z) = \prod_{\delta|N} \eta(\delta z)^{r(\delta)}$ where $\delta > 0$ and $r(\delta) \in \mathbb{Z}$. Let $w = \frac{1}{2}\sum_{\delta|N} r(\delta)$.

Theorem 2 *The function $h(z)$ is an automorphic form on $\Gamma_0(N)$ if and only if the following conditions are satisfied:*

(i) *24 divides $\sum_{\delta|N} \delta r(\delta)$,*
(ii) *24 divides $\sum_{\delta|N} (\frac{N}{\delta}) r(\delta)$,*
(iii) *w is a positive integer.*

If $h(z)$ satisfies these conditions, then it has weight w and multiplier

$$\nu\begin{pmatrix} a & b \\ cN & d \end{pmatrix} = \chi(d) = \left(\frac{(-1)^w D}{d}\right),$$

where $D = \prod_{\delta|N} \delta^{r(\delta)}$. In particular $h(z)$ is an automorphic form with trivial character if and only if it satisfies conditions (i) *and* (ii)*, w is an even positive integer, and D is a square in $\mathbb{Q}$.*

This result is essentially due to Newman [9, 10]; see also [5] and [2]. This particular formulation is taken from Gordon and Ono [6].

Now let f be a positive, squarefree integer and define

$$\Gamma_0(f)^+ = \{e^{-1/2}\left(\begin{smallmatrix} ae & b \\ cf & de \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{R}) | a,b,c,d,e \in \mathbb{Z},\ e|f,\ ade^2 - bcf = e\}.$$

These groups are of particular importance, since Helling [7] has shown that if G is a subgroup of $\mathrm{SL}(2,\mathbb{R})$ which is commensurable with $\mathrm{SL}(2,\mathbb{Z})$, then G is conjugate to a subgroup of $\Gamma_0(f)^+$ for some squarefree f. For this reason we call these groups Helling groups. Conway [3] has given a nice proof of Helling's Theorem. Note that $\Gamma_0(f)^+$ has one cusp.

In the rest of this paper, by *cusp form* we will mean a cusp form with a trivial multiplier system. The aim of this paper is to describe the analogues of the cusp form

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Δ for the Helling groups. We shall call these forms Δ_f. For $\mathrm{SL}(2,\mathbb{Z})$, up to a nonzero multiplicative constant, Δ is the unique cusp form of smallest-weight. We could also define Δ, up to a nonzero multiplicative constant, as the cusp form of smallest weight which is an η product or as the cusp form of smallest weight which does not vanish on $\mathcal{H}$. For a general Helling group G the last two conditions are equivalent and we will define Δ_f to be the smallest weight cusp form on G which is an η product. It is in this sense that Δ_f is a cusp form like Δ. In general Δ_f is not a cusp form of smallest weight.

A complete characterization of the cusp forms Δ_f is given in Theorem 6. The difficulty in obtaining this result is the complexity of the expression for ν in Theorem 1. This problem was also faced by Newman in obtaining Theorem 2. Newman observed that $\Gamma_0(N)$ is generated by matrices satisfying additional congruence conditions and inequalities, and that with these additional conditions the multiplier system simplifies:

Lemma 3 *If $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is an element of $\mathrm{SL}(2,\mathbb{Z})$ with $a > 0$, $c > 0$ and $\gcd(a,6) = 1$, then*

$$\eta(Az) = (-i)^{1/2}\exp(-\pi i\alpha(A))(cz+d)^{(1/2)}\eta(z),$$

and

$$\alpha(A) \equiv \tfrac{1}{12}a(c-b-3) - \tfrac{1}{2}\left(1 - \left(\tfrac{c}{a}\right)\right) \pmod 2.$$

Note that if $c > 0$ then $cz + d \in \mathcal{H}$ and so $0 < \arg(cz+d) < \pi$, so that this lemma holds for our nonstandard choice of argument.

In this paper we will make use of Theorem 1, Lemma 3, the structure of $\Gamma_0(f)^+$ and a congruence argument inspired by Newman to prove Theorem 6. First we need an explicit description of the generators of $\Gamma_0(f)^+$ over $\Gamma_0(f)$; see for example Atkin–Lehner [1]:

Let $f > 1$ be a squarefree integer and p a prime divisor of f and let

$$W_p = p^{-1/2}\begin{pmatrix} ap & b \\ cf & dp \end{pmatrix}$$

where a, b, c, d are integers chosen so that $adp^2 - cfb = p$. Different choices of a, b, c, d give rise to matrices in the same coset of $\Gamma_0(f)$. The Helling group $\Gamma_0(f)^+$ is generated by $\Gamma_0(f)$ together with the W_p for all primes p dividing f. Also W_p normalizes $\Gamma_0(f)$. By an abuse of notation we will refer to any element of the coset $W_p\Gamma_0(f)$ as an Atkin–Lehner element. The proof of Theorem 6 will depend on making a suitable choice of W_p, just as Newman's proof of Theorem 2 depends on making a suitable choice of generators of $\Gamma_0(N)$.

Lemma 4 *Let δ be a positive divisor of f. Then $\eta_\delta | W_p = (p/g^2)^{1/4}\nu\eta_{\delta\circ p}$, where ν is a 24-th root of unity (that depends on W_p), $\delta \circ p = (\delta p)/g^2$ with $g = \gcd(\delta, p)$, and $\eta_\delta(z) = \eta(\delta z)$.*

Proof First note that

$$\begin{pmatrix}\delta & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}ap & b\\ cf & dp\end{pmatrix} = \begin{pmatrix}g & 0\\ 0 & g\end{pmatrix}\begin{pmatrix}ag & b\delta/g\\ fgc/p\delta & dp/g\end{pmatrix}\begin{pmatrix}\delta\circ p & 0\\ 0 & 1\end{pmatrix}$$
$$= \begin{pmatrix}g & 0\\ 0 & g\end{pmatrix} A \begin{pmatrix}\delta\circ p & 0\\ 0 & 1\end{pmatrix}$$

and so $\eta(\delta W_p z) = \eta(A(\delta\circ p)z)$. Hence we have

$$\begin{aligned}\eta(\delta W_p z) &= \nu(A)((fgc/p\delta)(\delta p/g^2)z + dp/g)^{1/2}\eta((\delta\circ p)z)\\ &= \nu(A)((fc/g)z + dp/g)^{1/2}\eta((\delta\circ p)z)\\ &= g^{-1/2}\nu(A)(fcz + dp)^{1/2}\eta((\delta\circ p)z)\\ &= (p/g^2)^{1/4}\nu(A)(p^{-1/2}fcz + p^{-1/2}dp)^{1/2}\eta_{\delta\circ p}.\end{aligned}$$

So $\eta_\delta|W_p = (p/g^2)^{1/4}\nu\eta_{\delta\circ p}$ and $\nu = \nu(A)$ is a 24-th root of unity by Theorem 1, as required. ■

Lemma 5 *If $h(z) = \prod_{\delta|f}\eta(\delta z)^{r(\delta)}$ is an automorphic form on $\Gamma_0(f)^+$, then $r(\delta) = r$ for some fixed r.*

Proof If $h(z)|W_p$ = const $\times\, h(z)$, then by the previous lemma $\prod_{\delta|f}\frac{\eta(\delta z)^{r(\delta)}}{\eta(\delta z)^{r(\delta\circ p)}}$ is a constant. But by [2, Theorem B] this implies that $r(\delta) = r(\delta\circ p)$ for all positive divisors δ of f and all primes p dividing f. But the positive divisors of f form a group of exponent 2 under the operation $\delta\circ\delta' = (\delta\delta'/\gcd(\delta,\delta')^2)$, which is generated by the prime divisors of f. Thus $r(\delta) = r(\delta')$ for all positive divisors δ, δ' of f. ■

Now let $\psi(f) = \prod_{p|f}(1+p)$, which is equal to $\sum_{\delta|f}\delta$ since f is squarefree. Then define

$$r_{\min} = \begin{cases} 24/\gcd(24,\psi(f)) & \text{if } 24/\gcd(24,\psi(f)) \text{ is even or } f \text{ is composite,}\\ 48/\gcd(24,\psi(f)) & \text{if } 24/\gcd(24,\psi(f)) \text{ is odd and } f \text{ is prime,}\end{cases}$$

or equivalently

$$r_{\min} = \begin{cases} 24/\gcd(24,\psi(f)) & \text{if } 8\nmid\psi(f) \text{ or } f \text{ is composite,}\\ 48/\gcd(24,\psi(f)) & \text{if } 8|\psi(f) \text{ and } f \text{ is prime.}\end{cases}$$

A simple calculation shows that if $r(\delta)$ is constant and f is squarefree, then $D = \prod_{\delta|f}\delta^r = f^{r2^{\#f-1}}$. So by Theorem 2 and Lemma 5, $d_f(z) = \prod_{\delta|f}\eta(\delta z)^{r_{\min}}$ is a cusp form with trivial multiplier system on $\Gamma_0(f)$ and is the smallest power of $\prod_{\delta|f}\eta(\delta z)$ that is a cusp form with trivial multiplier system on $\Gamma_0(f)$. Since the square of the Atkin–Lehner element W_p is in $\Gamma_0(f)$, we have $d_f|W_p = \pm d_f$. Set $\Delta_f = d_f$ if $d_f|W_p = d_f$ for all primes p dividing f and $\Delta_f = d_f^2$ otherwise. Then Δ_f is the smallest power of $\prod_{\delta|f}\eta(\delta z)$ that is a cusp form with trivial multiplier system on $\Gamma_0(f)^+$, and it is not difficult to see that every such power is a multiple of Δ_f.

Let $\#f$ be the number of prime factors of f. The following theorem characterizes the two cases $\Delta_f = d_f$ and $\Delta_f = d_f^2$.

Theorem 6 $\Delta_f = d_f$ *if and only if one of the following conditions holds:*

(i) $\#f \geq 3$;

(ii) $\#f = 2$ *and either*

(a) f *is even and either* $p \equiv 1 \pmod 4$ *or* $p \equiv 3 \pmod 8$ *where* p *is the odd factor of* f *or*

(b) f *is odd and* $p \equiv 1 \pmod 4$ *for all factors* p *of* f;

(iii) $\#f = 1$ *and* $f \equiv 1 \pmod 4$ *or* $f = 2$.

To prove this theorem we first derive a transformation rule for η for a particular choice of W_p.

Lemma 7 *Let* f *be a squarefree integer and* p *a divisor of* f. *Let* S *be any finite set of primes excluding* p, *fix a positive integer* m, *and set* $Q = \prod_{q \in S} q^m$. *Then we can take the Atkin–Lehner transformation* W_p *to have the form* $p^{-1/2}\left(\begin{smallmatrix} ap & b \\ f & p \end{smallmatrix}\right)$ *with* $a > 0$, $pa \equiv 1 \pmod Q$, *and* $a \equiv 1 \pmod p$.

Proof If $f' = f/p$, then since f is squarefree, p and f' are coprime. So we can find a and b so that $ap - bf' = 1$ so that $p^{-1/2}\left(\begin{smallmatrix} ap & b \\ f & p \end{smallmatrix}\right)$ is in $\Gamma_0(f)^+$ and so is a possible Atkin–Lehner element. By the Chinese Remainder Theorem, we can find arbitrarily large solutions to the congruences:

$$k \equiv (1-a)f'^{-1} \pmod p,$$
$$k \equiv (p'-a)f'^{-1} \pmod{q^m}, \quad q \nmid f',$$
$$k \equiv \Big(\frac{(p'-a)}{q}\Big)\Big(\frac{f'}{q}\Big)^{-1} \pmod{q^{m-1}}, \quad q \mid f',$$

where p' is some integer such that $p'p \equiv 1 \pmod Q$

Then replacing a by $a' = a + kf'$ and b by $b + kp$ we obtain another Atkin–Lehner element. From the congruence mod p we have that $a' \equiv 1 \pmod p$. The second two congruences imply that $pa' \equiv 1 \pmod Q$ and since we can take kf' to be arbitrarily large we can also arrange for a' to be positive. ∎

Using this lemma we can give the transformation rule in Lemma 8 below. Although it is possible in principle to prove this result using Petersson's formula, a direct application leads to an explosion of special cases. In [10] Newman used a "congruence trick", which makes use of Lemma 3 to simplify the proof of Theorem 2. The proof of Lemma 8 uses a similar strategy to reduce the number of cases that have to be considered, although we still have to use the general formula for some of the cases.

Lemma 8 *Let* $h(z) = \prod_{\delta \mid f} \eta(\delta z)$. *Then with a choice of* W_p *such that* $\gcd(a, 6) = 1$ *(which is possible by Lemma 7):*

$$h|W_p = \Big[\frac{f'}{a}\Big] \exp\Big(\frac{\pi i}{12}[2(a-1)2^{\#f-1} + \psi(f')(a+1)(b-1)]\Big)\, h$$

where $[\frac{f'}{a}]$ *is equal to the Jacobi symbol* $\left(\frac{f'}{a}\right)$ *if* f' *is a prime and* 1 *otherwise.*

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Proof We start by computing $\eta_\delta|W_p$. By Lemma 4 this is equal to

$$(p/g^2)^{1/4}\nu(A)\eta((\delta \circ p)z)$$

where $A = \begin{pmatrix} ag & b\delta/g \\ fg/\delta & p/g \end{pmatrix}$ and $g = \gcd(p, \delta)$. There are two cases: $g = p$ and $g = 1$. If $g = p$ then $A = \begin{pmatrix} ap & b\delta' \\ f'/\delta' & 1 \end{pmatrix}$ where $f' = f/p$ and $\delta' = \delta/p$. For this case we use Petersson's formula, and so we have to consider two subcases, f'/δ' odd and f'/δ' even. However, since the lower right entry is 1, it turns out that these two subcases give the same result:

$$\nu(A) = \exp\Big(\frac{\pi i}{12}\big[\frac{f'}{\delta'}(ap - bf' - 2) + b\delta'\big]\Big).$$

In $h(z)$ we get one such contribution for all the terms in the product for which δ is divisible by p, and so the total contribution to the transformation from these terms is

$$\prod_{\delta'|f'} p^{-1/4}\exp\Big(\frac{\pi i}{12}\big[\frac{f'}{\delta'}(ap - bf' - 2) + b\delta'\big]\Big)$$
$$= p^{-2^{\#f-3}}\exp\Big(\frac{\pi i}{12}\psi(f')[ap - b(f'-1) - 2]\Big).$$

The second case is $g = 1$, and in this case $A = \begin{pmatrix} a & b\delta \\ f'/\delta & p \end{pmatrix}$. Since we have chosen W_p such that $\gcd(a, 6) = 1$, we can use Lemma 3, which gives

$$\nu(A) = \exp\Big(\frac{\pi i}{12}\big[3(a-1) - a\frac{f'}{\delta} + ab\delta + 6(1 - (\frac{f'\delta}{a}))\big]\Big)$$
$$= \Big(\frac{f'/\delta}{a}\Big)\exp\Big(\frac{\pi i}{12}[3(a-1) - a\frac{f'}{\delta} + ab\delta]\Big).$$

In $h(z)$ there is one such contribution for all the terms in the product for which δ is not divisible by p, and so the total contribution to the transformation from these terms is

$$p^{2^{\#f-3}}\Big(\prod_{\delta|f'}(\frac{\delta}{a})\Big)\exp\Big(\frac{\pi i}{12}[3(a-1)2^{\#f-1} + \psi(f')(ab - a)]\Big).$$

Now $\prod_{\delta|f'}\delta$ is a square except in the case that f' is a prime and so $\prod_{\delta|f'}(\frac{\delta}{a}) = \big[\frac{f'}{a}\big]$. So the total constant term in the transformation is

$$\Big[\frac{f'}{a}\Big]\exp\Big(\frac{\pi i}{12}[3(a-1)2^{\#f-1} + \psi(f')(ap - a - b(f'-1) + ab - 2)]\Big).$$

Finally, using the fact that $ap - bf' = 1$ we obtain the expression:

$$\Big[\frac{f'}{a}\Big]\exp\Big(\frac{\pi i}{12}[3(a-1)2^{\#f-1} + \psi(f')(a+1)(b-1)]\Big)$$

as required. ■

Using this lemma we can now complete the proof of Theorem 6.

Proof of Theorem 6 Since d_f is a cusp form for $\Gamma_0(f)$ by Theorem 2, it is sufficient to evaluate the transformation of $d_f|W_p$ for any representative Atkin–Lehner element W_p for all primes p dividing f. As noted above, we either have $d_f|W_p = d_f$ or $d_f|W_p = -d_f$. To determine this sign factor we will use the transformation rule found in Lemma 8 together with the special form of W_p found in Lemma 7. To do this, in Lemma 7 we take S and m to be such that $\gcd(a, 6) = 1$ and $pa \equiv 1 \pmod{p+1}$, which is possible since p and $p + 1$ are coprime. Then $a + 1 \equiv 0 \pmod{p+1}$, so that

$$r_{\min}\psi(f')(a+1)(b-1) = r_{\min}\psi(f)\frac{(a+1)}{(p+1)}(b-1),$$

and the definition of $r_{\min}$ tells us that this is divisible by 24. Thus, from Lemma 8, $d_f|W_p = \nu d_f$ with

$$\nu = \Big[\frac{f'}{a}\Big]^{r_{\min}} \exp\Big(\frac{\pi i}{4} r_{\min}(a-1)2^{\#f-1}\Big).$$

Since a is odd, this implies that if $\#f \geq 3$ then $\nu = 1$.

Suppose next that $\#f = 1$; then $\nu = \exp(\frac{\pi i}{4} r_{\min}(a-1))$. If $f = 2$, then $r_{\min} = 8$ and so $\nu = 1$. Otherwise f is odd. If $f \equiv 1 \pmod 4$, then $4|r_{\min}$ and $\nu = 1$, while if $f \equiv 3 \pmod 4$, then 2 exactly divides $r_{\min}$ (recall that if $24/\gcd(24, p+1)$ is odd then $r_{\min}$ has an extra factor of 2). Also $a - 1 \equiv -2 \pmod{f+1}$ so $a - 1 \equiv 2 \pmod 4$ and hence in this case $\nu = -1$.

The remaining case is $\#f = 2$, say $f = pq$ with q prime, so

$$\begin{aligned}\nu &= \Big(\frac{q}{a}\Big)^{r_{\min}} \exp\Big(\frac{\pi i}{4} r_{\min} 2(a-1)\Big)\\ &= \Big(\frac{q}{a}\Big)^{r_{\min}} (-1)^{r_{\min}(a-1)/2}.\end{aligned}$$

Consider the case that f is even. Take $q = 2$; then $\nu = (-1)^{\frac{a^2-1}{8} r_{\min}} (-1)^{\frac{a-1}{2} r_{\min}}$. If $p \equiv 7 \pmod 8$, then $r_{\min}$ is odd and since $a \equiv -1 \pmod{p+1}$ we have $a \equiv 7 \pmod 8$, which gives $\nu = -1$. If $p \equiv 1, 3, 5 \pmod 8$, then $r_{\min}$ is even and $\nu = 1$. We also have to consider $p = 2$. In this case

$$\nu = \Big(\frac{q}{a}\Big)^{r_{\min}} (-1)^{r_{\min}(a-1)/2}.$$

If q is not congruent to 7 modulo 8 then $r_{\min}$ is even and ν is 1, while if $q \equiv 7 \pmod 8$ then by quadratic reciprocity, $\nu = (\frac{q}{a})(-1)^{(a-1)/2} = (\frac{a}{q}) = (\frac{2}{q})$, since $2a - bq = 1$ gives $2a \equiv 1 \pmod q$. But $(\frac{2}{q}) = 1$ since $q \equiv 7 \pmod 8$, and so ν is one in this case also. This deals with all the cases when $\#f = 2$ and f is even.

Finally consider $\#f = 2$ and f odd. If $p \equiv 1 \pmod 4$ and $q \equiv 1 \pmod 4$ then $r_{\min}$ is even and ν is one. So $\Delta_f = d_f$ in this case.

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f	R	e	w	A	P	f	R	e	w	A	P
1	$[3,2]$	1	12	$1/6$	1^{24}	35	$[2^6]$	4	4	2	$1^2 5^2 7^2 35^2$
2	$[4,2]$	1	8	$1/4$	$1^8 2^8$	38	$[2^7]$	5	4	$5/2$	$1^2 2^2 19^2 38^2$
3	$[6,2]$	2	12	$1/3$	$1^{12} 3^{12}$	39	$[6,2^5]$	14	12	$7/3$	$1^6 3^6 13^6 39^6$
5	$[2^3]$	1	4	$1/2$	$1^4 5^4$	41	$[2^9]$	7	4	$7/2$	$1^4 41^4$
6	$[2^3]$	1	4	$1/2$	$1^2 2^2 3^2 6^2$	42	$[2^6]$	4	4	2	$1^1 2^1 3^1 6^1 7^1 14^1 21^1 42^1$
7	$[3,2^2]$	4	12	$2/3$	$1^{12} 7^{12}$	46	$[2^8]$	6	4	3	$1^2 2^2 23^2 46^2$
10	$[4,2^2]$	3	8	$3/4$	$1^4 2^4 5^4 10^4$	47	$[2^{10}]$	8	4	4	$1^4 47^4$
11	$[2^4]$	2	4	1	$1^4 11^4$	51	$[2^8]$	6	4	3	$1^2 3^2 17^2 51^2$
13	$[3,2^3]$	7	12	$7/6$	$1^{12} 13^{12}$	55	$[2^8]$	6	4	3	$1^2 5^2 11^2 55^2$
14	$[2^4]$	2	4	1	$1^2 2^2 7^2 14^2$	59	$[2^{12}]$	10	4	5	$1^4 59^4$
15	$[2^4]$	2	4	1	$1^2 3^2 5^2 15^2$	62	$[2^{10}]$	8	4	4	$1^2 2^2 31^2 62^2$
17	$[2^5]$	3	4	$3/2$	$1^4 17^4$	66	$[2^8]$	6	4	3	$1^1 2^1 3^1 6^1 11^1 22^1 33^1 66^1$
19	$[3,2^4]$	10	12	$5/3$	$1^{12} 19^{12}$	69	$[2^{10}]$	8	4	4	$1^2 3^2 23^2 69^2$
21	$[6,2^3]$	8	12	$4/3$	$1^6 3^6 7^6 21^6$	70	$[2^8]$	6	4	3	$1^1 2^1 5^1 7^1 10^1 14^1 35^1 70^1$
22	$[2^5]$	3	4	$3/2$	$1^2 2^2 11^2 22^2$	71	$[2^{14}]$	12	4	6	$1^4 71^4$
23	$[2^6]$	4	4	2	$1^4 23^4$	78	$[2^9]$	7	4	$7/2$	$1^1 2^1 3^1 6^1 13^1 26^1 39^1 78^1$
26	$[4,2^4]$	7	8	$7/4$	$1^4 2^4 13^4 26^4$	87	$[2^{12}]$	10	4	5	$1^2 3^2 29^2 87^2$
29	$[2^7]$	5	4	$5/2$	$1^4 29^4$	94	$[2^{14}]$	12	4	6	$1^2 2^2 47^2 94^2$
30	$[2^5]$	3	4	$3/2$	$1^1 2^1 3^1 5^1 6^1 10^1 15^1 30^1$	95	$[2^{12}]$	10	4	5	$1^2 5^2 19^2 95^2$
31	$[3,2^6]$	16	12	$8/3$	$1^{12} 31^{12}$	105	$[2^{10}]$	8	4	4	$1^1 3^1 5^1 7^1 15^1 21^1 35^1 105^1$
33	$[2^6]$	4	4	2	$1^2 3^2 11^2 33^2$	110	$[2^{11}]$	9	4	$9/2$	$1^1 2^1 5^1 10^1 11^1 22^1 55^1 110^1$
34	$[4,2^5]$	9	8	$9/4$	$1^4 2^4 17^4 34^4$	119	$[2^{14}]$	12	4	6	$1^2 7^2 17^2 119^2$

Table 1: f is a squarefree integer such that $\Gamma_0(f)^+$ is genus zero. R is a list of the orders of the classes of elliptic fixed points in partition notation, e is the leading exponent of the q expansion of Δ_f, and w is the weight of Δ_f. A is the area of a fundamental domain of $\Gamma_0(f)^+$ in multiples of 2π, and P is the expression for Δ_f as an η product written in partition notation, for example $1^8 2^8$ in the entry for $f = 2$ means $\Delta_2(z) = \eta(z)^8 \eta(2z)^8$.

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Next suppose both $p \equiv 3 \pmod 4$ and $q \equiv 3 \pmod 4$ then $r_{\min}$ is odd. Also $a \equiv -1 \pmod{p+1}$ implies that $a \equiv 3 \pmod 4$. So $\nu = (\frac{q}{a})(-1)^{(a-1)/2} = -(-1)^{(q-1)/2(a-1)/2}(\frac{a}{q}) = (\frac{p}{q})$, using $ap \equiv 1 \pmod q$ and quadratic reciprocity. By symmetry, the sign factor for W_q is $(\frac{q}{p})$, but by quadratic reciprocity, one of $(\frac{q}{p})$ and $(\frac{p}{q})$ is -1 and $\Delta_f = d_f^2$ in this case. Finally suppose $p \equiv 3 \pmod 4$ and $q \equiv 1 \pmod 4$. Once again, $r_{\min}$ is odd. So the sign factor for W_p is $-(\frac{q}{a}) = -(\frac{a}{q}) = -(\frac{p}{q})$. The sign factor for W_q is $(\frac{p}{\alpha})(-1)^{(\alpha-1)/2}$ (where we have used α rather than a to avoid confusion, since a depends on the prime). Since $p \equiv 3 \pmod 4$, by quadratic reciprocity $(\frac{p}{\alpha})(-1)^{(\alpha-1)/2} = (\frac{\alpha}{p})$ and since $\alpha q \equiv 1 \pmod p$ this gives a sign of $\left(\frac{q}{p}\right)$ for W_q. Thus the product of the sign factors for W_p and W_q is $-(\frac{p}{q})(\frac{q}{p}) = -1$ by quadratic reciprocity. So $\Delta_f = d_f^2$ in this case.

Since all the cases listed in Theorem 6 give $\Delta_f = d_f$ and all the remaining cases give $\Delta_f = d_f^2$, the result follows. ■

If $h \neq 0$ is a cusp form on $\Gamma_0(f)^+$ of weight k and trivial multiplier system, then for suitable integers s and t, h^a/Δ^b is a modular function with divisor supported only at the one cusp of $\Gamma_0(f)^+$. This is only possible if h^a/Δ^b is a constant, so that h is an η product and so $h = \text{const} \times \Delta_f^u$ for some positive integer u. Thus, as mentioned previously, Δ_f is also characterized, up to a nonzero multiplicative constant, as the cusp form of smallest weight on $\Gamma_0(f)^+$ that does not vanish on $\mathcal{H}$.

If the genus of $G = \Gamma_0(f)^+$ is not zero, then there are cusp forms of weight 2 on G. A simple calculation shows that the weight of Δ_f is always divisible by 4, and so in this case Δ_f is never a cusp form of smallest weight. The cases when the genus of G is zero are given in Table 1, together with the expression for Δ_f. Since the signatures of these groups are known, see for example Cummins [4] and the references therein, we can use Shimura's expression for the dimensions of spaces of cusp forms [13, Theorem 2.24] to conclude that Δ_f is, up to a nonzero multiplicative constant, the unique cusp form of smallest weight only in the cases $f = 1, 2, 5, 6$.

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