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# ON HARMONIC BLOCH SPACES IN THE UNIT BALL OF $\mathbb{C}^n$

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**Abstract**

In this paper, our main aim is to discuss the properties of harmonic mappings in the unit ball $\mathbb{B}^n$. First, we characterize the harmonic Bloch spaces and the little harmonic Bloch spaces from $\mathbb{B}^n$ to $\mathbb{C}$ in terms of weighted Lipschitz functions. Then we prove the existence of a Landau–Bloch constant for a class of vector-valued harmonic Bloch mappings from $\mathbb{B}^n$ to $\mathbb{C}^n$.

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## 1. Introduction and preliminaries

Let $\mathbb{C}$ denote the complex plane and let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Also we let $\mathbb{C}^n = \{z = (z_1, \ldots, z_n) : z_1, \ldots, z_n \in \mathbb{C}\}$ and for $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$,

$$\mathbb{B}^n(a, r) = \left\{z \in \mathbb{C}^n : |z - a| = \sqrt{\sum_{k=1}^{n} |z_k - a_k|^2} < r\right\}.$$

Especially, we use $\mathbb{B}^n$ to denote the unit ball $\mathbb{B}^n(0, 1)$ and for $a \in \mathbb{R}^n$,

$$\mathbb{B}^n_R(a, r) = \left\{x \in \mathbb{R}^n : |x - a| = \sqrt{\sum_{k=1}^{n} |x_k - a_k|^2} < r\right\}.$$

A function $f = u + iv$ of an open subset $\Omega \subset \mathbb{C}^n$ into $\mathbb{C}$ is called a *harmonic mapping* if both $u$ and $v$ are real harmonic in $\Omega$, that is, $\Delta u = 0$ and $\Delta v = 0$, where $\Delta$ represents the complex Laplacian operator (see [10, 15–17])

$$\Delta = 4 \sum_{k=1}^{n} \frac{\partial^2}{\partial z_k \partial \overline{z}_k} = \sum_{k=1}^{n} \left(\frac{\partial^2}{\partial x_k^2} + \frac{\partial^2}{\partial y_k^2}\right)$$

and for each $k \in \{1, \ldots, n\}$, $z_k = x_k + iy_k$.

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A planar harmonic mapping $f$ in $\mathbb{D}$ is called a *harmonic Bloch mapping* if and only if *the Lipschitz number*

$$\beta_f = \sup_{z,w\in\mathbb{D},z\neq w} \frac{|f(z)-f(w)|}{\rho(z,w)} < +\infty,$$

where

$$\rho(z,w) = \frac{1}{2}\log\left(\frac{1+\left|\frac{z-w}{1-\overline{z}w}\right|}{1-\left|\frac{z-w}{1-\overline{z}w}\right|}\right) = \operatorname{arctanh}\left|\frac{z-w}{1-\overline{z}w}\right|$$

denotes the hyperbolic distance between $z$ and $w$ in $\mathbb{D}$. In [7], Colonna proved that

$$\beta_f = \sup_{z\in\mathbb{D}}(1-|z|^2)[|f_z(z)|+|f_{\overline{z}}(z)|] \quad \text{(see also [3–6])}.$$

DEFINITION 1.1. The *harmonic Bloch space* $\mathcal{HB}$ consists of all harmonic mappings $f$ of $\mathbb{B}^n$ into $\mathbb{C}$ such that

$$\|f\|_{\mathcal{HB}} = \sup_{z\in\mathbb{B}^n}\{(1-|z|^2)[|\nabla f(z)|+|\nabla\overline{f}(z)|]\} < \infty,$$

where $\nabla f = (\partial f/\partial z_1, \ldots, \partial f/\partial z_n)$ denotes the complex gradient of $f$ and $\nabla\overline{f} = (\partial f/\partial\overline{z}_1, \ldots, \partial f/\partial\overline{z}_n)$.

DEFINITION 1.2. The *little harmonic Bloch space* $\mathcal{HB}_0$ consists of all mappings $f \in \mathcal{HB}$ such that

$$\lim_{|z|\to 1-}\{(1-|z|^2)[|\nabla f(z)|+|\nabla\overline{f}(z)|]\} = 0.$$

For any $z \neq w \in \mathbb{B}^n$, let

$$\mathcal{L}_f(z,w) = \frac{(1-|z|^2)^{1/2}(1-|w|^2)^{1/2}|f(z)-f(w)|}{|z-w|}$$

denote the *weighted Lipschitz function*, where $f:\mathbb{B}^n\to\mathbb{C}$ is a harmonic mapping. The relationship between $\mathcal{L}_f(z,w)$ and the Bloch space (or the little Bloch space) has attracted much attention (see [7, 9, 13, 17]). Recently, many authors have also discussed the relationship between Lipschitz continuity and harmonic quasi-conformal (or quasi-regular) mappings in $\mathbb{B}^n$ (see [1, 2, 11, 12, 15]). In this paper, we use the weighted Lipschitz functions to characterize the harmonic Bloch spaces and the little harmonic Bloch spaces in $\mathbb{B}^n$. Our main results are Theorems 2.2 and 2.4. Their proofs will be presented in Section 2.

Let $f = (f_1, \ldots, f_n)$ be a *vector-valued harmonic mapping* from $\mathbb{B}^n$ into $\mathbb{C}^n$, that is, for each $i \in \{1, 2, \ldots, n\}$, $f_i$ is a harmonic mapping from $\mathbb{B}^n$ into $\mathbb{C}$. Let $H(\mathbb{B}^n)$ denote all harmonic mappings of $\mathbb{B}^n$ into $\mathbb{C}^n$. For any $f = (f_1, \ldots, f_n) \in H(\mathbb{B}^n)$,

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denote by

$$f_z = (\nabla f_1, \dots, \nabla f_n)^T$$

the matrix formed by the complex gradients $\nabla f_1, \dots, \nabla f_n$, where $T$ denotes matrix transposition, and let

$$f_{\overline{z}} = (\nabla \overline{f}_1, \dots, \nabla \overline{f}_n)^T.$$

For an $n \times n$ matrix $A$, we introduce the operator norm

$$|A| = \sup_{x \neq 0} \frac{|Ax|}{|x|} = \max\{|A\theta| : \theta \in \partial\mathbb{B}^n\}.$$

Here and in the following, we always treat any $z \in \mathbb{C}^n$ as a column vector, that is, $z = (z_1, \dots, z_n)^T$, unless otherwise stated.

DEFINITION 1.3. The *vector-valued harmonic Bloch space* $\mathcal{HB}(n)$ consists of all mappings $f \in H(\mathbb{B}^n)$ such that

$$\|f\|_{\mathcal{HB}(n)} = \sup_{z \in \mathbb{B}^n} \{(1 - |z|^2)[|f_z(z)| + |f_{\overline{z}}(z)|]\} < \infty.$$

In Section 3, we prove the existence of the Landau constant for a class of mappings in $\mathcal{HB}(n)$, which is stated as Theorem 3.6.

## 2. The relationship between weighted Lipschitz functions and harmonic Bloch spaces

We shall make use of the group consisting of all biholomorphic mappings of $\mathbb{B}^n$ onto itself, which is denoted by $\mathrm{Aut}(\mathbb{B}^n)$. The following results are from [16].

(i) For any $a \in \mathbb{B}^n$, let

$$\phi_a(z) = \frac{a - P_a z - (1 - |a|^2)^{1/2} Q_a z}{1 - \langle z, a \rangle}.$$

Then $\phi_a \in \mathrm{Aut}(\mathbb{B}^n)$, where

$$\langle z, a \rangle = z_1 \overline{a}_1 + \cdots + z_n \overline{a}_n, \qquad P_a z = \frac{a \langle z, a \rangle}{\langle a, a \rangle}$$

and $Q_a z = z - P_a z$.

(ii) For any $\phi_a$,

$$\phi_a(0) - a = \phi_a(a) = 0, \qquad \phi_a = \phi_a^{-1}$$

and

$$1 - |\phi_a(z)|^2 = \frac{(1 - |z|^2)(1 - |a|^2)}{|1 - \langle z, a \rangle|^2}. \tag{2.1}$$

By using similar reasoning as in the proof of [15, Lemma 2.5], we have the following lemma.

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LEMMA 2.1. *Suppose that* $f : \overline{\mathbb{B}}^n_R(a, r) \to \mathbb{R}$ *is a continuous mapping in* $\overline{\mathbb{B}}^n_R(a, r)$ *and harmonic in* $\mathbb{B}^n_R(a, r)$. *Then*

$$|\nabla f(a)| \leq \frac{\sqrt{n}}{V(n)r^n} \int_{\partial \mathbb{B}^n_R(a,r)} |f(a) - f(y)| \, d\sigma(y),$$

*where* $d\sigma$ *denotes the surface measure on* $\partial \mathbb{B}^n_R(a, r)$ *and* $V(n)$ *the volume of the unit ball in* $\mathbb{R}^n$.

PROOF. Without loss of generality, we may assume that $a = 0$ and $f(0) = 0$. Let

$$K(x, y) = \frac{r^2 - |x|^2}{nrV(n)|x - y|^n}.$$

Then

$$f(x) = \int_{\partial \mathbb{B}^n_R(0,r)} K(x, t) f(t) \, d\sigma(t), \quad x \in \mathbb{B}^n_R(0, r),$$

where $d\sigma$ denotes the surface measure on $\partial \mathbb{B}^n_R(0, r)$. Calculations lead to

$$\frac{\partial}{\partial x_i} K(x, t) = \frac{1}{nrV(n)} \left[ \frac{-2x_i}{|x - t|^n} - \frac{n(r^2 - |x|^2)(x_i - t_i)}{|x - t|^{n+2}} \right]$$

which yields

$$\frac{\partial}{\partial x_i} K(0, t) = \frac{t_i}{V(n)r^{n+1}},$$

whence

$$\begin{aligned}
|\nabla f(0)| &= \left[ \sum_{i=1}^n \left| \int_{\partial \mathbb{B}^n_R(0,r)} \frac{\partial}{\partial x_i} K(0, t) f(t) \, d\sigma(t) \right|^2 \right]^{1/2} \\
&\leq \sum_{i=1}^n \left| \int_{\partial \mathbb{B}^n_R(0,r)} \frac{\partial}{\partial x_i} K(0, t) f(t) \, d\sigma(t) \right| \\
&\leq \int_{\partial \mathbb{B}^n_R(0,r)} |f(t)| \sum_{i=1}^n \left| \frac{\partial}{\partial x_i} K(0, t) \right| d\sigma(t) \\
&\leq \sqrt{n} \int_{\partial \mathbb{B}^n_R(0,r)} |f(t)| \left( \sum_{i=1}^n \left| \frac{\partial}{\partial x_i} K(0, t) \right|^2 \right)^{1/2} d\sigma(t) \\
&= \frac{\sqrt{n}}{V(n)r^n} \int_{\partial \mathbb{B}^n_R(0,r)} |f(t)| \, d\sigma(t).
\end{aligned}$$

The proof of the lemma is complete. □

THEOREM 2.2. *Let* $f$ *be a harmonic mapping in* $\mathbb{B}^n$. *Then* $f \in \mathcal{HB}$ *if and only if*

$$\sup_{z, w \in \mathbb{B}^n, z \neq w} \mathcal{L}_f(z, w) < +\infty.$$

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PROOF. First we prove the sufficiency. Let $f(z) = u(z) + iv(z)$, where $u$ and $v$ are real harmonic functions. Fix $r \in (0, 1)$. Then by (2.1),

$$\frac{|\phi_a(z)|}{|z-a|} = \sqrt{\frac{|z-a|^2 + |\langle z, a\rangle|^2 - |z|^2|a|^2}{|z-a|^2|1-\langle z, a\rangle|^2}} \leq \frac{1}{|1-\langle z, a\rangle|},$$

which gives

$$|\phi_a(z)| \leq \frac{|z-a|}{|1-\langle z, a\rangle|} \leq \frac{|z-a|}{1-|a|}, \tag{2.2}$$

whence for any $a \in \mathbb{B}^n$,

$$\mathbb{B}^n\left(a, \frac{r(1-|a|^2)}{2}\right) \subset E(a, r),$$

where

$$E(a, r) = \{z \in \mathbb{B}^n : |\phi_a(z)| < r\}.$$

By Lemma 2.1,

$$\begin{aligned}(1-|z|^2)|\nabla u(z)| &\leq \frac{\sqrt{2n}(1-|z|^2)}{V(2n)\left[\frac{r(1-|z|^2)}{2}\right]^{2n}} \int_{\partial\mathbb{B}^n(z, r(1-|z|^2)/2)} |u(\zeta) - u(z)|\, d\sigma(\zeta)\\ &= M(|z|, r)\int_{\partial\mathbb{B}^n(z, r(1-|z|^2)/2)} |u(\zeta) - u(z)|\, d\sigma(\zeta),\end{aligned}$$

where $V(2n)$ denotes the volume of the unit ball in $\mathbb{R}^{2n}$ and

$$M(|z|, r) = \frac{2^{2n}\sqrt{2n}}{V(2n)(1-|z|^2)^{2n-1}r^{2n}}.$$

Similarly, we obtain

$$(1-|z|^2)|\nabla v(z)| \leq M(|z|, r)\int_{\partial\mathbb{B}^n(z, r(1-|z|^2)/2)} |v(\zeta) - v(z)|\, d\sigma(\zeta).$$

Cauchy's inequality and chain rules of derivation show that

$$|\nabla f(z)| \leq \tfrac{1}{2}(|\nabla u(z)| + |\nabla v(z)|) \quad \text{and} \quad |\nabla \overline{f}(z)| \leq \tfrac{1}{2}(|\nabla u(z)| + |\nabla v(z)|),$$

which implies that

$$\begin{aligned}(1-|z|^2)(|\nabla f(z)| + |\nabla \overline{f}(z)|) &\leq (1-|z|^2)(|\nabla u(z)| + |\nabla v(z)|)\\ &\leq M(|z|, r)\int_{\partial\mathbb{B}^n(z, r(1-|z|^2)/2)} (|u(\zeta) - u(z)|\\ &\qquad + |v(\zeta) - v(z)|)\, d\sigma(\zeta)\\ &\leq \sqrt{2}M(|z|, r)M_1\int_{\partial\mathbb{B}^n(z, r(1-|z|^2)/2)} d\sigma(\zeta)\\ &= \frac{8M_1 n^{3/2}}{r},\end{aligned}$$

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where $M_1 = \sup\{|f(z) - f(w)| : w \in E(z, r)\}$. Hence for any $w \in \mathbb{B}^n(z, r(1 - |z|^2)/2) \subset E(z, r)$, it follows from (2.2) that

$$\begin{aligned}\frac{(1-|z|^2)^{1/2}(1-|w|^2)^{1/2}}{|z-w|} &= \frac{(1-|z|^2)^{1/2}(1-|w|^2)^{1/2}}{|1-\langle z, w\rangle|}\cdot\frac{|1-\langle z, w\rangle|}{|z-w|}\\ &= \sqrt{1-|\phi_z(w)|^2}\cdot\frac{|1-\langle z, w\rangle|}{|z-w|}\\ &\geq \sqrt{1-r^2}\cdot\frac{|1-\langle z, w\rangle|}{|z-w|}\\ &\geq \frac{\sqrt{1-r^2}}{r}.\end{aligned}$$

Therefore, there exists a positive constant $M_2(n, r)$ such that

$$(1-|z|^2)[|\nabla f(z)| + |\nabla\overline{f}(z)|] \leq M_2(n, r) \sup_{w\in E(z,r), w\neq z} \mathcal{L}_f(z, w),$$

from which we see that $f \in \mathcal{HB}$.

We now prove the necessity. For any $z \neq w \in \mathbb{B}^n$,

$$\begin{aligned}|f(z) - f(w)| &= \left|\int_0^1 \frac{df}{dt}(zt + (1-t)w)\,dt\right|\\ &= \left|\sum_{k=1}^n (z_k - w_k)\int_0^1 \frac{df}{dz_k}(zt + (1-t)w)\,dt\right.\\ &\qquad \left.+ \sum_{k=1}^n (\overline{z}_k - \overline{w}_k)\int_0^1 \frac{df}{d\overline{z}_k}(zt + (1-t)w)\,dt\right|\\ &\leq \sum_{k=1}^n |z_k - w_k|\cdot\left|\int_0^1 \frac{df}{dz_k}(zt + (1-t)w)\,dt\right|\\ &\qquad + \sum_{k=1}^n |\overline{z}_k - \overline{w}_k|\cdot\left|\int_0^1 \frac{df}{d\overline{z}_k}(zt + (1-t)w)\,dt\right|\\ &\leq \left(\sum_{k=1}^n |z_k - w_k|^2\right)^{1/2}\left\{\left[\sum_{k=1}^n\left(\int_0^1\left|\frac{\partial f}{\partial z_k}(zt + (1-t)w)\right|dt\right)^2\right]^{1/2}\right.\\ &\qquad \left.+ \left[\sum_{k=1}^n\left(\int_0^1\left|\frac{\partial f}{\partial \overline{z}_k}(zt + (1-t)w)\right|dt\right)^2\right]^{1/2}\right\}\\ &\leq \sqrt{n}|z-w|\left[\int_0^1 |\nabla f(tz + (1-t)w)|\,dt\right.\\ &\qquad \left.+ \int_0^1 |\nabla\overline{f}(tz + (1-t)w)|\,dt\right],\end{aligned}$$

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from which we infer that

$$\begin{aligned}\frac{|f(z)-f(w)|}{|z-w|} &\le \sqrt{n}\int_0^1 \frac{[|\nabla f(\psi(t))|+|\nabla \overline{f}(\psi(t))|](1-|\psi(t)|^2)}{1-|\psi(t)|^2}\,dt\\ &\le \sqrt{n}\|f\|_{\mathcal{HB}}\int_0^1 \frac{dt}{1-|\psi(t)|^2}\\ &\le \sqrt{n}\|f\|_{\mathcal{HB}}\int_0^1 \frac{dt}{[(1-t)(1-|z|)]^{1/2}[t(1-|w|)]^{1/2}}\\ &= \frac{\pi\sqrt{n}\|f\|_{\mathcal{HB}}}{(1-|z|)^{1/2}(1-|w|)^{1/2}},\end{aligned}$$

where $\psi(t)=tz+(1-t)w$. Thus

$$\sup_{z,w\in\mathbb{B}^n, z\neq w} \mathcal{L}_f(z,w) \le 2\pi\sqrt{n}\|f\|_{\mathcal{HB}}. \tag{2.3}$$

Hence the proof is complete. □

REMARK 2.3. When $n=1$ (respectively $n=1$ and $f_{\overline{z}}\equiv 0$), Theorem 2.2 coincides with [7, Theorem 1] (respectively [9, Theorem 3]).

THEOREM 2.4. *Let $f$ be a harmonic mapping in $\mathbb{B}^n$. Then $f\in\mathcal{HB}_0$ if and only if*

$$\lim_{|z|\to 1-}\ \sup_{z,w\in\mathbb{B}^n, z\neq w} \mathcal{L}_f(z,w) = 0. \tag{2.4}$$

PROOF. In order to prove the sufficiency, we assume that (2.4) holds. Then for any $\varepsilon>0$, there exists $\delta\in(0,1)$ such that

$$\sup_{z,w\in\mathbb{B}^n, z\neq w} \mathcal{L}_f(z,w) < \varepsilon,$$

whenever $\delta<|z|<1$. Similar arguments to the proof of the sufficiency of Theorem 2.2 show that

$$\begin{aligned}(1-|z|^2)[|\nabla f(z)|+|\nabla\overline{f}(z)|] &\le \frac{\sqrt{2}M(|z|,r)r}{\sqrt{1-r^2}}\int_{\partial\mathbb{B}^n(z,r(1-|z|^2)/2)} \mathcal{L}_f(z,w)\,d\sigma(w)\\ &\le \frac{\sqrt{2}M(|z|,r)r}{\sqrt{1-r^2}}\varepsilon\int_{\partial\mathbb{B}^n(z,r(1-|z|^2)/2)} d\sigma(w)\\ &= \frac{8n^{3/2}}{\sqrt{1-r^2}}\varepsilon,\end{aligned}$$

whenever $\delta<|z|<1$. Hence

$$\lim_{|z|\to 1-}(1-|z|^2)[|\nabla f(z)|+|\nabla\overline{f}(z)|]=0.$$

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We now prove the necessity. For $r \in (0, 1)$, let $f_r(z) = f(rz)$. Similar reasoning to the proof of (2.3) shows that there exist positive constants $M_3$ and $M_4$ such that for any $z \neq w \in \mathbb{B}^n$,

$$(1-|z|^2)^{1/2}(1-|w|^2)^{1/2}\frac{|(f(z)-f_r(z))-(f(w)-f_r(w))|}{|z-w|} \le M_3\|f-f_r\|_{\mathcal{HB}}$$

and

$$\begin{aligned}
&(1-|z|^2)^{1/2}(1-|w|^2)^{1/2}\frac{|f_r(z)-f_r(w)|}{|z-w|}\\
&= \frac{r(1-|z|^2)^{1/2}(1-|w|^2)^{1/2}}{(1-|zr|^2)^{1/2}(1-|wr|^2)^{1/2}}(1-|zr|^2)^{1/2}(1-|wr|^2)^{1/2}\frac{|f_r(z)-f_r(w)|}{|rz-rw|}\\
&\le \frac{M_4 r(1-|z|^2)^{1/2}}{(1-r^2)}\|f\|_{\mathcal{HB}}.
\end{aligned}$$

These yield that

$$\sup_{z,w\in\mathbb{B}^n, z\neq w} \mathcal{L}_f(z,w) \le M_3\|f-f_r\|_{\mathcal{HB}} + \frac{M_4 r(1-|z|^2)^{1/2}}{(1-r^2)}\|f\|_{\mathcal{HB}}.$$

In the above inequality, by letting $|z| \to 1-$ and then $r \to 1-$, we obtain the desired result. $\square$

## 3. Landau constant for a class of harmonic Bloch mappings

We introduce a version of the Schwarz lemma for planar harmonic mappings, which is from [8].

Lemma 3.1 [8, Lemma]. *Let $f$ be a harmonic mapping of $\mathbb{D}$ such that $f(0) = 0$ and $f(\mathbb{D}) \subset \mathbb{D}$. Then*

$$|f(z)| \le \frac{4}{\pi}\arctan|z| \le \frac{4}{\pi}|z| \quad \text{for } z \in \mathbb{D}.$$

The following result is a direct generalization of Lemma 3.1 to the setting of harmonic mappings from $\mathbb{B}^n$ to $\mathbb{C}$.

Lemma 3.2. *Let $f$ be a harmonic mapping from $\mathbb{B}^n$ to $\mathbb{C}$ satisfying $f(0) = 0$ and $|f| < M$, where $M$ is a positive constant. Then*

$$|f(z)| \le \frac{4M}{\pi}\cdot\frac{|z|}{\sqrt{1-|z|^2}}.$$

Corollary 3.3. *Let $f \in H(\mathbb{B}^n)$ such that $f(0) = 0$ and $|f| < M$, where $M$ is a positive constant. Then*

$$|f(z)| \le \frac{4M\sqrt{n}}{\pi}\cdot\frac{|z|}{\sqrt{1-|z|^2}}.$$

PROOF. Let $f = (f_1, \ldots, f_n)$. Then Lemma 3.2 implies that

$$|f(z)| = \left(\sum_{k=1}^{n} |f_k(z)|^2\right)^{1/2} \leq \frac{4M\sqrt{n}}{\pi} \cdot \frac{|z|}{\sqrt{1-|z|^2}},$$

which shows that our corollary holds. □

COROLLARY 3.4. *Let $A = (a_{i,j}(z))_{n\times n}$ be a matrix-valued harmonic mapping of $\mathbb{B}^n(0, r)$ into the space of all $n \times n$ complex matrices, that is, each $a_{i,j}(z)$ is a harmonic mapping of $\mathbb{B}^n(0, r)$ into $\mathbb{C}$. If $A(0) = 0$ and $|A(z)| \leq M$ for $z \in \mathbb{B}^n(0, r)$, then*

$$|A(z)| \leq \frac{4M\sqrt{n}}{\pi} \cdot \frac{|z|}{\sqrt{r^2-|z|^2}}.$$

PROOF. For any fixed $\theta \in \partial\mathbb{B}^n$, let $P(z) = A(z)\theta$. Then $P \in H(\mathbb{B}^n)$ and $|P(z)| \leq M$ for $z \in \mathbb{B}^n(0, r)$. By Corollary 3.3,

$$|P(z)| \leq \frac{4M\sqrt{n}}{\pi} \cdot \frac{|z|}{\sqrt{r^2-|z|^2}}.$$

The arbitrariness of $\theta$ implies that

$$|A(z)| \leq \frac{4M\sqrt{n}}{\pi} \cdot \frac{|z|}{\sqrt{r^2-|z|^2}},$$

which completes the proof. □

The following lemma due to Liu is from [14], which is crucial for the proof of our next main result.

LEMMA 3.5 [14, Lemma 4]. *Let $A$ be an $n \times n$ complex matrix. Then for any unit vector $\theta \in \partial\mathbb{B}^n$, the inequality*

$$|A\theta| \geq \frac{|\det A|}{|A|^{n-1}}$$

*holds.*

THEOREM 3.6. *Let $f$ be a vector-valued harmonic mapping of $\mathbb{B}^n$ into $\mathbb{C}^n$ with $f(0) = 0$, $|\det f_z(0)| - \alpha = |f_{\overline{z}}(0)| = 0$ and $\|f\|_{\mathcal{HB}(n)} \leq M$, where $M$ and $\alpha$ are positive constants. Then $f$ is univalent in $\mathbb{B}^n(0, \rho_0/2)$, where*

$$\rho_0 = \frac{t}{\sqrt{1+t^2}} \quad \text{and} \quad t = \frac{3\alpha\pi}{44\sqrt{n}M^n}.$$

*Moreover, the range $f(\mathbb{B}^n(0, \rho_0))$ contains a univalent ball $\mathbb{B}^n(0, R)$, where*

$$R \geq \frac{\rho_0}{2}\left\{\frac{\alpha}{M^{n-1}} - \frac{22M\sqrt{n}}{3\pi} \cdot \frac{\rho_0}{\sqrt{1-\rho_0}}\right\}.$$

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PROOF. For $\zeta \in \mathbb{B}^n$, let $F(\zeta) = 2f(1/2\zeta)$. Then

$$|F_\zeta(\zeta)| + |F_{\overline{\zeta}}(\zeta)| \le \frac{M}{1 - \frac{|\zeta|^2}{4}} \le \frac{4M}{3}$$

and

$$|F_\zeta(\zeta) - F_\zeta(0)| \le |F_\zeta(\zeta)| + |F_\zeta(0)| \le \frac{7M}{3}.$$

Corollary 3.4 implies that

$$|F_\zeta(\zeta) - F_\zeta(0)| \le \frac{28M\sqrt{n}}{3\pi} \cdot \frac{|\zeta|}{\sqrt{1 - |\zeta|^2}}.$$

Since for any $\zeta \in \mathbb{B}^n$,

$$|F_{\overline{\zeta}}(\zeta) - F_{\overline{\zeta}}(0)| = |F_{\overline{\zeta}}(\zeta)| \le \frac{4M}{3},$$

Corollary 3.4 again implies that

$$|F_{\overline{\zeta}}(\zeta) - F_{\overline{\zeta}}(0)| \le \frac{16M\sqrt{n}}{3\pi} \cdot \frac{|\zeta|}{\sqrt{1 - |\zeta|^2}}.$$

On the other hand, for any $\theta \in \partial\mathbb{B}^n$, we infer from Lemma 3.5 that

$$|F_\zeta(0)\theta| \ge \frac{\alpha}{|F_\zeta(0)|^{n-1}} \ge \frac{\alpha}{M^{n-1}}.$$

In order to prove the univalence of $F$ in $\mathbb{B}^n(0, \rho)$, we choose two distinct points $\zeta', \zeta'' \in \mathbb{B}^n(0, \rho)$ and let $[\zeta', \zeta'']$ denote the segment from $\zeta'$ to $\zeta''$ with the endpoints $\zeta'$ and $\zeta''$, where $\rho = t/\sqrt{1 + t^2}$ and $t = 3\alpha\pi/44\sqrt{n}M^n$. Set $d\zeta = (d\zeta_1, \ldots, d\zeta_n)^T$ and $(d\overline{\zeta} = (d\overline{\zeta}_1, \ldots, d\overline{\zeta}_n)^T$. Then we have

$$\begin{aligned}
|F(\zeta') - F(\zeta'')| &\ge \left| \int_{[\zeta',\zeta'']} F_\zeta(0)\, d\zeta + F_{\overline{\zeta}}(0)\, d\overline{\zeta} \right| \\
&\quad - \left| \int_{[\zeta',\zeta'']} (F_\zeta(\zeta) - F_\zeta(0))\, d\zeta + (F_{\overline{\zeta}}(\zeta) - F_{\overline{\zeta}}(0))\, d\overline{\zeta} \right| \\
&\ge |\zeta' - \zeta''| \left\{ \frac{\alpha}{M^{n-1}} - \frac{44M\sqrt{n}}{3\pi} \cdot \frac{\rho}{\sqrt{1 - \rho^2}} \right\} \\
&> 0.
\end{aligned}$$

This shows that $F$ is univalent in $\mathbb{B}^n(0, \rho)$.

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Furthermore, for any $z$ with $|\zeta| = \rho$,

$$
\begin{aligned}
|F(\zeta) - F(0)| &\geq \left| \int_{[0,\zeta]} F_\zeta(0)\, d\zeta + F_{\overline{\zeta}}(0)\, d\overline{\zeta} \right| \\
&\quad - \left| \int_{[0,\zeta]} (F_\zeta(\zeta) - F_\zeta(0))\, d\zeta + (F_{\overline{\zeta}}(\zeta) - F_{\overline{\zeta}}(0))\, d\overline{\zeta} \right| \\
&\geq \rho \left\{ \frac{\alpha}{M^{n-1}} - \frac{22M\sqrt{n}}{3\pi} \cdot \frac{\rho}{\sqrt{1-\rho^2}} \right\}.
\end{aligned}
$$

Hence the range $f(\mathbb{B}^n(0, \rho_0))$ contains a univalent ball $\mathbb{B}^n(0, R)$, where

$$R \geq \frac{\rho}{2} \left\{ \frac{\alpha}{M^{n-1}} - \frac{22M\sqrt{n}}{3\pi} \cdot \frac{\rho}{\sqrt{1-\rho^2}} \right\}.$$

The proof of this theorem is complete. □

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