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MINIMAL PLAT REPRESENTATIONS OF PRIME KNOTS AND LINKS ARE NOT UNIQUE

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1. Introduction. Let $\tilde{L}$ denote the 2-fold cyclic covering space branched over a link L in S^3. We wish to describe an infinite family of prime knots and links in which each member L exhibits two minimal 6-plat representations, where the associated Heegaard splittings of $\tilde{L}$ are minimal and inequivalent. Thus each knot or link of that family admits at least two equivalence classes of 6-plat representations which are minimal.

Recently, Joan Birman has proved in [**5**] that all plat representations of a link are *stably* equivalent. In the same paper, Birman shows that the adjective "stably" cannot be deleted for composite knots. Birman asks (see Problem 32 of page 220 of [**4**]) if all $2n$-plat representations of a *prime* link are equivalent. We will see in this note that that is not the case.

The result of this paper is similar to that of [**3**]. Reidemeister [**9**] and Singer [**11**] proved that all Heegaard representations of a closed, orientable 3-manifold are *stably* equivalent. Engmann [**6**], and also Birman [**2**], have found connected sums of lens spaces which exhibit inequivalent Heegaard splittings. In [**3**], an infinite family of *prime* 3-manifolds is described. Each of these exhibits inequivalent Heegaard splittings. Note that we present in this paper new examples of these manifolds, namely the 2-fold covering spaces branched over the links studied here.

2. Preliminaries. If we represent S^3 as $R^3 + \infty$, then the x, y plane separates S^3 in two 3-balls D_1 and D_2, D_1 containing the positive part of axis z. Let A be a collection of n circles in the x, z plane, of radii 2 and centers at points $(2 + 8i, 0, 0)$, where $0 \leqq i \leqq n - 1$. Let $\tau : D_1 \rightarrow D_2$ be the symmetry with respect to the x, y plane. Let $p_i : \tilde{D}_i \rightarrow D_i$ be the 2-fold cyclic covering branched over $D_i \cap A$, $i = 1, 2$. Note that $\tilde{D}_1$ and $\tilde{D}_2$ are handlebodies of genus $n - 1$, and let $\tilde{\tau} : \tilde{D}_1 \rightarrow \tilde{D}_2$ be a homeomorphism such that $p_2\tilde{\tau} = \tau p_1$. We orient S^3, $\tilde{D}_1$ and $\tilde{D}_2$ so that p_1 and p_2 are orientation-preserving.

Each link L in S^3 has a $2n$-*plat representation* for some $n \geqq 1$. By definition this is a triad (S^3, L, S), where S is a 2-sphere which separates S^3 into two 3-balls B_1 and B_2 so that $B_i \cap L$ is a collection of n unknotted and unlinked arcs with $\partial(B_i \cap L)$ a set of $2n$ points on ∂B_i, for $i = 1, 2$. The *plat number* of L is the smallest integer n so that L admits such a representation. Two such $2n$-plats (S^3, L, S) and (S^3, L', S') are *equivalent* if (S^3, L, S) and (S^3, L', S')

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are of the same topological type. This topological type is fully described by an element of the classical braid group B_{2n}. Concretely, an orientation-preserving homeomorphism $\phi : \partial D_1 \rightarrow \partial D_1$, which keeps $\partial D_1 \cap A$ fixed as a set, defines a plat $(D_1 \cup_\phi D_2, (A \cap D_1) \cup_\phi (A \cap D_2), \partial D_1)$, where the topological type does not depend on the particular choice of D_1, D_2 or A. Conversely, given a plat (S^3, L, S), where S separates S^3 into two 3-balls B_1 and B_2, there are orientation-preserving homeomorphisms $\alpha_i : B_i \rightarrow D_i$ with $\alpha_i(B_i \cap L) = D_i \cap A$ for $i = 1, 2$; it then follows that there is a homeomorphism from (S^3, L, S) onto $(D_1 \cup_\phi D_2, (A \cap D_1) \cup_\phi (A \cap D_2), \partial D_1)$, where ϕ is defined by $(\alpha_2|\partial B_2)(\alpha_1|\partial B_1)^{-1}$.

Each closed, orientable 3-manifold M has a *Heegaard splitting of genus g*. By definition this is a pair (M, F_g), where F_g is a closed, orientable surface of genus g which separates M into two handlebodies X_1 and X_2. The *genus of M* is the smallest integer g so that M admits such a representation. Two such Heegaard splittings (M, F_g) and (M', F_g') are *equivalent* if (M, F_g) and (M', F_g') are of the same topological type. This topological type is fully described by an element of the homeotopy group of a closed, orientable surface of genus g. Concretely, an orientation-preserving autohomeomorphism ψ of $\partial \tilde{D}_1$ defines a Heegaard splitting $(\tilde{D}_1 \cup_{\tilde{\tau}\psi} \tilde{D}_2, \partial \tilde{D}_1)$, where the topological type does not depend on the special choice of $\tilde{D}_1$, $\tilde{D}_2$ or $\tilde{\tau}$. Conversely, given a Heegaard splitting (M, F_g), where F_g separates M into two handlebodies X_1 and X_2, there are orientation-preserving homeomorphisms $\beta_i : X_i \rightarrow \tilde{D}_i$ for $i = 1, 2$; it then follows that there is a homeomorphism from (M, F_g) onto $(\tilde{D}_1 \cup_{\tilde{\tau}\psi} \tilde{D}_2, \partial \tilde{D}_1)$, where ψ is defined by $\tilde{\tau}^{-1}(\beta_2|\partial X_2)(\beta_1|\partial X_1)^{-1}$.

To each equivalence class of $2n$-plat representations of the link L, there is uniquely defined an equivalence class of Heegaard splittings of the 2-fold cyclic covering space $\tilde{L}$ branched over L. Concretely, given a representative braid $\phi : (\partial D_1, \partial D_1 \cap A) \rightarrow (\partial D_1, \partial D_1 \cap A)$ of the equivalence class of (S^3, L, S), there is an orientation-preserving homeomorphism $\tilde{\phi} : \partial \tilde{D}_1 \rightarrow \partial \tilde{D}_1$ which covers ϕ, and such that $\tilde{L}$ is homeomorphic to $\tilde{D}_1 \cup_{\tilde{\tau}\tilde{\phi}} \tilde{D}_2$. The homeomorphism $\tilde{\phi}$ is uniquely defined up to composition with an involution of $\partial \tilde{D}_1$ which extends to $\tilde{D}_1$; this proves that the topological type of $(\tilde{D}_1 \cup_{\tilde{\tau}\tilde{\phi}} \tilde{D}_2, \partial \tilde{D}_1)$ does not depend on the choice of $\tilde{\phi}$.

In order to visualize a representative plat of the class defined by a braid $\phi : (\partial D_1, \partial D_1 \cap A) \rightarrow (\partial D_1, \partial D_1 \cap A)$ note that ϕ is isotopic to the identity map in ∂D_1. Let us consider a homeomorphism $F'': \partial D_1 \times [0, 1] \rightarrow \partial D_1 \times [0, 1]$ such that $F''(x, t) = (\bar{x}, t)$, $F''(x, 1) = (x, 1)$ and $F''(x, 0) = (\phi x, 0)$. Then F'' is extended by the identity map outside $\partial D_1 \times [0, 1]$ to an autohomeomorphism F' of D_1. The homeomorphism F from $D_1 \cup_\phi D_2$ onto $D_1 \cup D_2$ defined by $F(x) = F'(x)$ for $x \in D_1$ and $F(x) = x$ for $x \in D_2$, maps $(D_1 \cup_\phi D_2, (D_1 \cap A) \cup_\phi (D_2 \cap A), \partial D_1)$ onto the plat

$$P(\phi) = (S^3, F'(A \cap D_1) \cup (A \cap D_2), \partial D_1).$$

Note that $F(A \cap (\partial D_1 \times [0, 1]))$ is a geometric braid on $2n$ strings. Thus we might visualize $P(\phi)$ as the geometric braid ϕ by joining the initial points in pairs and by doing the same with the terminal points.

The classical braid group B_{2n} is generated by the homeomorphisms $\sigma_1, \sigma_2, \ldots, \sigma_{2n-1}$ defined as follows: Let E_i be the disc of radius 3, in the x, y plane, with its center at $(4i - 2, 0, 0)$, $1 \leqq i \leqq 2n - 1$. Let us consider a fixed orientation on ∂D_1. Inside E_i, σ_i is a positive twist, holding ∂E_i fixed, which exchanges the points of $A \cap E_i$; outside E_i, σ_i is the identity map.

Let us suppose, as we may, that $p_1|\partial \tilde{D}_1 : \partial \tilde{D}_1 \rightarrow \partial D_1$ is the covering projection that is induced by the axial symmetry with respect to the axis E of Figure 1. We orient $\partial \tilde{D}_1$ so that $p_1|\partial \tilde{D}_1$ is orientation-preserving. Then σ_i lifts to a positive Dehn-twist $\tilde{\sigma}_i$ around the curve C_i shown in Figure 1, for $1 \leqq i \leqq 2n - 1$.

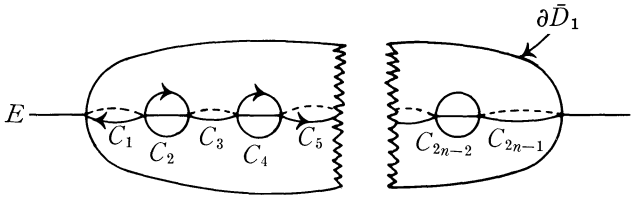

FIGURE 1

Next, let us suppose that $\partial \tilde{D}_1$ has genus 2, and we recall in the following the action of $\tilde{\sigma}_i$ in the generators w_1, w_2, w_3, w_4 of $H_1(\partial \tilde{D}_1)$ which are represented, respectively, by C_2, C_4, C_1, C_5 of Figure 1. This action is given by a 4×4 matrix of integers $\alpha(\tilde{\sigma}_i) = ||\epsilon_{mn}||$, where ϵ_{mn} is the coefficient of w_n in $\tilde{\sigma}_i(w_m)$. These matrices are the following (see [**1**, page 109]):

$$\tilde{\sigma}_1 = \left[\begin{array}{c|cc} I & -1 & 0 \\ & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{I} \end{array}\right]; \quad \tilde{\sigma}_2 = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{I} & 0 \\ \hline 1 & 0 & \\ 0 & 0 & I \end{array}\right]; \quad \tilde{\sigma}_3 = \left[\begin{array}{c|cc} I & -1 & 1 \\ & 1 & -1 \\ \hline 0 & \multicolumn{2}{c}{I} \end{array}\right];$$

$$\tilde{\sigma}_4 = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{I} & 0 \\ \hline 0 & 0 & \\ 0 & 1 & I \end{array}\right]; \quad \tilde{\sigma}_5 = \left[\begin{array}{c|cc} I & 0 & 0 \\ & 0 & -1 \\ \hline 0 & \multicolumn{2}{c}{I} \end{array}\right].$$

3. The examples. Let $P(\phi_\alpha)$ and $P(\phi_\alpha')$ be the 6-plats which are defined by the braids $\phi_\alpha = \sigma_2^{7\alpha}\sigma_3\sigma_4\sigma_3^{-1}\sigma_4^{3}\sigma_2\sigma_1^{-1}\sigma_2^{3}$ and $\phi_\alpha' = \sigma_2^{7\alpha}\sigma_3\sigma_4^{3}\sigma_3^{-1}\sigma_4\sigma_2\sigma_1^{-1}\sigma_2^{3}$ respectively. The 6-plats $P(\phi_\alpha)$ and $P(\phi_\alpha')$ are illustrated in Figures 2a and 2b respectively if $\alpha > 0$, and the same plats have the bracketed crossings going in the opposite direction if $\alpha < 0$.

THEOREM. (i) *The plats $P(\phi_\alpha)$ and $P(\phi_\alpha')$ are representatives of the same link type L_α.*

(ii) *The manifold $\tilde{L}_\alpha$ is prime if and only if $\alpha \neq 0$.*

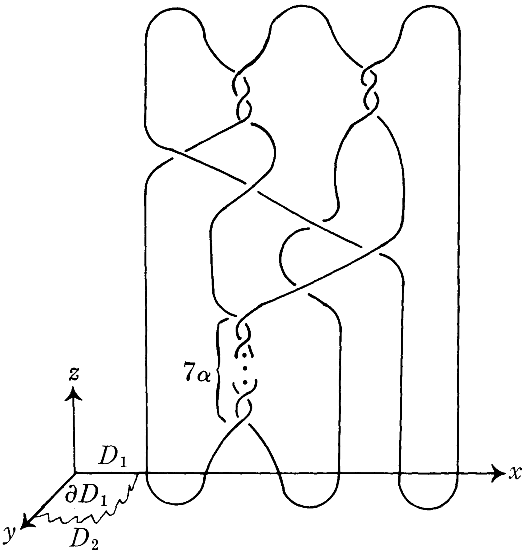

FIGURE 2a

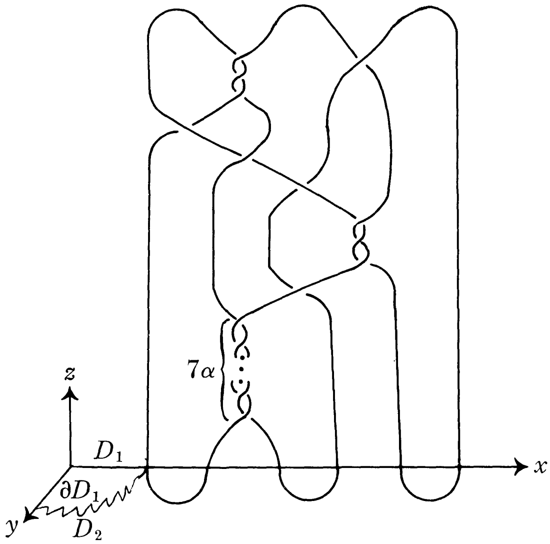

FIGURE 2b

(iii) *The link* L_α *is prime if and only if* $\alpha \neq 0$, *and is a knot if and only if* α *is even.*

(iv) *The manifold* $\tilde{L}_\alpha$ *has Heegaard genus* 2.

(v) *The link* L_α *has plat number* 3.

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(vi) *The manifold* $\tilde{L}_\alpha$ *admits at least two equivalence classes of genus* 2 *Heegaard splittings.*

(vii) *The link* L_α *admits at least two equivalence classes of* 6-*plat representations.*

Proof. The plats $P(\phi_\alpha)$ and $P(\phi_\alpha')$ are easily recognized as representatives of the link L_α of Figure 3, having the bracketed crossing going in the opposite direction if $\alpha < 0$. Note that the link L_α of Figure 3 is the one defined by the schematic diagram of page 6 of [**8**], with $(1, b) = (1, -2)$, $(\alpha_1, \beta_1) = (7, 3)$, $(\alpha_2, \beta_2) = (|7\alpha|, |7\alpha - 1|)$, $(\alpha_3, \beta_3) = (7, 3)$. It then follows from the Theorem in § 2 of [**8**] that $\tilde{L}_\alpha$, when $\alpha \neq 0$, is the Seifert fiber space $(O \circ 0| - 2; (7, 3), (7, 3), (|7\alpha|, |7\alpha - 1|))$. By Theorem 7.1 and Lemma 10.2 of [**13**], it then follows that $\tilde{L}_\alpha$ is a prime 3-manifold when $\alpha \neq 0$. Hence by Section 3.7 of [**12**], or Theorem V.5.3 of [**7**], and the main result of [**14**], L_α is a prime link when $\alpha \neq 0$. Since L_0 is a composite knot and $\tilde{L}_0$ is a connected sum of two lens spaces, parts (i), (ii) and (iii) are established.

In order to prove (iv) and (v) note that the homeomorphism $\tilde{\phi}_\alpha$ defines a Heegaard splitting of genus 2 of $\tilde{L}_\alpha$. Since $\tilde{L}_\alpha$ has a non-cyclic fundamental group [**10**] its Heegaard genus cannot be less than 2, establishing (iv) and (v).

Finally, we shall prove (vi) and (vii). In order to demonstrate that the Heegaard splittings of $\tilde{L}_\alpha$ which are defined by $\tilde{\phi}_\alpha$ and $\tilde{\phi}_\alpha'$ are inequivalent, we

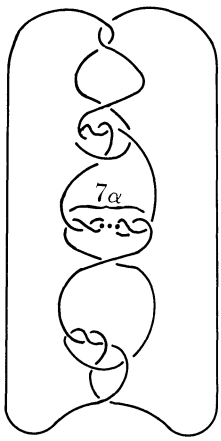

FIGURE 3

shall apply Theorem 2 of [**2**]. Observe that the action of $\tilde{\phi}_\alpha$ in $H_1(\partial\tilde{D}_1)$ is given by the matrix

$$\left[\begin{array}{c|c} R & S \\ \hline P & Q \end{array}\right] = \left[\begin{array}{cc|cc} 2 - 7\alpha & -2 & & * \\ 21\alpha & 4 & & \\ \hline 7 - 21\alpha & -7 & -3 & 7 \\ 28\alpha & 7 & 4 & -3 \end{array}\right],$$

and the action of $\tilde{\phi}_\alpha{}'$ is given by the matrix

$$\left[\begin{array}{c|c} R' & S' \\ \hline P' & Q' \end{array}\right] = \left[\begin{array}{cc|cc} 2-35\alpha & -6 & & * \\ 7\alpha & 2 & & \\ \hline 7-119\alpha & -21 & -17 & 21 \\ 42\alpha & 7 & 6 & -5 \end{array}\right].$$

Then, following the notation of Theorem 2 of [**2**], $p = 7$, $\det Q = -19$, $\det R = 4$, $\det Q' = -41$, $\det R' = 8 - 14\alpha$. As none of the congruences (30)–(33) of [**2**] is fulfilled, it follows that the Heegaard splittings defined by $\tilde{\phi}_\alpha$ and $\tilde{\phi}_\alpha{}'$ are inequivalent. Therefore $P(\phi_\alpha)$ and $P(\phi_\alpha{}')$ are inequivalent 6-plat representations of L_α. This proves (vi) and (vii).

Remarks. 1. We conjecture that the $2(n+3)$-plats $P(\phi_{n\alpha})$ and $P(\phi_{n\alpha}{}')$ which are defined by the braids

$$\phi_{n\alpha} = \sigma_2{}^{7\alpha}(\sigma_4\sigma_3\sigma_5\sigma_4)\ldots(\sigma_{2n+2}\sigma_{2n+1}\sigma_{2n+3}\sigma_{2n+2})\sigma_{2n+3}\sigma_{2n+4}$$
$$\sigma_{2n+3}{}^{-1}\sigma_{2n+4}{}^3(\sigma_{2n+2}\sigma_{2n+1}{}^{-1}\sigma_{2n+2}{}^3)\ldots(\sigma_2\sigma_1{}^{-1}\sigma_2{}^3)$$
$$\phi_{n\alpha}{}' = \sigma_2{}^{7\alpha}(\sigma_4\sigma_3\sigma_5\sigma_4)\ldots(\sigma_{2n+2}\sigma_{2n+1}\sigma_{2n+3}\sigma_{2n+2})\sigma_{2n+3}\sigma_{2n+4}{}^3$$
$$\sigma_{2n+3}{}^{-1}\sigma_{2n+4}(\sigma_{2n+2}\sigma_{2n+1}{}^{-1}\sigma_{2n+2}{}^3)\ldots(\sigma_2\sigma_1{}^{-1}\sigma_2{}^3)$$

respectively, are inequivalent and minimal plat representations of the same link type. This would show that the example studied in this paper is widespread.

2. Let Φ be a $2n$-plat for a link L with two components, K_1 and K_2, which are of different type. We obtain a $2(n+1)$-plat Φ_1 (resp. Φ_2) by adding a "trivial loop" (see Figure 3 of [**5**]) to K_1 (resp. K_2). It is obvious that Φ_1 and Φ_2 are *inequivalent* representatives of the link L. But, of course, they are *not* minimal.

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