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EXISTENCE OF POSITIVE SOLUTION FOR INDEFINITE KIRCHHOFF EQUATION IN EXTERIOR DOMAINS WITH SUBCRITICAL OR CRITICAL GROWTH

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Abstract

Using variational methods and depending on a parameter λ we prove the existence of solutions for the following class of nonlocal boundary value problems of Kirchhoff type defined on an exterior domain $\Omega \subset \mathbb{R}^3$:

$$\begin{cases} M(\|u\|^2)[-\Delta u + u] = \lambda a(x)g(u) + \gamma|u|^4 u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

for the subcritical case ($\gamma = 0$) and also for the critical case ($\gamma = 1$).

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1. Introduction

The purpose of this article is to investigate the existence of positive solutions to the following class of nonlocal boundary value problems of the Kirchhoff type:

$$(P_{\lambda,\gamma}) \qquad \begin{cases} M(\|u\|^2)[-\Delta u + u] = \lambda a(x)g(u) + \gamma|u|^4 u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where λ is a positive real parameter and Ω is an exterior domain in $\mathbb{R}^3$, that is, $\Omega = \mathbb{R}^3 \backslash \overline{\Theta}$, with Θ a bounded smooth domain in $\mathbb{R}^3$. In this paper we study two cases of the problem $(P_{\lambda,\gamma})$; the first case is when $\gamma = 0$ (subcritical case) and the second case is when $\gamma = 1$ (critical case). The functions $M : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+$, $a : \Omega \to \mathbb{R}$ and

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$g : \mathbb{R} \to \mathbb{R}$ are continuous functions that satisfy some conditions to be established later, and

$$\|u\|^2 = \int_\Omega |\nabla u|^2\, dx + \int_\Omega |u|^2\, dx.$$

Before stating our main result, we need the following hypotheses on the function $M : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+$:

(M_1) The function M is increasing and $0 < M(0) := m_0$.
(M_2) The function $t \mapsto M(t)/t$ is decreasing.

More information about the physical motivation behind Kirchhoff problems can be found in [1–3, 6–8, 12] and references therein.

The hypothesis (M_1) allows us to deal with the problem ($P_{\lambda,\gamma}$) via variational methods. The hypothesis (M_2) gives an important growth criterion to be used throughout the paper.

A typical example of a function satisfying conditions (M_1) and (M_2) is given by

$$M(t) = m_0 + bt,$$

for a real constant $b > 0$ and for all $t \geq 0$. This function is the one originally considered in the Kirchhoff equation in the seminal paper [10]. However, our hypotheses on the function M include different classes of functions, such as $M(t) = m_0 + bt + \sum_{i=1}^{k} b_i t^{d_i}$ with $b_i \geq 0$, $d_i \in (0, 1)$ for all $i \in \{1, 2, \ldots, k\}$, and also some more special functions, not involving powers, such as $M(t) = ln(t + m_0)$, for some $m_0 > e$ (base of the natural logarithm) and $t \geq 0$.

The hypotheses on the function $g : \mathbb{R} \to \mathbb{R}$ are as follows:

(g_1)

$$\lim_{t\to 0} \frac{g(t)}{|t|} = 0.$$

(g_2)

$$\lim_{|t|\to +\infty} \frac{g(t)}{t^5} = 0.$$

(g_3) There exists $\theta \in (4, 6)$ such that

$$0 < \theta G(t) \leq t g(t) \quad \text{for all } |t| > 0,$$

where $G(t) = \int_0^t g(s)\, ds$.

A typical example of a function satisfying conditions (g_1)–(g_3) is given by

$$g(t) = \sum_{i=1}^{k} C_i t_+^{q_i}$$

with $k \in \mathbb{N}$, $1 < q_i < 5$, $C_i > 0$ and $t_+ = \max\{t, 0\}$.

The first hypothesis on the function a is:

(a_1) $a \in C(\Omega, \mathbb{R})$ changes sign in Ω.

To state the next hypotheses on the function a, let us define

$$\Omega^+ = \{x \in \Omega : a(x) > 0\} \quad \text{and} \quad \Omega^- = \{x \in \Omega : a(x) < 0\}.$$

It is known that there is a function $\zeta \in C_0^\infty(\Omega)$ such that $0 \le \zeta(x) \le 1$ in Ω, $\zeta(x) = 1$ in Ω^+ and $\zeta(x) = 0$ in Ω^-. Depending on the distance $\mathrm{dist}(\overline{\Omega^+}, \overline{\Omega^-})$ between the sets $\overline{\Omega^+}$ and $\overline{\Omega^-}$, it is possible to take this function with $\widetilde{K} := \sup_\Omega |\nabla\zeta|$ as small as we need (see (1.1)).

The next hypotheses on a are as follows:

(a_2) The distance

$$\mathrm{dist}(\overline{\Omega^+}, \overline{\Omega^-}) = \delta > 0$$

is such that

$$\widetilde{K} < \frac{\theta}{2} - 2, \tag{1.1}$$

where θ is the constant that appears in hypothesis (g_3).

(a_3) There is $R_0 > 0$ such that

$$a(x) < 0 \quad \text{for } |x| \ge R_0 \quad \text{and} \quad \sup_{|x|\ge R} |a(x)||x|^2 < \infty \quad \text{for all } R \ge R_0.$$

The hypothesis (a_1) characterizes the problem ($P_{\lambda,\gamma}$) as indefinite, as can be seen in [4, 9, 13].

The previous function ζ will be essential to overcome some difficulties such as the boundedness of the Palais–Smale sequence.

The hypothesis (a_3) appeared in [13] and is used to overcome the lack of compactness of the embedding $H_0^1(\Omega) \hookrightarrow L^s(\Omega)$, for $2 < s \le 6$, due to the unboundedness of the domain Ω.

The main result of this paper is the following theorem.

THEOREM 1.1. *Assume that conditions (M_1), (M_2), (g_1)–(g_3) and (a_1)–(a_3) hold. If $\gamma = 0$, the problem ($P_{1,0}$) has a positive solution for all $\lambda > 0$. If $\gamma = 1$, there exists $\lambda_* > 0$ such that the problem ($P_{\lambda,1}$) has a positive solution for all $\lambda > \lambda_*$.*

In the literature, we did not find any works about Kirchhoff equations on exterior unbounded domains as studied in this paper.

The scheme of this paper is as follows: in Section 2, we build up the variational framework and prove some technical results; in Section 3, we prove the subcritical case of the problem; and in Section 4, we prove the critical case.

2. The variational framework and some technical lemmas

Since we intend to find a positive solution for the problem ($P_{\lambda,\gamma}$), throughout this paper let us assume that

$$g(t) = 0 \quad \forall t \le 0.$$

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We recall that $u \in H_0^1(\Omega)$ is a weak solution of ($P_{\lambda,\gamma}$) if it satisfies

$$M(\|u\|^2)\Big[\int_\Omega \nabla u \nabla \phi\,dx + \int_\Omega u\phi\,dx\Big] = \lambda \int_\Omega a(x)g(u)\phi\,dx + \gamma \int_\Omega |u|^4 u\phi\,dx$$

for all $\phi \in H_0^1(\Omega)$.

We shall look for positive solutions as critical points of the C^1-functional $I_{\lambda,\gamma} : H_0^1(\Omega) \to \mathbb{R}$, given by

$$I_{\lambda,\gamma}(u) = \frac{1}{2}\widehat{M}(\|u\|^2) - \lambda \int_\Omega a(x)G(u)\,dx - \frac{\gamma}{6}\int_\Omega u_+^6\,dx,$$

where $\widehat{M}(t) = \int_0^t M(s)\,ds$ and $u_+ := \max\{u, 0\}$.

Note that

$$I'_{\lambda,\gamma}(u)(\phi) = M(\|u\|^2)\Big[\int_\Omega \nabla u \nabla \phi\,dx + \int_\Omega u\phi\,dx\Big] - \lambda \int_\Omega a(x)g(u)\phi\,dx - \gamma \int_\Omega u_+^5\phi\,dx$$

for all $\phi \in H_0^1(\Omega)$. Moreover, if the critical point is nontrivial, by a maximum principle, we conclude that it is a positive solution for ($P_{\lambda,\gamma}$).

In order to use variational methods, we first derive some results related to the well-known Palais–Smale compactness condition ((PS) for short).

In the sequel, we prove that the functional $I_{\lambda,\gamma}$ has the mountain pass geometry. This fact is proved in the next two lemmas.

Lemma 2.1. *Assume that conditions* (M_1), (a_1)–(a_3), (g_1) *and* (g_2) *hold. Then there exist positive numbers* ρ *and* α *such that*

$$I_{\lambda,\gamma}(u) \ge \alpha > 0 \quad \forall u \in H_0^1(\Omega); \|u\| = \rho.$$

Proof. It follows from (g_1) and (g_2) that, for each $\epsilon > 0$, there is a positive constant C_ϵ such that

$$g(t) \le \epsilon|t| + C_\epsilon|t|^5. \tag{2.1}$$

By (a_3) we conclude that Ω^+ is bounded. Defining $C_0 = \sup_{\overline{\Omega^+}} a$ and using (2.1),

$$\int_\Omega a(x)G(u)\,dx \le \int_{\Omega^+} a(x)G(u)\,dx \le C_0\frac{\epsilon}{2}\int_\Omega |u|^2\,dx + C_0\frac{C_\epsilon}{6}\int_\Omega |u|^6\,dx. \tag{2.2}$$

By (2.2) and (M_1),

$$I_{\lambda,\gamma}(u) \ge \frac{m_0}{2}\|u\|^2 - C_0\frac{\epsilon}{2}\int_\Omega |u|^2\,dx - \frac{\gamma + C_\epsilon}{6}\int_\Omega u_+^6\,dx.$$

Finally, using the Sobolev embedding theorem, there is a positive constant $C > 0$ such that

$$I_{\lambda,\gamma}(u) \ge \frac{m_0 - C\cdot\epsilon}{2}\|u\|^2 - C\|u\|^6.$$

For a sufficiently small ϵ the result follows choosing $\rho > 0$ small enough. □

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Lemma 2.2. *Assume that conditions* (M_1), (M_2), (a_1)–(a_3) *and* (g_1)–(g_3) *hold. Then there exists a function* $e \in H_0^1(\Omega)$ *such that* $I_{\lambda,\gamma}(e) < 0$ *and* $\|e\| > \rho$, *where* $\rho > 0$ *appears in Lemma 2.1.*

Proof. Notice that using (g_3) there exist constants $C, D > 0$ such that

$$G(t) \geq C|t|^\theta - D. \tag{2.3}$$

Moreover, by (M_2),

$$M(t) \leq M(1)t, \tag{2.4}$$

for all $t \geq 1$.

Let us consider $v_0 \in C_0^\infty(\Omega^+) \setminus \{0\}$ with $v_0 \geq 0$ in Ω^+ and $\|v_0\| = 1$. Using $C_0 := \sup_{\overline{\Omega^+}} a$, (2.3) and (2.4),

$$I_{\lambda,\gamma}(tv_0) \leq \frac{M(1)}{2}t^2 - Ct^\theta \int_{\mathrm{supp}\, v_0} a(x)v_0^\theta \, dx + DC_0|\mathrm{supp}\, v_0| - \gamma\frac{t^6}{6}\int_\Omega v_0^6 \, dx,$$

where $|\mathrm{supp}\, v_0|$ denotes the Lebesgue measure of the set $\mathrm{supp}\, v_0$. Since $4 < \theta < 6$, the result follows picking $e = t_* v_0$, for some large enough $t_* > 0$. □

Due to the two previously proved lemmas, we may employ a version of the mountain pass theorem, without the (PS) condition (see [14, page 12]), due to Ambrosetti and Rabinowitz [5], and conclude the existence of a sequence $(u_n) \subset H_0^1(\Omega)$ satisfying

$$I_{\lambda,\gamma}(u_n) \to c_{\lambda,\gamma} \quad \text{and} \quad I'_{\lambda,\gamma}(u_n) \to 0, \tag{2.5}$$

where

$$c_{\lambda,\gamma} = \inf_{\eta\in\Gamma} \max_{t\in[0,1]} I_{\lambda,\gamma}(\eta(t)) > 0 \tag{2.6}$$

and

$$\Gamma := \{\eta \in C([0,1], H_0^1(\Omega)) : \eta(0) = 0, \eta(1) = e\}. \tag{2.7}$$

Let us prove the following lemma about the sequence (u_n).

Lemma 2.3. *Let* $(u_n) \subset H_0^1(\Omega)$ *be a sequence satisfying (2.5). Then* (u_n) *is bounded.*

Proof. By straightforward calculations, there is a positive constant C such that

$$I_{\lambda,\gamma}(u_n) - \frac{1}{\theta}I'_{\lambda,\gamma}(u_n)(\zeta u_n) \leq \|u_n\| + C, \tag{2.8}$$

where ζ is the function whose supremum (least upper bound), $\widetilde{K}$, appears in hypothesis (a_2).

On the other hand, since $\zeta = 0$ in Ω^-, by (a_3),

$$I_{\lambda,\gamma}(u_n) - \frac{1}{\theta}I'_{\lambda,\gamma}(u_n)(\zeta u_n) \geq \frac{1}{2}\widehat{M}(\|u_n\|^2) - \frac{1}{\theta}M(\|u_n\|^2)\|u_n\|^2 - \frac{1}{\theta}\widetilde{K}M(\|u_n\|^2)\int_\Omega |u_n||\nabla u_n| \, dx.$$

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From Young's inequality,

$$I_{\lambda,\gamma}(u_n) - \frac{1}{\theta} I'_{\lambda,\gamma}(u_n)(\zeta u_n) \geq \frac{1}{2}\widehat{M}(\|u_n\|^2) - \frac{1}{\theta} M(\|u_n\|^2)\|u_n\|^2 - \frac{1}{2\theta}\widetilde{K} M(\|u_n\|^2)\|u_n\|^2. \tag{2.9}$$

By (2.8) and (2.9),

$$\frac{1}{2}\widehat{M}(\|u_n\|^2) - \frac{1}{\theta} M(\|u_n\|^2)\|u_n\|^2 - \frac{1}{2\theta}\widetilde{K} M(\|u_n\|^2)\|u_n\|^2 \leq \|u_n\| + C. \tag{2.10}$$

Recall that by the definition of $\widehat{M}$ and by (M_2), it follows that

$$\widehat{M}(t) \geq \tfrac{1}{2} M(t)t \quad \text{for all } t \geq 0. \tag{2.11}$$

Thus, using (M_1), (2.10) and (2.11),

$$m_0\Big(\frac{1}{4} - \frac{1}{\theta} - \frac{\widetilde{K}}{2\theta}\Big)\|u_n\|^2 \leq \|u_n\| + C.$$

Since δ in (a_2) was taken to satisfy (1.1), we conclude that the sequence (u_n) is bounded. □

Lemma 2.3 guarantees the existence of a convergent subsequence $(u_n) \in H_0^1(\Omega)$ such that

$$u_n \rightharpoonup u \quad \text{in } H_0^1(\Omega) \tag{2.12}$$

and

$$u_n \to u \quad \text{in } L^s_{loc}(\Omega), \tag{2.13}$$

for $2 \leq s < 6$.

The following lemma will also be very useful.

Lemma 2.4. *Let $(u_n) \subset H_0^1(\Omega)$ be as in (2.12) and (2.13). Then*

$$\int_\Omega a(x)g(u_n)u_n\,dx \to \int_\Omega a(x)g(u)u\,dx \tag{2.14}$$

and

$$\int_\Omega a(x)g(u_n)u\,dx \to \int_\Omega a(x)g(u)u\,dx. \tag{2.15}$$

Proof. We shall prove (2.14); the proof of the other limit (2.15) is analogous.

Notice that for all $R \geq R_0$, where R_0 is the positive constant that appears in hypothesis (a_3), since g has subcritical growth and (2.13) holds, applying the Lebesgue dominated convergence theorem,

$$\int_{\Omega \cap B_R} a(x)g(u_n)u_n\,dx \to \int_{\Omega \cap B_R} a(x)g(u)u\,dx.$$

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The proof is completed if we prove that

$$\lim_{R\to\infty}\int_{\Omega\setminus B_R} a(x)g(u_n)u_n\,dx = 0 \tag{2.16}$$

uniformly in n.

It follows from (g_1) and (g_2) that, for $\epsilon > 0$ and a fixed $q \in (4, 6)$, there is a constant C_ϵ such that

$$g(t) \le \epsilon|t| + C_\epsilon|t|^{q-1} + \epsilon|t|^5. \tag{2.17}$$

Using (2.17),

$$\begin{aligned}\int_{\Omega\setminus B_R} a(x)g(u_n)u_n\,dx &\le \epsilon\sup_{|x|\ge R}|a(x)|\int_{\Omega\setminus B_R}|u_n|^2\,dx\\ &\quad + C_\epsilon\sup_{|x|\ge R}|a(x)||x|^2\int_{\Omega\setminus B_R}\frac{|u_n|^q}{|x|^2}\,dx\\ &\quad + \epsilon\sup_{|x|\ge R}|a(x)|\int_{\Omega\setminus B_R}|u_n|^6\,dx.\end{aligned} \tag{2.18}$$

Applying Hölder's inequality,

$$\int_{\Omega\setminus B_R}\frac{|u_n|^q}{|x|^2}\,dx \le \left(\int_{\Omega\setminus B_R}\frac{dx}{|x|^{2r}}\right)^{1/r}\|u_n\|^q \tag{2.19}$$

where $r = 6/(6-q)$. Since $r > 3/2$, then, given $\epsilon > 0$, there exists $R = R(\epsilon) \ge R_0$ such that

$$\left(\int_{\Omega\setminus B_R}\frac{dx}{|x|^{2r}}\right)^{1/r} < \epsilon. \tag{2.20}$$

Employing inequality (2.20) in (2.19) and (2.19) in (2.18), considering that (a_3) implies $\sup_{|x|\ge R}|a(x)| < \infty$, and taking into account the boundedness of the sequence (u_n), the limit (2.16) is proved and the proof is completed. □

3. The subcritical case

In the subcritical case, we consider $\gamma = 0$ and no restriction on the parameter λ is made, so it can be absorbed by the function a. Then, the problem $(P_{\lambda,\gamma})$ reduces to

$$(P_{1,0})\qquad \begin{cases} M(\|u\|^2)[-\Delta u + u] = a(x)g(u) & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega,\end{cases}$$

with the associated functional given by

$$I_0(u) = \frac{1}{2}\widehat{M}(\|u\|^2) - \int_\Omega a(x)G(u)\,dx.$$

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Proof of Theorem 1.1 in the subcritical case ($\gamma = 0$).

Let us show that the sequence (u_n) that satisfies (2.5) has, indeed, a convergent subsequence. By Lemma 2.3, the sequence is bounded and has a weak convergent subsequence converging to u. Hence, up to subsequences, since $\|u\| \le \liminf_{n\to\infty} \|u_n\|$, we get $\|u_n\|^2 - \|u\|^2 \ge 0$ for sufficiently large n. Thus, the weak convergence of (u_n), (2.14) and (2.15) implies that for large n,

$$\begin{aligned} o_n &= I_0'(u_n)(u_n) - I_0'(u_n)(u) \\ &= M(\|u_n\|^2)[\|u_n\|^2 - \|u\|^2] + \int_\Omega a(x)g(u_n)u\,dx - \int_\Omega a(x)g(u_n)u_n\,dx \\ &\ge m_0[\|u_n\|^2 - \|u\|^2] + o_n. \end{aligned}$$

Hence, since $m_0 > 0$, we conclude that $\lim_{n\to\infty} \|u_n\| = \|u\|$, which implies the convergence $\lim_{n\to\infty} u_n = u$ in the space $H_0^1(\Omega)$.

The functional I_0 is of C^1 class, and by (2.5) and the above convergences it follows that $I_0'(u) = 0$. Therefore, u is a weak solution of the problem $(P_{1,0})$.

4. The critical case

In the critical case, we consider $\gamma = 1$, and the problem $(P_{\lambda,\gamma})$ takes the form

$$(P_{\lambda,1}) \qquad \begin{cases} M(\|u\|^2)[-\Delta u + u] = \lambda a(x)g(u) + |u|^4 u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with the associated functional given by

$$I_\lambda(u) = \frac{1}{2}\widehat{M}(\|u\|^2) - \lambda \int_\Omega a(x)G(u)\,dx - \frac{1}{6}\int_\Omega u_+^6\,dx.$$

To prove Theorem 1.1 in the case $\gamma = 1$, we need to establish some definitions. Let us denote by S the best Sobolev constant of the embedding

$$H_0^1(\Omega) \hookrightarrow L^6(\Omega),$$

which is given by

$$S := \inf_{u\in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2\,dx}{(\int_\Omega |u|^6\,dx)^{1/3}}.$$

It is well known that S is independent of the set Ω and it is never achieved, except when $\Omega = \mathbb{R}^3$. Moreover,

$$S := \frac{\int_{\mathbb{R}^3} |\nabla U|^2\,dx}{(\int_{\mathbb{R}^3} |U|^6\,dx)^{1/3}},$$

where $U(x) = C_3/(|x|^2 + 1)$ and C_3 is a constant such that

$$-\Delta U = U^5 \quad \text{in } \mathbb{R}^3.$$

From now on, we shall prove an estimate for $c_{\lambda,1}$ defined in (2.6). For economy of notation let us write $c_{\lambda,1} := c_\lambda$.

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Lemma 4.1. *If conditions* (M_1)–(M_3), (a_1)–(a_3) *and* (g_1)–(g_3) *hold, then*

$$\lim_{\lambda\to\infty} c_\lambda = 0. \tag{4.1}$$

Proof. Let $v_0 \in H_0^1(\Omega)$ be the function given in the proof of Lemma 2.2. Since the functional I_λ has the mountain pass theorem geometry, there exists t_λ such that

$$I_\lambda(t_\lambda v_0) = \max_{t\geq 0} I_\lambda(t v_0).$$

Hence, since v_0 is normalized,

$$t_\lambda^2 M(t_\lambda^2) = \lambda \int_\Omega a(x) g(t_\lambda v_0) t_\lambda v_0 + t_\lambda^6 \int_\Omega {v_0}^6\, dx, \tag{4.2}$$

and then,

$$t_\lambda^2 M(t_\lambda^2) \geq t_\lambda^6 \int_\Omega {v_0}^6\, dx.$$

Using (2.4),

$$t_\lambda^4 M(1) \geq t_\lambda^6 \int_\Omega {v_0}^6\, dx,$$

and, therefore, for any sequence $\lambda_n \to \infty$ there is a sequence $t_{\lambda_n} \to t_0$, for some real number $t_0 \geq 0$.

Let us prove that $t_0 = 0$. If $t_0 > 0$, we would have a contradiction. Indeed, (4.2) implies that the expression

$$\lambda_n \int_\Omega a(x) g(t_{\lambda_n} v_0) t_{\lambda_n} v_0 + t_{\lambda_n}^6 \int_\Omega {v_0}^6\, dx$$

is bounded. This, in turn, yields that

$$\lambda_n \int_\Omega a(x) g(t_{\lambda_n} v_0) t_{\lambda_n} v_0 \leq \int_\Omega a(x) g(t_{\lambda_n} v_0) t_{\lambda_n} v_0 + t_{\lambda_n}^6 \int_\Omega {v_0}^6\, dx$$

is also bounded, but this cannot happen because

$$\lim_{n\to\infty} \lambda_n \int_\Omega a(x) g(t_{\lambda_n} v_0) t_{\lambda_n} v_0 = +\infty.$$

Therefore, $t_0 = 0$.

Using the notation and results of Lemma 2.2, let us define the path $\eta_*(t) =: te = t t_* v_0$. Note that $\eta_*(0) = 0$, $I_\lambda(\eta_*(1)) < 0$ and, consequently, $\eta_*(t) \in \Gamma$, as defined in (2.7).

Finally,

$$0 < c_\lambda \leq \max_{t\in[0,1]} I_\lambda(\eta_*(t)) = I_\lambda(t_\lambda v_0) \leq \tfrac{1}{2}\widehat{M}(t_\lambda^2)$$

and the continuity of the function $\widehat{M}$, together with the limit $t_{\lambda_n} \to 0$, imply that $\lim_{\lambda\to+\infty} c_\lambda = 0$, as we wished to prove. □

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Proof of Theorem 1.1 in the critical case ($\gamma = 1$).

Let us show that the Palais–Smale sequence (u_n) that satisfies (2.5) has a subsequence such that

$$\lim_{n\to\infty} \int_\Omega |u_n|^6\,dx = \int_\Omega |u|^6\,dx \tag{4.3}$$

and also that

$$\lim_{n\to\infty} \|u_n\|^2 = \|u\|^2. \tag{4.4}$$

Indeed, in order to prove (4.3), taking a subsequence, we may suppose that

$$|\nabla u_n|^2 \rightharpoonup |\nabla u|^2 + \mu \quad \text{and} \quad |u_n|^6 \rightharpoonup |u|^6 + \nu \quad \text{(in the weak}^* \text{ sense of measures).}$$

Using the concentration compactness principle due to Lions (see [11, Lemma 2.1]), we obtain an at most countable index set Λ and sequences $(x_i) \subset \mathbb{R}^3$, $(\mu_i), (\nu_i) \subset [0,\infty)$, such that

$$\nu = \sum_{i\in\Lambda} \nu_i \delta_{x_i}, \quad \mu \ge \sum_{i\in\Lambda} \mu_i \delta_{x_i} \quad \text{and} \quad S\,\nu_i^{1/3} \le \mu_i, \tag{4.5}$$

for all $i \in \Lambda$, where δ_{x_i} is the Dirac mass at $x_i \in \Omega$.

Now we claim $\Lambda = \emptyset$. Arguing by contradiction, assume that $\Lambda \ne \emptyset$. Consider a function ϕ such that $\phi \in C_0^\infty(\Omega, [0,1])$, $\phi \equiv 1$ on $B_1(0)$, $\phi \equiv 0$ on $B_2(0)$ and $|\nabla\phi|_\infty \le 2$. Let us fix $i \in \Lambda$. Defining $\psi_\varrho(x) := \phi((x - x_i)/\varrho)$ where $\varrho > 0$, we have that $(\psi_\varrho u_n)$ is bounded. Thus $I_1'(u_n)(\psi_\varrho u_n) \to 0$, that is,

$$\begin{aligned} &M(\|u_n\|^2)\Big[\int_\Omega u_n \nabla u_n \cdot \nabla\psi_\varrho\,dx + \int_\Omega |u_n|^2\psi_\varrho\,dx\Big] \\ &\quad = -M(\|u_n\|^2)\int_\Omega \psi_\varrho |\nabla u_n|^2\,dx + \lambda\int_\Omega a(x)g(u_n)\psi_\varrho u_n\,dx + \int_\Omega \psi_\varrho |u_n|^6\,dx + o_n(1). \end{aligned}$$

Since the support of ψ_ϱ is $B_{2\varrho}(x_i)$, we obtain

$$\Big|\int_\Omega u_n \nabla u_n \cdot \nabla\psi_\varrho\,dx\Big| \le \int_{B_{2\rho}(x_i)} |\nabla u_n||u_n \nabla\psi_\varrho|\,dx.$$

By the Hölder inequality and the fact that the sequence (u_n) is bounded in $H_0^1(\Omega)$,

$$\Big|\int_\Omega u_n \nabla u_n \cdot \nabla\psi_\varrho\,dx\Big| \le C\Big(\int_{B_{2\varrho}(x_i)} |u_n \nabla\psi_\varrho|^2\,dx\Big)^{1/2}.$$

By the dominated convergence theorem, $\int_{B_{2\varrho}(x_i)} |u_n\nabla\psi_\varrho|^2\,dx \to 0$ as $n \to +\infty$ and $\varrho \to 0$. Therefore,

$$\lim_{\varrho\to 0}\Big[\lim_{n\to\infty}\int_\Omega u_n \nabla u_n \cdot \nabla\psi_\varrho\,dx\Big] = 0.$$

By (M_1),

$$\lim_{\varrho\to 0}\lim_{n\to\infty}\Big[M_a(\|u_n\|^2)\int_\Omega u_n \nabla u_n \cdot \nabla\psi_\varrho\,dx\Big] = 0.$$

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Moreover, similar reasoning yields

$$\lim_{\varrho\to 0}\lim_{n\to\infty}\Big[\int_\Omega a(x)g(u_n)\psi_\varrho u_n\,dx\Big]=0.$$

Thus,

$$m_0\int_\Omega \psi_\varrho d\mu \le \int_\Omega \psi_\varrho d\nu + o_\varrho(1).$$

Letting $\varrho\to 0$, and using the standard theory of Radon measures and (4.5), the following inequality is achieved:

$$\nu_i \ge (m_0S)^{3/2}.$$

Now we shall prove that the above inequality cannot occur, and therefore that the set Λ is empty. Indeed, arguing by contradiction, let us suppose that $\nu_i\ge (m_0S)^{3/2}$ for some $i\in\Lambda$. Thus, some known standard arguments imply that

$$c_\lambda \ge \Big(\frac{1}{\theta}-\frac{1}{2^*}\Big)(m_0S)^{3/2}. \tag{4.6}$$

By (4.1), there exists $\lambda^*>0$ such that

$$c_\lambda < \Big(\frac{1}{\theta}-\frac{1}{2^*}\Big)(m_0S)^{3/2} \quad \forall\lambda\ge\lambda^*. \tag{4.7}$$

But inequality (4.6) contradicts (4.7) above. Hence, the index set $\Lambda=\emptyset$ and thus (4.3) holds.

In order to prove (4.4), by Lemma 2.3, again up to subsequences, we may assume that $\lim_{n\to\infty}\|u_n\|^2=A$, for some real number $A\ge 0$.

By (4.3) and Lemma 2.4,

$$\lim_{n\to\infty} M(\|u_n\|^2)\|u_n\|^2=\lambda\int_\Omega a(x)g(u)u\,dx+\int_\Omega |u|^6\,dx. \tag{4.8}$$

Using (M_1),

$$M(A)\Big[\int_\Omega \nabla u\nabla\phi\,dx+\int_\Omega u\phi\,dx\Big]=\lambda\int_\Omega a(x)g(u)\phi\,dx+\int_\Omega |u|^4u\phi\,dx, \tag{4.9}$$

for all $\phi\in H_0^1(\Omega)$.

The limits in (4.8) and (4.9) yield

$$\lim_{n\to\infty} M(\|u_n\|^2)\|u_n\|^2 = M(A)\|u\|^2$$

and (4.4) is valid.

For $\lambda\ge\lambda^*$, the rest of the proof follows the same steps made at the end of the proof of Theorem 1.1 in the subcritical case. □

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