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Part 2. Lévy processes

A PIECEWISE LINEAR STOCHASTIC DIFFERENTIAL EQUATION DRIVEN BY A LÉVY PROCESS

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A PIECEWISE LINEAR STOCHASTIC DIFFERENTIAL EQUATION DRIVEN BY A LÉVY PROCESS

BY JOSH REED AND BERT ZWART

Abstract

We consider a stochastic differential equation (SDE) with piecewise linear drift driven by a spectrally one-sided Lévy process. We show that this SDE has some connections with queueing and storage models, and we use this observation to obtain the invariant distribution.

Keywords: Lévy process; queues with many servers; storage model; stochastic differential equation; martingale; time change

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1. Introduction

Let $X(\cdot)$ be a spectrally positive Lévy process, i.e. a process with stationary and independent increments for which we can write

$$\mathrm{E}[\mathrm{e}^{-sX(t)}] = \mathrm{e}^{t\phi(s)},$$

where

$$\phi(s) = ds + \frac{1}{2}\sigma^2 s^2 + \int_0^\infty (\mathrm{e}^{-sx} - 1 + sx\,\mathbf{1}(x < 1))\nu(\mathrm{d}x) \tag{1}$$

for $d \in \mathbb{R}$ and a measure (the jump measure) ν such that

$$\int_0^\infty \min\{1, x^2\}\nu(\mathrm{d}x) < \infty.$$

We are mainly interested in the following stochastic differential equation (SDE) driven by $X(\cdot)$:

$$\mathrm{d}W(t) = -c_+ W(t)^+\,\mathrm{d}t - c_- W(t)^-\,\mathrm{d}t + \mathrm{d}X(t). \tag{2}$$

Here $x^+ = \max\{x, 0\}$ and $x^- = \min\{x, 0\}$. It is assumed that c_- and c_+ are nonnegative constants such that $c_+ + c_- > 0$. Since the drift function $b(x) = -c_+x^+ - c_-x^-$ is Lipschitz continuous, it follows from Theorem 4.1 of [19] that there exists a unique, strong solution to (2).

The SDE (2) is an extension of the generalized Ornstein–Uhlenbeck process, which is obtained when $c_+ = c_-$, and as such is relevant to recent developments in mathematical finance [7]. Our motivation for studying this SDE is that it can arise as a limit of a sequence of many-server queues, as explained in the recent work of Pang and Whitt [18]. The Brownian case arises in both application areas as well; for related queueing literature, we refer the reader to [10], [19], and the references therein. The additional jumps can arise as a consequence of heavy-tailed interarrival times, or large-scale service interruptions—we refer the reader to [18] for details.

Contrary to the above references, in this paper we concentrate almost exclusively on the SDE itself, rather than formalizing the appearance of the SDE as the limit of a sequence of queueing models. A survey of the results in the Brownian case is given in Browne and Whitt [5], who proposed the terminology 'piecewise linear diffusion processes' (leading to our title choice). Intuitively, it is clear that on a single half-space, the process behaves as a storage/dam model with affine release rate. This model is well understood; see, for example, the chapter on this topic in [2]. Making this connection formal using purely probabilistic tools such as martingales is one of the key aims of this paper. Once this connection is established, it is possible to use existing results for reflected spectrally negative Lévy processes, as well as Ornstein–Uhlenbeck processes, to obtain qualitative and quantitative results on the invariant distribution.

As mentioned, the work in this paper covers Lévy processes, storage models, many-server queues, and martingales. Thanks to Søren Asmussen and his impact on applied probability, these topics are now well connected. Indeed, Søren's books, papers, and lectures have been inspiring and have made an important impact on our careers. It is therefore an honor and pleasure to contribute to this volume.

The paper is organized as follows. In Section 2 we focus on structural results, such as stability and the process restricted to the negative half-space. Examples are considered in Section 3. Section 4 closes with comments on time-dependent results.

2. Structural results

We exclude the case that $X(\cdot)$ is a subordinator, so that $\liminf_{s\to\infty}\phi(s)/s > 0$. Note that $X(\cdot)$ is a strong Markov process with generator $\mathcal{L}$. The function $f_s(x) = \mathrm{e}^{-sx}$ is in the domain of $\mathcal{L}$, and $\mathcal{L}(f_s)(x) = \mathrm{e}^{-sx}\phi(s)$. Consequently, the generator $\mathcal{A}$ of $W(\cdot)$ is given by

$$\mathcal{A}(f)(x) = -c_+ x\,\mathbf{1}(x>0)f'(x) - c_- x\,\mathbf{1}(x<0)f'(x) + \mathcal{L}(f)(x).$$

We assume that $c_-, c_+ \geq 0$ and $c_+ + c_- > 0$. To ensure stability, we also assume that $\mathrm{E}[X(1)] > 0$ if $c_- = 0$ and that $\mathrm{E}[X(1)] < 0$ if $c_+ = 0$.

Proposition 1. *The process $W(\cdot)$ is strong Markov. Moreover, there exists an almost-sure finite random variable W such that $W(t) \xrightarrow{\mathrm{D}} W$ as $t\to\infty$.*

Proof. The fact that W is strong Markov follows from Theorem 32 of [20, Chapter V]. In order to show that W is ergodic, we rely on results from [15], [16], and [17]. Let $\mathcal{O}_m = (-m, m)$, $m \geq 1$, be a sequence of open sets, and let T^m denote the first entrance time of W to the closed set $\mathcal{O}_m^{\mathrm{c}}$. Let $\mathcal{A}_m$ denote the extended generator of the truncated process $W^m(\cdot) = W(\cdot\wedge T^m)$, and let $\mathcal{D}(\mathcal{A}_m)$ be its domain. Note that, by (1), (2), and the optional sampling theorem, $\mathrm{E}_x[W^m(t)] - x$ is equal to

$$\mathrm{E}_x\biggl[\int_0^{t\wedge T_m}(-c_+(W^m(s))^+ - c_-(W^m(s))^- + \mathrm{E}[X(1)])\,\mathrm{d}s\biggr].$$

Moreover, by (2) and the fact that X is spectrally positive,

$$\int_0^t \mathrm{E}_x[|-c_+(W^m(s))^+ - c_-(W^m(s))^- + \mathrm{E}[X(1)]|]\,\mathrm{d}s < \infty.$$

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Hence, the identity function $e(x) = x$ is in $\mathcal{D}(\mathcal{A}_m)$ and

$$\mathcal{A}_m e(x) = -c_+ x\,\mathbf{1}(x > 0) - c_- x\,\mathbf{1}(x < 0) + \mathrm{E}[X(1)].$$

It is then clear that, under each of the assumptions above, there exist a closed interval I, positive numbers $c, d > 0$, and a linear function $f \geq 1$ such that

$$\mathcal{A}_m e(x) = -cf(x) + d\,\mathbf{1}(x \in I)$$

for $x \in \mathcal{O}_m$. Combining Theorem 5.1 of [17], Theorem 4.1 of [16], and Theorem 3.4 of [15], we see that it suffices to show that the skeleton chain W^1 obtained by sampling W at integer time points is φ-irreducible for some measure φ whose support has nonempty interior and that W^1 has the Feller property.

It is clear from Proposition 2 of [21] that W^1 has the Feller property. Now let $x_\star$ be such that $c_- x_\star^- + c_+ x_\star^+ = \mathrm{E}[X_1]$. It is also clear that, under the above assumptions, such a point must exist. If $x_\star < 0$, let φ be a probability measure with support equal to $(-\infty, x^\star)$; if $x_\star > 0$, let φ be a probability measure with support equal to $(x^\star, \infty)$; and, if $x_\star = 0$, let the support of φ be $\mathbb{R}$. It is straightforward to show that W^1 is φ-irreducible, which completes the proof.

The main goal of this work is to obtain information about the distribution of W. Our plan to obtain this information is as follows. Suppose that $\mathrm{E}[\mathrm{e}^{-sW}] < \infty$ on some open interval D_ϕ. For $s \in D_\phi$, we see that the limiting random variable W satisfies

$$\phi(s)\,\mathrm{E}[\mathrm{e}^{-sW}] = -c_+ s\,\mathrm{E}[W\,\mathbf{1}(W > 0)\mathrm{e}^{-sW}] - c_- s\,\mathrm{E}[W\,\mathbf{1}(W < 0)\mathrm{e}^{-sW}].$$

Let $p = \mathrm{P}(W > 0)$.

Suppose that $\mathrm{E}[\mathrm{e}^{-sW} \mid W < 0] =: G_-(s)$ is known (from which we would be able to infer that D_ϕ is nonempty). Then we can write $\mathrm{E}[\mathrm{e}^{-sW}] = pG_+(s) + (1-p)G_-(s)$, and we obtain a tractable differential equation for G_+:

$$\phi(s)(pG_+(s) + (1-p)G_-(s)) = c_+ psG'_+(s) + c_-(1-p)sG'_-(s).$$

To find G_-, we will consider the process constrained in $(-\infty, 0]$. Call this process $W_{R-}(\cdot)$. Formally, define $J_-(t) = \int_0^t \mathbf{1}(W(s) < 0)\,\mathrm{d}s$. We can set $W_{R-}(t) = W(J_-^{-1}(t))$. Since $W(\cdot)$ has only positive jumps and the driving process $X(\cdot)$ has stationary and independent increments, the following holds.

Theorem 1. *The process $W_{R-}(\cdot)$ has the same law as $W(\cdot)$ restricted on $(-\infty, 0]$ and reflected at 0. In addition, $W_{R-}(t) \xrightarrow{\mathrm{D}} W_{R-}$ for some random variable W_{R-} which has Laplace–Stieltjes transform (LST) $G_-(s)$.*

Proof. We modify the proof of Theorem 6.3.1 of [12] to account for jumps and a state-dependent drift. By (2) and the generalized Itô formula for convex functions (see Theorem 9.46 of [11]), it follows that

$$\begin{aligned} W(t)^- = W(0)^- &+ \int_0^t \mathbf{1}(W(s-) \leq 0)\,\mathrm{d}X(s) - c_- \int_0^t W(s)^-\,\mathrm{d}s \\ &+ \sum_{0<s\leq t} [W(s)^- - W(s-)^- - \mathbf{1}(W(s-) \leq 0)\Delta W(s)] - \frac{1}{2}L(t), \end{aligned} \tag{3}$$

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where $L(t)$ denotes the local time of W at 0. In addition, note that

$$K(t) = \sum_{0<s\le t} [W(s)^- - W(s-)^- - \mathbf{1}(W(s-) \le 0)\Delta W(s)] - \frac{1}{2}L(t)$$

is a nonincreasing process as a function of t and that, by Corollary 9.45 of [11],

$$\int_0^\infty \mathbf{1}(W(s)^- < 0)\,\mathrm{d}K(s) = 0.$$

Now note that $J_-(J_-^{-1}(t)) = t < J_-(J_-^{-1}(t) + \delta)$ for $\delta > 0$ and so, since X is spectrally positive, this implies that $W_{R-}(t) \le 0$. Thus, $W_{R-}(t) = W(J_-^{-1}(t))^-$. By (3), it then follows that

$$\begin{aligned} W_{R-}(t) = W_{R-}(0) &+ \int_0^{J_-^{-1}(t)} \mathbf{1}(W(s-) \le 0)\,\mathrm{d}X(s) - c_- \int_0^{J_-^{-1}(t)} W(s)^-\,\mathrm{d}s \\ &+ \sum_{0<s\le t} [W_{R-}(s)^- - W_{R-}(s-)^- - \mathbf{1}(W_{R-}(s-) \le 0)\Delta W(J_-^{-1}(s))] \\ &- \tfrac{1}{2}L(J_-^{-1}(t)). \end{aligned}$$

Next note that, since $W_{R-}(t) = W(J_-^{-1}(t))^-$ (see above), it follows by a change of variables and the fact that $\mathrm{d}J_-(t) = \mathbf{1}(W(t) < 0)\,\mathrm{d}t$ that

$$\begin{aligned} c_- \int_0^{J_-^{-1}(t)} W(s)^-\,\mathrm{d}s &= c_- \int_0^{J_-^{-1}(t)} W_{R-}(J_-(s))\,\mathrm{d}s \\ &= c_- \int_0^{J_-^{-1}(t)} W_{R-}(J_-(s))\,\mathbf{1}(W(s) < 0)\,\mathrm{d}s \\ &= c_- \int_0^t W_{R-}(s)\,\mathrm{d}s. \end{aligned}$$

Moreover, note that

$$B(t) := \sum_{0<s\le t} [W_{R-}(s)^- - W_{R-}(s-)^- - \mathbf{1}(W_{R-}(s-) \le 0)\Delta W(J_-^{-1}(s))] - \frac{1}{2}L(J_-^{-1}(t))$$

is nonincreasing as a function of t and, again by Corollary 9.45 of [11],

$$\int_0^\infty W_{R-}(s)\,\mathrm{d}B(s) = 0.$$

By Proposition 2 of [21], it remains to show that

$$\int_0^{J_-^{-1}(t)} \mathbf{1}(W(s-) \le 0)\,\mathrm{d}X(s), \qquad t \ge 0,$$

is a Lévy process with Laplace exponent ϕ.

Let $0 < \varepsilon < 1$. By the Lévy–Itô decomposition for Lévy processes (see Theorem 2.1 of [14]) we may write

$$X(t) = X^{(1)}(t) + X_\varepsilon^{(2)}(t) + X_\varepsilon^{(3)}(t) - \left(\int_\varepsilon^\infty x\nu(\mathrm{d}x)\right)t,$$

where $X^{(1)}(t) = \sigma B(t) - (d + \int_1^\infty x\nu(\mathrm{d}x))t$, B is a standard Brownian motion, $X_\varepsilon^{(2)}$ is a

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compound Poisson process with jump rate $\nu(\varepsilon,\infty)$, where the distribution of the jump sizes is $\nu(\mathrm{d}x)/\nu(\varepsilon,\infty)$ for $x>\varepsilon$ and 0 otherwise, and $X_\varepsilon^{(3)}$ is a square-integrable martingale with maximum jump size ε and quadratic variation

$$\langle\langle X_\varepsilon^{(3)}\rangle\rangle_t=\biggl(\int_0^\varepsilon x^2\nu(\mathrm{d}x)\biggr)t$$

(see Exercise 2.4.14 of [1]). In addition, $X^{(1)}$, $X_\varepsilon^{(2)}$, and $X_\varepsilon^{(3)}$ are independent of one another for a given ε. We may therefore write

$$\begin{aligned}\int_0^{J_-^{-1}(t)}&\mathbf{1}(W(s-)\le 0)\,\mathrm{d}X(s)\\ &=\int_0^{J_-^{-1}(t)}\mathbf{1}(W(s-)\le 0)\,\mathrm{d}X^{(1)}(s)+\sum_{i=2}^{3}\int_0^{J_-^{-1}(t)}\mathbf{1}(W(s-)\le 0)\,\mathrm{d}X_\varepsilon^{(i)}(s)\\ &\quad-\int_0^{J_-^{-1}(t)}\mathbf{1}(W(s-)\le 0)\,\mathrm{d}\biggl(\biggl(\int_\varepsilon^\infty x\nu(\mathrm{d}x)\biggr)s\biggr).\end{aligned}$$

We proceed term by term in analyzing the right-hand side above. First note that

$$\int_0^{J_-^{-1}(t)}\mathbf{1}(W(s-)\le 0)\,\mathrm{d}X^{(1)}(s)=\int_0^{J_-^{-1}(t)}\mathbf{1}(W(s-)\le 0)\,\mathrm{d}(\sigma B(s))-\biggl(d+\int_1^\infty \nu(\mathrm{d}x)\biggr)t.$$

In addition, since $J_-^{-1}(t)$ is a stopping time for each $t\ge 0$, it follows by the optional sampling theorem that

$$\int_0^{J_-^{-1}(t)}\mathbf{1}(W(s-)\le 0)\,\mathrm{d}(\sigma B(s)),\qquad t\ge 0,\tag{4}$$

is a continuous martingale with quadratic variation σt, and so, by Lévy's martingale characterization of Brownian motion (see Theorem 3.3.16 of [12]), it is a Brownian motion. Next consider

$$\int_0^{J_-^{-1}(t)}\mathbf{1}(W(s-)\le 0)\,\mathrm{d}X_\varepsilon^{(3)}(s),\qquad t\ge 0.$$

In the same manner as was shown above it is a martingale with quadratic variation $\int_0^\varepsilon x^2\nu(\mathrm{d}x)t$. However, since $\lim_{\varepsilon\to 0}\int_0^\varepsilon x^2\nu(\mathrm{d}x)=0$, it follows by the martingale invariance principle (see Theorem 7.1.4 of [8]) that

$$\int_0^{J_-^{-1}(t)}\mathbf{1}(W(s-)\le 0)\,\mathrm{d}X_\varepsilon^{(3)}(s)\Rightarrow 0\quad\text{as }\varepsilon\to 0,$$

where we use '$\Rightarrow$' to denote weak convergence. Finally, consider

$$\int_0^{J_-^{-1}(t)}\mathbf{1}(W(s-)\le 0)\,\mathrm{d}X_\varepsilon^{(2)}(s)-\int_0^{J_-^{-1}(t)}\mathbf{1}(W(s-)\le 0)\,\mathrm{d}\biggl(\biggl(\int_\varepsilon^\infty x\nu(\mathrm{d}x)\biggr)s\biggr)$$

for $t\ge 0$. Let $N_\varepsilon\colon \mathcal{B}[0,\infty)\times\mathcal{B}(\mathbb{R}^+)\mapsto\mathbb{N}\cup\infty$ be the Poisson random measure associated with $X_\varepsilon^{(2)}$ (see Lemma 2.2 of [14]), and define the new random measure

$$\hat{N}_\varepsilon\colon \mathcal{B}[0,\infty)\times\mathcal{B}(\mathbb{R}^+)\mapsto\mathbb{N}\cup\infty$$

by setting

$$\hat{N}_\varepsilon(\mathrm{d}t\times\mathrm{d}x)=\mathbf{1}(W(J_-^{-1}(t)-)\le 0)N_\varepsilon(\mathrm{d}J_-^{-1}(t)\times\mathrm{d}x).\tag{5}$$

We claim that $\hat{N}_\varepsilon$ is a Poisson random measure on $([0,\infty)\times\mathbb{R}, \mathcal{B}([0,\infty))\times\mathcal{B}(\mathbb{R}), \Pi_\varepsilon)$, independent of the process in (4), and where $\Pi_\varepsilon = \mathrm{d}t \times \nu(\mathrm{d}x)$ for $x > \varepsilon$ and 0 otherwise. It suffices to show that, for disjoint sets $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{B}(\mathbb{R})$, the processes

$$\int_{[0,t]\times\mathcal{A}_1} \hat{N}_\varepsilon(\mathrm{d}t\times\mathrm{d}x), \quad t\geq 0, \quad \text{and} \quad \int_{[0,t]\times\mathcal{A}_2} \hat{N}_\varepsilon(\mathrm{d}t\times\mathrm{d}x), \quad t\geq 0, \tag{6}$$

are independent Poisson processes with rates $\nu(\mathcal{A}_1\cap(x,\infty))$ and $\nu(\mathcal{A}_2\cap(x,\infty))$, respectively. First note that each of the processes in (6) are counting processes. Next, for $i=1,2$, we have, by (5),

$$\int_{[0,t]\times\mathcal{A}_i} \hat{N}_\varepsilon(\mathrm{d}s\times\mathrm{d}x) = \int_{[0,J_-^{-1}(t)]} \mathbf{1}(W(s-)\leq 0)\int_{\mathcal{A}_i} N_\varepsilon(\mathrm{d}s\times\mathrm{d}x).$$

However,

$$\int_{[0,t]}\int_{\mathcal{A}_i} N_\varepsilon(\mathrm{d}s\times\mathrm{d}x), \qquad t\geq 0,$$

is a Poisson process with rate $\nu(\mathcal{A}_i\cap(x,\infty))$ and so

$$\int_{[0,t]}\biggl(\int_{\mathcal{A}_i} N_\varepsilon(\mathrm{d}s\times\mathrm{d}x) - \nu(\mathcal{A}_i\cap(x,\infty))\,\mathrm{d}s\biggr), \qquad t\geq 0,$$

is a martingale. Thus,

$$\int_{[0,J_-^{-1}(t)]} \mathbf{1}(W(s-)\leq 0)\biggl(\int_{\mathcal{A}_i} N_\varepsilon(\mathrm{d}s\times\mathrm{d}x) - \nu(\mathcal{A}_i\cap(x,\infty))\,\mathrm{d}s\biggr)$$

is a martingale as well. Thus, since

$$\int_{[0,J_-^{-1}(t)]} \mathbf{1}(W(s-)\leq 0)\nu(\mathcal{A}_i\cap(x,\infty))\,\mathrm{d}s = \nu(\mathcal{A}_i\cap(x,\infty))t,$$

it follows by Watanabe's characterization of Poisson processes (see Theorem 1.8.2 of [3]) that each of the processes in (6) is a Poisson process with rate $\nu(\mathcal{A}_i\cap(x,\infty))$. Independence follows from the fact that since N_ε is a Poisson random measure, the set of jump times of the two processes in (6) are disjoint from one another and so, since we already know they are Poisson processes, they must be independent (see [4]). Independence from the Brownian term follows from Corollary 11.5.3 of [22]. Since

$$\int_0^{J_-^{-1}(t)} \mathbf{1}(W(s-)\leq 0)\,\mathrm{d}X_\varepsilon^{(2)}(s) = \int_{[0,t]\times\mathbb{R}} x\hat{N}_\varepsilon(\mathrm{d}s\times\mathrm{d}x),$$

it now follows from Lemma 2.8 of [14] that

$$\int_0^{J_-^{-1}(t)} \mathbf{1}(W(s-)\leq 0)\,\mathrm{d}X_\varepsilon^{(2)}(s), \qquad t\geq 0,$$

is a compound Poisson process with arrival rate $\nu(\varepsilon,\infty)$ and jump distribution $\nu(\mathrm{d}x)/\nu(\varepsilon,\infty)$. Now considering the compensated process

$$\hat{X}_\varepsilon^{(2)}(t) = \int_0^{J_-^{-1}(t)} \mathbf{1}(W(s-)\leq 0)\,\mathrm{d}X_\varepsilon^{(2)}(s) - t\int_\varepsilon^\infty x\nu(\mathrm{d}x),$$

it follows from Theorem 2.10 of [14] that $\hat{X}^{(2)}_\varepsilon \Rightarrow \hat{X}^{(2)}$ as $\varepsilon \to 0$, where $\hat{X}^{(2)}$ is a martingale with the desired properties. This completes the first part of the proof.

It remains to show that $W_{R-}(t) \xrightarrow{\mathrm{D}} W_{R-}$ for some random variable W_{R-} which has LST $G_-(s)$. First note that since W is ergodic by Proposition 1, it follows by the ergodic theorem that, for each $f \in C_b(\mathbb{R})$,

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f(W(s)^-)\,\mathrm{d}s = \mathrm{E}[f(W^-)] \quad \text{P-almost surely (P-a.s.).}$$

Next, recall that (see above)

$$\int_0^{J_-(t)} f(W_{R-}(s))\,\mathrm{d}s = \int_0^t W(s)^-\,\mathrm{d}s$$

for $t \ge 0$, and so, since $J(t) \to \infty$ as $t \to \infty$, it follows that

$$\begin{aligned}\lim_{t\to\infty} \frac{1}{J_-(t)} \int_0^{J_-(t)} f(W_{R-}(s))\,\mathrm{d}s &= \lim_{t\to\infty} \frac{t}{J_-(t)} \frac{\int_0^t f(W(s)^-)\,\mathrm{d}s}{t} \\ &= \frac{1}{1-p}\,\mathrm{E}[f(W^-)] \quad \text{P-a.s.}\end{aligned}$$

Using basic storage process theory (see [2]), $W_{R-}(\cdot)$ is ergodic as well, so the result follows.

3. Examples

In this section we focus on several particular cases. To compute the invariant distribution, we make use of Theorem 1, and the general outline before that.

3.1. The case $c_+ > 0$, $c_- = 0$

If $c_- = 0$, Theorem 1 implies that $-W_{R-}(\cdot)$ is a reflected at 0 Lévy process in $[0, \infty)$ with negative jumps. Consequently, $-W_{R-}$ is exponentially distributed with parameter $\gamma > 0$ satisfying $\phi(\gamma) = 0$; see, for example, Proposition 1 of [13]. Thus, $G_-(s) = \gamma/(\gamma - s)$. The differential equation for $G_+(s)$ becomes

$$\frac{\phi(s)}{c_+ s} G_+(s) - G_+'(s) = \frac{1-p}{c_+ p} \frac{\phi(s)}{s} \frac{\gamma}{s-\gamma}.$$

Here $(\phi(s)/(s-\gamma))(\gamma/s)$ is well defined for all values of s using l'Hôpital's rule. This differential equation can be solved using standard methods. Since $G_+(0) = 1$, we obtain

$$\begin{aligned}G_+(s) = \exp&\left\{\frac{1}{c_+}\int_0^s \frac{\phi(u)}{u}\,\mathrm{d}u\right\} \\ &\times \left(1 - \frac{1-p}{p}\int_0^s \frac{1}{c_+}\frac{\phi(u)}{u-\gamma}\frac{\gamma}{u}\exp\left\{-\frac{1}{c_+}\int_0^u \frac{\phi(y)}{y}\,\mathrm{d}y\right\}\mathrm{d}u\right).\end{aligned}$$

Since $\liminf_{s\to\infty} \phi(s)/s > 0$ and $G_+(s) \to 0$ as $s \to \infty$, we necessarily have

$$\frac{p}{1-p} = \int_0^\infty \frac{1}{c_+}\frac{\phi(u)}{u-\gamma}\frac{\gamma}{u}\exp\left\{-\frac{1}{c_+}\int_0^u \frac{\phi(y)}{y}\,\mathrm{d}y\right\}\mathrm{d}u.$$

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Example 1. If we assume a Brownian motion, we obtain $\phi(s) = \frac{1}{2}\sigma^2 s^2 - \beta s$, with $\beta > 0$. If we further take $\sigma^2 = 2$ then $1 - p$ has the interpretation of being the limiting waiting probability for the M/M/k queue in the Halfin–Whitt regime, where $\mu = 1$ and $\lambda = k - \sqrt{k} + o(\sqrt{k})$. Indeed, it can be shown straightforwardly that our equation for p (which is the main quantity of interest from a practical standpoint) simplifies to

$$1 - p = \frac{1}{1 + \beta \mathrm{e}^{\beta^2/2} \int_{-\infty}^{\beta} \mathrm{e}^{-s^2/2}\, \mathrm{d}s},$$

using the fact that $\gamma = (2\beta)/\sigma^2$.

Example 2. Assume now a stable process with negative drift, i.e. $\phi(s) = Cs^\alpha - \beta s$, $\alpha \in (1, 2)$. This arises in a G/M/k queue with heavy-tailed interarrival times (i.e. the interarrival times are regularly varying with index $-\alpha$; see again [18] for details). In this case, we have $\gamma = (\beta/C)^{1/(\alpha-1)}$ and

$$1 - p = \left(1 + \int_0^\infty \frac{1}{c_+} \frac{Cu^\alpha - \beta u}{u - \gamma} \frac{\gamma}{u} \exp\left\{-\frac{1}{c_+}\left(\left(\frac{C}{\alpha}\right)u^\alpha - \beta u\right)\right\} \mathrm{d}u\right)^{-1}.$$

3.2. The case $c_- > 0$

Let V be a random variable given by

$$v(s) = \mathrm{E}[\mathrm{e}^{-sV}] = \exp\left\{\frac{1}{c}\int_0^s \frac{\phi(u)}{u}\, \mathrm{d}u\right\}, \qquad s \geq 0.$$

Note that this coincides with the LST of W if $c = c_- = c_+$, which is a well-known result; see, for example, [6]. Define $F_V(x) = \mathrm{P}(V \leq x)$.

Proposition 2. *If $c_- > 0$ then*

$$\mathrm{P}(W \leq x \mid W \leq 0) = \frac{F_V(x)}{F_V(0)}, \qquad x \leq 0,$$

with $c = c_-$.

Proof. This result is trivial if $c_+ = c_-$ and Theorem 1 implies that the left-hand side is independent of c_+.

We now review some examples for which F_V can be computed. It is easy to see that, since $V \stackrel{\mathrm{D}}{=} \int_0^\infty \mathrm{e}^{-t}\, \mathrm{d}X(t)$, if $X(t)$ is the sum of N independent Lévy processes with Laplace exponent ϕ_i and associated F_{V_i}, then $F_V = F_{V_1} * \cdots * F_{V_N}$, with '$*$' denoting the convolution. This can be used to compute explicit examples. Some basic examples/building blocks are as follows.

1. Brownian motion: $\phi(u) = \frac{1}{2}\sigma^2 u^2$, leading to $v(s) = \mathrm{e}^{\sigma^2 s^2/4c}$ so that

$$F_V(x) = \Phi\left(\frac{\sigma x}{\sqrt{2c}}\right).$$

2. (Right-skewed) α-stable process: $\phi(u) = \kappa u^\alpha$ leading to $v(s) = \mathrm{e}^{s^\alpha \kappa/(\alpha-1)c}$, so that V is α-stable as well.

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3. Compound poisson process with exponential jumps:

$$\phi(u) = -\lambda\left(1 - \frac{\mu}{\mu + s}\right),$$

leading to

$$\frac{\mathrm{d}}{\mathrm{d}x} V(x) = \frac{\mu^{\lambda/c}}{\Gamma(\lambda/c)} x^{\lambda/c-1} \mathrm{e}^{-\mu x}.$$

We are now ready to analyze the case $c_+ \neq c_-$. We first treat $c_+ = 0$, and let V be as above with $c = c_-$. Recall that $X(\cdot)$ has a negative drift in this case, and that the transform $G(s) = \mathrm{E}[\mathrm{e}^{-sW}]$ satisfies

$$G(s) = c_-(1-p)\frac{s}{\phi(s)} G'_-(s).$$

Write

$$G'_-(s) = \frac{1}{V(0)} \int_0^\infty x\mathrm{e}^{sx}\,\mathrm{d}\bar{F}_V(x),$$

where $\bar{F}_V(x) = \mathrm{P}(-V \leq x)$. Define

$$\bar{v}_0 = \int_0^\infty x\,\mathrm{d}\bar{F}_V(x),$$

and let V_0 be a random variable with density $x\,\mathrm{d}\bar{F}_V(x)/v_0$ so that

$$G(s) = \frac{v_0 c_-(1-p)}{F_V(0)\phi'(0)} \frac{s\phi'(0)}{\phi(s)} \mathrm{E}[\mathrm{e}^{sV_0}].$$

We recognize $s\phi'(0)/\phi(s)$ to be the generalized Pollaczek–Khintchine formula, i.e. the Laplace transform of the supremum M of $X(\cdot)$. This implies that $v_0 c_-(1-p)/F_V(0)\phi'(0) = 1$ (yielding the unknown value of p) and that $W \stackrel{\mathrm{D}}{=} M - V_0$.

The case $c_+ > 0$ is less tractable. Note that

$$\phi(s)\,\mathrm{E}[\mathrm{e}^{-sW}] = -c_+ s\,\mathrm{E}[W\mathrm{e}^{-sW}] + (c_+ - c_-)s\,\mathrm{E}[W(W < 0)\mathrm{e}^{-sW}],$$

i.e.

$$\frac{\phi(s)}{c_+ s} G(s) = G'(s) + \frac{c_- - c_+}{c_+}(1-p)G'_-(s).$$

This is a standard differential equation. Consequently,

$$G(s) = \exp\left\{\frac{1}{c_+}\int_0^s \frac{\phi(u)}{u}\,\mathrm{d}u\right\} \times \left(1 - \frac{c_- - c_+}{c_+}(1-p)\int_0^s G'_-(u)\exp\left\{-\frac{1}{c_+}\int_0^u \frac{\phi(y)}{y}\,\mathrm{d}y\right\}\mathrm{d}u\right).$$

Let us look at the case of a Brownian motion with drift β and infinitesimal variance σ^2. This example arises when considering Markovian many-server queues with abandonments (cf. [9]), known as the Erlang A model. The case $c_- = c_+$ would correspond to the case where the service rate equals the abandonment rate, in which case we would get the M/M/∞ queue.

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Here $\phi(s) = \frac{1}{2}s^2\sigma^2 - \beta s$. This is Example 1, i.e. the density of the limiting distribution is $(1/\sqrt{\pi\sigma^2/c_-})\mathrm{e}^{-(x-\beta/c_-)^2\sigma^2/c_-}$. Hence,

$$G_-(s) = \frac{1}{V(0)}\frac{1}{\sqrt{\pi\sigma^2/c_-}}\int_0^\infty \mathrm{e}^{-sx}\mathrm{e}^{-(x-\beta/c_-)^2c_-/\sigma^2}\,\mathrm{d}x.$$

It is now possible to proceed as before and derive an expression for p. We omit the computational details and refer the reader to [9].

4. Comments on the time-dependent analysis

We finally briefly outline how to obtain results for the time-dependent behavior, i.e. the law of $W(e_q)$, where e_q is exponential with rate q.

Lemma 1. *Let x_0 be a real-valued constant. Consider the Markov process with generator $\mathcal{A}_q$ given by*

$$\mathcal{A}_q f(x) = \mathcal{A}f(x) + q(f(x_0) - f(x)).$$

This Markov process is positive recurrent, and the unique invariant distribution is the same as that of W_{e_q} given that $W_0 = x_0$.

Set $W_{q,x_0}(s) = \mathrm{E}_{x_0}[\mathrm{e}^{-sW_{e_q}}]$. From the lemma, we obtain, with $f_s(x) = \mathrm{e}^{-sx}$,

$$\begin{aligned}(q - \phi(s))W_{x_0,q}(s) = q\mathrm{e}^{-sx_0} &+ sc_+\,\mathrm{E}_{x_0}[W_q\mathrm{e}^{-sW_q}(W_q > 0)] \\ &+ sc_-\,\mathrm{E}_{x_0}[W_q\mathrm{e}^{-sW_q}(W_q < 0)].\end{aligned}$$

In what follows, we focus on the case $c_- = 0$, $c_+ > 0$. In the steady-state analysis of this case, a key argument consisted of the fact that the negative part of the steady-state random variable must be exponentially distributed, since you can only enter the negative half-plane through 0. In the case considered here, this is still true if $x_0 \geq 0$. In particular, if $x_0 \geq 0$,

$$G^-_{x_0,q}(s) := \mathrm{E}_{x_0}[\mathrm{e}^{-sW_q} \mid W_q < 0] = \frac{\gamma_q}{\gamma_q - s}.$$

We can write, as before, $W_{x_0,q}(s) = P_qG^+_{x_0,q}(s) + (1 - P_q)G^-_{x_0,q}(s)$. As before, this leads to an ordinary differential equation for $G_{x_0,q}(s)$ that can be solved explicitly. If $x_0 < 0$, the process behaves like a Lévy process up to the point of upcrossing of level 0—this is τ_0. The idea is to split between the events $e_q < \tau_0$ and $e_q > \tau_0$ using the memoryless property:

$$W_{x_0,q}(s) = \mathrm{E}[\mathrm{e}^{-sX(e_q)}; e_q < \tau_0] + \int_0^\infty W_{y,q}(s)\,\mathrm{d}\mathrm{P}(X(\tau_0) \leq y; \tau_0 < e_q).$$

The behavior of a spectrally positive process starting from $x_0 < 0$ up to time τ_0 is pretty well understood; all functionals in this expression related to Lévy processes can be found in, for example, [14]. So essentially, we reduced it to a calculation issue, since we can solve in principle for $W_{y,q}(s)$ if $y \geq 0$. These computations seem tractable in the Brownian case.

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