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ON SOLUBLE GROUPS OF AUTOMORPHISM OF RIEMANN SURFACES

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ABSTRACT. Let G be a soluble group of derived length 3. We show in this paper that if G acts as an automorphism group on a compact Riemann surface of genus $g \neq 3, 5, 6, 10$ then it has at most $24(g-1)$ elements. Moreover, given a positive integer n we show the existence of a Riemann surface of genus $g = n^4 + 1$ that admits such a group of automorphisms of order $24(g-1)$, whilst a surface of specified genus can admit such a group of automorphisms of order $48(g-1)$, $40(g-1)$, $30(g-1)$ and $36(g-1)$ respectively.

1. **Introduction.** It was shown by Hurwitz [7] that a compact Riemann surface of genus $g \geq 2$ has at most $84(g-1)$ automorphisms. Although this bound is attained for infinitely many values of g [9], there is not a compact Riemann surface of genus g that admits a soluble group of order $84(g-1)$ as the group of automorphisms. As a result a soluble group of automorphisms of such a surface has order $\leq 48(g-1)$ [2]. It was shown by Chetiya [2] (see also [3]) that for every positive integer n there is a compact Riemann surface of genus $2n^6 + 1$ which admits a soluble group of automorphisms of order $96n^6$.

A soluble group G is said to be *n-soluble*, if its derived length is n, i.e., n is the smallest integer such that the n-th derived group $G^{(n)}$ of G vanishes. Thus one can look on abelian and metabelian groups as 1 and 2-soluble groups respectively. It is known that an abelian group of automorphisms of a compact Riemann surface of genus $g \geq 2$ has order $\leq 4(g+1)$, and this bound is attained for every g [12] (the direct proof is given in [5]). Recently it was shown by Chetiya and Patra [4] that a metabelian group of automorphisms of such a surface of genus $g \neq 3, 5$ has at most $16(g-1)$ elements, whilst surfaces of genus 3 and 5 can admit metabelian groups of orders 48 and 80 respectively as groups of automorphisms. As we will see they did not consider all possible cases and it turns out that their main result holds true for $g \geq 3$ only. At the end of this paper we give an example of a Riemann surface of genus $g = 2$ that admits a metabelian group of order 24 as a group of automorphisms.

All soluble groups of order > 96 constructed by Chetiya in [2] are 4-soluble. The group of order 96 is 3-soluble, and it is easy to observe that there is no bigger 3-soluble group for which the bound $48(g-1)$ is attained. Thus it is natural to ask for a bound for the order of 3-soluble groups of automorphisms of a compact Riemann surface of genus

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$g \geq 4$. We show in this paper that such a group of automorphisms of a compact Riemann surface of genus $g \neq 5, 6, 10$ has at most $24(g-1)$ elements. Moreover, we show that this bound is attained for infinitely many values of g. We also examine the remaining values of g showing that a surface of the indicated genus may have a bigger 3-soluble group of automorphisms and we describe these groups. Similar results concerning other classes of finite groups can be found in [1], [5], [6], [8], [10], [14], [15], [16].

2. **Preliminaries.** We are going to prove results stated in the introductory section by means of Fuchsian groups. A *Fuchsian group* is a discrete subgroup Γ of orientation-preserving isometries of the upper half plane D with hyperbolic structure. If Γ is cocompact, i.e., D/Γ is a compact surface then Γ has a presentation of the form:

$$(2.1) \qquad \langle x_1, \ldots, x_r, a_1, b_1, \ldots, a_g, b_g : x_i^{m_i} (i = 1, \ldots, r), \prod_{i=1}^{r} x_i \prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} \rangle$$

and is said to have *signature*

$$(2.2) \qquad (g; m_1, \ldots, m_r)$$

or to be a Fuchsian $(g; m_1, \ldots, m_r)$-*group*. The integers $m_1, \ldots, m_r$ are called *periods* of Γ, and g the *orbit genus*. If the orbit genus of a Fuchsian group Γ is zero we will write the signature shortly as

$$(m_1, \ldots, m_r).$$

Every Fuchsian group Γ has a *fundamental region* whose hyperbolic area depends only on the group and thus can be denoted by $\mu(\Gamma)$ and called the *area* of the group. If Γ has signature (2.2) then

$$(2.3) \qquad \mu(\Gamma) = 2\pi[2g - 2 + \sum_{i=1}^{r}(1 - 1/m_i)].$$

In addition for any subgroup Γ_1 of Γ of finite index the following Riemann-Hurwitz index formula holds:

$$(2.5) \qquad \mu(\Gamma_1)/\mu(\Gamma) = [\Gamma : \Gamma_1].$$

It is well-known that a compact Riemann surface of genus $g \geq 2$ can be represented as D/Γ, where Γ is a Fuchsian group with signature $(g; -)$ called a *surface group of genus g*. Moreover, a finite group G is then a group of its automorphisms if and only if there exists a Fuchsian group Λ and a homomorphism θ from Λ onto G having Γ as the kernel. Following Macbeath and Zomorrodian (see [9], [14]) such a homomorphism is said to be a *smooth homomoprhism* and a group G represented in such a form, a *smooth factor group of* Λ, while Λ is said to *admit G as a smooth factor group*.

It is also well-known that a homomorphism from Λ onto G is a smooth homomorphism if and only if it preserves the periods of Λ.

Summing up, in order to find an upper bound for the order of a 3-soluble group of automorphisms of a compact Riemann surface of genus $g \geq 2$ we have to find a lower bound for the area of Fuchsian groups that admit a 3-soluble group as a smooth factor group.

3. **3-soluble groups of automorphisms of compact Riemann surfaces.** Let X be a compact Riemann surface of genus $g \geq 2$ represented as a factor D/Γ, where Γ is a surface group of genus g, and let $G = \Lambda/\Gamma$ be a group of its autmorphisms. Since $G^{(3)} = \Lambda^{(3)}\Gamma/\Gamma$ it follows that $G^{(3)} = \{1\}$ if and only if $\Lambda^{(3)} \subseteq \Gamma$. Thus since Γ is a surface group we find that a necessary condition for a Fuchsian group Λ to admit a finite 3-soluble group as a smooth factor group is that $\Lambda^{(3)}$ has no elements of finite order. In calculations of the derived groups of a given Fuchsian group one can use the classical Reidemeister-Schreier algorithm as well as a method of Singerman [13] exploited in [4]. In certain cases another quick method of Maclachlan [11] can be applied. In order to state it we need a definition. A Fuchsian group Λ is said to satisfy the *l.c.m. condition* if every period of it divides the l.c.m. of the remaining periods. Let Λ be a Fuchsian group for which Λ/Λ' is finite. Then the derived group Λ' of Λ is a surface group if and only if Λ satisfies the l.c.m. condition. The corresponding orbit genus can be read off from the Hurwitz-Riemann formula.

It is easy to check that the Fuchsian groups listed below are all groups with the area $\leq \pi/6$.

$$(2,3,k),\ k \leq 12,$$
$$(2,4,k),\ 5 \leq k \leq 6,$$
$$(3,3,4).$$

Clearly groups $(2,3,7)$ and $(2,3,11)$ are perfect and thus do not admit any 3-soluble smooth factor.

Considering the area of the remaining groups, listed above, and using the Hurwitz-Riemann formula we obtain at once

COROLLARY. *Let G be a 3-soluble group of automorphisms of genus $g \geq 2$. Then*

$|G| \leq 48(g-1)$. *If* $|G| < 48(g-1)$ *then*
$|G| \leq 40(g-1)$. *If* $|G| < 40(g-1)$ *then*
$|G| \leq 36(g-1)$. *If* $|G| < 36(g-1)$ *then*
$|G| \leq 30(g-1)$. *If finally* $|G| < 30(g-1)$ *then*
$|G| \leq 24(g-1)$.

We shall examine all the above mentioned cases.

PROPOSITION 3.1. *A Fuchsian group Λ with signature* $(2,3,8)$ *admits one and only one finite 3-soluble smooth factor group. This factor is of order* 96.

PROOF. It was shown by Chetiya [3] that

$$\begin{aligned} \Lambda/\Lambda' &\cong Z_2, \quad \Lambda' = (3,3,4), \\ \Lambda'/\Lambda'' &\cong Z_3, \quad \Lambda'' = (4,4,4), \\ \Lambda''/\Lambda^{(3)} &\cong Z_4 \oplus Z_4, \quad \Lambda^{(3)} = (3,-). \end{aligned} \tag{3.1.1}$$

Thus $\Lambda/\Lambda^{(3)}$ is a 3-soluble smooth factor group of order 96. Now assume to get a contradiction that $G \cong \Lambda/\Gamma$ is another such factor. As we showed before $\Lambda^{(3)} \subseteq \Gamma$ and as a result $|G| \leq 96$. On the other hand $|G| = 48(g-1)$. Thus $|G| = 48$ or $|G| = 96$; but in the latter case $\Gamma = \Lambda^{(3)}$. Hence we have to show that $|G| \neq 48$ only.

Assume to get a contradiction that there exists such a factor G. From (3.1.1) follows that G'' is a group of order 8 and is generated by two elements of order 4 whose product has also order 4. Thus G'' cannot be abelian and so $G^{(3)} \neq 1$.

COROLLARY. *There exists a compact Riemann surface X of genus $g \geq 2$ having a 3-soluble group of automorphisms of order $48(g-1)$ if and only if $g = 3$; this group is unique up to isomorphism.*

PROPOSITION 3.2. *A Fuchsian group Λ with signature $(2,3,9)$ admits one and only one 3-soluble smooth factor group. This factor is of order 4×3^4.*

PROOF. $\Lambda/\Lambda' = Z_3$ and Chetiya [3] proved that $\Lambda' = (2,2,2,3)$. Clearly $\Lambda'/\Lambda'' = Z_2 \oplus Z_2$. If θ is the regular representation of Λ' as a group of permutations of the four cosets of Λ'' in Λ', then

$$\theta(x_1) = (1,2)(3,4), \quad \theta(x_2) = (2,3)(1,4),$$
$$\theta(x_3) = (1,3)(2,4), \quad \theta(x_4) = (1)(2)(3)(4).$$

Thus by a theorem of Singerman [13] Λ'' has signature $(3,3,3,3)$. Thus $\Lambda^{(3)}$ is a surface group since Λ'' satisfies l.c.m. condition. As a result $\Lambda/\Lambda^{(3)}$ is a 3-soluble smooth factor of Λ of order 4×3^4. Let $G = \Lambda/K$ be another such factor. Then as was noted at the beginning of the section $\Lambda^{(3)} \subseteq K$ and so $|G|$ divides 4×3^4. Moreover $|G| = 36(g-1)$ and thus $|G| = 4 \times 3^k$, where $k = 2, 3$, or 4. For $k = 4$, $G = \Lambda/\Lambda^{(3)}$. Thus it remains to show there are no such factors for $k = 2$ and $k = 3$.

Assume, to get a contradiction, that G is a 3-soluble smooth factor group of Λ of order 36. Since $\Lambda/\Lambda' = Z_3$ and $\Lambda'/\Lambda'' = Z_2 \oplus Z_2$, G'' has order 6 or 3. The first case is impossible since G'' is a smooth factor group of $\Lambda'' = (3,3,3,3)$. Thus $|G/G''| = 12$. Now $(xy)^3$ belongs to G'' since G/G'' has no elements of order 9. Thus G'' is a subgroup of G generated by $(xy)^3$. As a result $x(xy)^3x = (xy)^3$ or $x(xy)^3x = (xy)^{-3}$ is a relation in G. We claim that each of these relations and the remaining ones imply that $(xy)^3 = 1$. In fact in the first case $(yx)^3 = (xy)^3$ and so $(xy)^4 = yxyxy^{-1}$ is an element of order 3, i.e., $(xy)^{12} = 1$. But the last and $(xy)^9 = 1$ imply $(xy)^3 = 1$. Similarly we argue in the second case. Therefore G is a factor group of a group generated by two elements of orders 2 and 3 whose product has order 3, i.e., G is a factor group of the alternating group A_4, a contradiction.

Now assume that a group G in question of order 4×3^3 exists. Clearly G'' is an abelian group of order 9 or 18. But the second case is impossible since G'' is a smooth factor of $\Lambda'' = (3,3,3,3)$. We shall show that G'' contains a subgroup of order 3 normal in the whole group G. It is obvious when G'' is cyclic, so assume that $G'' = Z_3 \oplus Z_3$. Let $\phi : G \longrightarrow \operatorname{Aut} G''$ be a homomorphism defining the action of G on G''. We have that G is generated by two elements x, and y of order 2 and 3 respectively whose product has order 9, whilst $\operatorname{Aut} G'' = Gl(2,3)$ has order 3×16. Thus $\phi\left((xy)^3\right) = \text{id}$. As a result ϕ can be factored through the alternating group A_4. But it is easy to check that $Gl(2,3)$ does not contain A_4. So ϕ is trivial or its image has order 3. Thus we are done since every automorphism of order 3 in $Gl(2,3)$ possesses in $Z_3 \oplus Z_3$ an invariant subgroup of order 3.

Now let $G_1 = G/H_1$, where H_1 is a normal subgroup of order 3 just constructed. Clearly G_1 is a 3-soluble group and has order 36. Let x and y be generators of G of order 2 and 3 respectively whose product has order 9. If one of x, y, xy or $(xy)^3$ belonged to H_1 then G_1 would be clearly a group of order ≤ 12, a contradiction. Thus x and y induce in G_1 elements of orders 2 and 3 respectively whose product has order 9. So G_1 is smooth factor of Λ, a contradiction with the previous part of the proof.

COROLLARY. *There exists a compact Riemann surface of genus $g \geq 2$ having a 3-soluble group of automorphisms of order $36(g-1)$ if and only if $g = 10$; this group is unique up to isomorphism.*

PROPOSITION 3.3. *A Fuchsian group Λ with signature $(2,3,10)$ admits one and only one finite 3-soluble smooth factor group. This factor is of order 150.*

PROOF. From [3] $\Lambda/\Lambda' = Z_2$ and $\Lambda' = (3,3,5)$. Clearly $\Lambda'/\Lambda'' = Z_3$. If θ is the regular representation of Λ' as a group of permutations of the three cosets of Λ'' in Λ', then

$$\theta(x_1) = (1,2,3), \quad \theta(x_2) = (1,3,2), \quad \text{and } \theta(x_3) = (1)(2)(3).$$

Thus by the theorem of Singerman $\Lambda'' = (5,5,5)$. As a result $\Lambda^{(3)}$ is a surface group and so $\Lambda/\Lambda^{(3)}$ is a 3-soluble smooth factor group of order 150.

In order to show the uniqueness assume, to get a contradiction, that G is another such factor. As in the previous Proposition we argue that $|G| = 150$ or 30 and in the first case $G \cong \Lambda/\Lambda^{(3)}$. Thus we have to rule out the second possibility. If $|G|$ were 30 then clearly $|G'| = 15$ and thus on the one hand G' is cyclic and on the other is generated by two elements of order 3 whose product has order 5, a contradiction.

COROLLARY. *There exists a compact Riemann surface of genus $g \geq 2$ having a 3-soluble group of automorphisms of order $30(g-1)$ if and only if $g = 6$; this group is unique up to isomorphism.*

PROPOSITION 3.4. *A Fuchsian group Λ with signature $(2,4,5)$ admits one and only one 3-soluble smooth factor. This factor is of order 160.*

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PROOF. Clearly $\Lambda/\Lambda' = Z_2$, and using, again the same theorem of Singerman it is easy to see that $\Lambda' = (2, 5, 5)$. Now we see that each 3-soluble smooth factor of Λ induces a metabelian smooth factor of Λ'. But Chetiya and Patra have shown [4] that there is only one such factor, the group $\Lambda'/\Lambda^{(3)}$ that has order 80. Thus $\Lambda/\Lambda^{(3)}$ is the unique 3-soluble smooth factor of Λ.

COROLLARY. *There exists a compact Riemann surface of genus $g \geq 2$ having a 3-soluble group of automorphisms of order $40(g-1)$ if and only if $g = 5$; this group is unique up to isomorphism.*

THEOREM. *A 3-soluble group of automorphisms of a compact Riemann surface of genus $g \neq 3, 5, 6, 10$ has order $\leq 24(g-1)$. Moreover given a positive integer n there exists a compact Riemann surface of genus $g = n^4 + 1$ that admits a 3-soluble group of order $24(g-1)$ as a group of automorphisms.*

PROOF. Let $X = D/\Gamma$ be a surface in question and let $G = \Lambda/\Gamma$ be its 3-soluble group of automorphisms. If $\mu(\Lambda) < \pi/6$ then Λ is one of the groups considered in the four previous propositions and thus D/Γ has genus 3, 5, 6, or 10, a contradiction. Thus $\mu(\Lambda) \geq \pi/6$ and by the Hurwitz-Riemann formula $|G| = \mu(\Gamma)/\mu(\Lambda) \leq 4\pi(g-1)/(\pi/6) = 24(g-1)$.

Now let $\Lambda = (2, 4, 6)$. Then $\Lambda/\Lambda' = Z_2 \oplus Z_2$. Again if θ is the regular representation of Λ' as a group of permutations of the four cosets of Λ' in Λ, then

$$\theta(x_1) = (1, 2)(3, 4), \quad \theta(x_2) = (1, 3)(2, 4), \quad \theta(x_3) = (1, 4)(2, 3).$$

Thus by the theorem of Singerman $\Lambda' = (2, 2, 3, 3)$, and so Λ'' is a surface group; by the Hurwitz-Riemann formula the genus of Λ'' is 2.

Given a positive integer n let $\Gamma_n = (\Lambda'')^n\Lambda^{(3)}$ be the subgroup of Λ'' generated by all its n-powers and commutators. Clearly Γ_n is a surface group normal in Λ and $G = \Lambda/\Gamma_n$ is a 3-soluble group of order $24 \times n^4$ that acts as a group of automorphisms on a surface $X = D/\Gamma_n$ which has the genus $g = n^4 + 1$.

REMARK. We showed in the proof of the above theorem that for $\Lambda = (2, 4, 6)$, Λ'' is a surface group of genus $g = 2$. Thus $G = \Lambda/\Lambda''$ is a metablian group of order 24 acting as a group of automorphisms on a Riemann surface $X = D/\Lambda''$ of genus $g = 2$. Thus the main result of Chetiya and Patra in [4] holds only for $g \geq 3$ and should be restated:

THEOREM. *A metabelian group of automorphisms of a compact Riemann surface of genus $g \neq 2, 3, 5$ has at most $16(g-1)$ elements.*

The remaining results of the paper in question hold true.

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