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ON THE MAPPING THEOREM FOR LUSTERNIK-SCHNIRELMANN CATEGORY II

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0. Introduction. Let X and Y be 1-connected spaces having the homotopy type of cw-complexes.

Definition 0.1. A continuous map $f:X \to Y$ is *Ω-split* if $\Omega f:\Omega X \to \Omega Y$ admits a retraction up to homotopy.

In [**6**] we prove the following "mapping theorem":

THEOREM 0.1. (a) *If f is Ω-split, then* $\text{cat}(X) \leqq \text{cat}(Y)$;
(b) *If $\pi_*(f)$ is split injective and ΩY has the homotopy type of a product of Eilenberg-MacLane spaces, then f is Ω-split.*

This result applies to the case when X and Y are rational or tame spaces because the loop space of such a space has the homotopy type of a weak product of Eilenberg-MacLane spaces: actually the proof of this statement in the tame case given in [**6**] is incomplete and we therefore give another one in an appendix to this paper.

Other examples of Ω-split maps have been given by H. Scheerer [**9**], who kindly pointed out that our argument for tame spaces did not suffice.

Also in [**6**] we define the T^r-category of a space as either the category of its "r-taming" or as the infimum of the categories of all spaces having the same tame homotopy type. Unfortunately these two definitions are not equivalent: for instance the three-sphere has category 1 while its 3-taming (which by definition has the same tame homotopy type!) is $K(\mathbf{Z}, 3)$ whose category is infinite. Indeed the category of a non contractible tame space "of finite type" is always infinite (Theorem 2.1 below). This justifies a new treatment of the tame case, which constitutes the first part of this paper; in the second we discuss an extension of the mapping theorem to non-simply connected spaces.

1. L.-S.-category of tame spaces. Let us first recall a few definitions. Denote R_k the ring $\mathbf{Z}[1/2, \ldots, 1/l]$ with $l = [(k + 3)/2]$. Let r be an integer $\geqq 3$. A map $f:X \to Y$ between $(r - 1)$-connected spaces is called an *r-tame equivalence* if and only if $\pi_{r+k}(f) \otimes R_k$ is an isomorphism for all k. An $(r - 1)$-connected space X is called *r-tame* if $\pi_{r+k}(X)$ is a R_k-module

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for all k. A space X is called *of finite tame type*, or for short a f.t.t. *space*, if $\pi_{r+k}(X) \otimes R_k$ is a finitely generated R_k-module for each k. We assume once and for all that all spaces we consider have the homotopy type of cw-complexes and have a non-degenerate base-point when needed.

Every $(r - 1)$-connected space X (of finite type) admits an "r-taming" $\alpha:X \to X_T^{(r)}$ (of finite tame type) unique up to homotopy equivalence [**3**], and a map is an r-tame equivalence if and only if its r-taming is a homotopy equivalence.

Let $A_q(X; R_k)$ denote the cokernel of the Hurewicz map

$$h_q:\pi_q(X) \otimes R_k \to H_q(X; R_k);$$

the following proposition gives a useful criterion for r-tame equivalences:

PROPOSITION 1.1 [Berlin]. *Let $f:X \to Y$ be a map between $(r - 1)$-connected* f.t.t. *spaces; then the following statements are equivalent*:

(a) *f is an r-tame equivalence.*

(b) *For all $k \geqq 0$,*

$$H_{r+k}(X; R_k) \to H_{r+k}(Y; R_k)$$

is an isomorphism and

$$A_{r+k+1}(X; R_k) \to A_{r+k+1}(Y; R_k)$$

is surjective.

If moreover X is an r-tame space, (a) *is equivalent to*

(c) *For all $k \geqq 0$ and finitely generated R_k-modules $\mathscr{W}_k$, the map*

$$H^{r+k}(Y; \mathscr{W}_k) \to H^{r+k}(X; \mathscr{W}_k)$$

is an isomorphism and the map

$$H^{r+k+1}(Y; \mathscr{T}_k) \to H^{r+k+1}(X; \mathscr{T}_k)$$

is injective for all finite P_k-torsion modules $\mathscr{T}_k$, where P_k is the set of primes p such that $1/p \in R_{k+1} \backslash R_k$.

The loop space on an r-connected r-tame f.t.t. space has the homotopy type of a weak product of Eilenberg-MacLane spaces (see the Appendix), thus Theorem 0.1 above implies the following

PROPOSITION 1.2. *Let X and Y be r-connected spaces and let $f:X \to Y$ be a map such that $\pi_*(f)$ is split injective. If Y is r-tame of finite tame type, then* $\mathrm{cat}(X) \leqq \mathrm{cat}(Y)$.

We use this result to show that ordinary L.-S.-category is irrelevant for tame spaces of finite tame type:

THEOREM 1.3. *Every non-contractible r-connected r-tame* f.t.t. *space type has infinite category.*

Proof. Let X be such a space; since it is both non-contractible and r-tame of finite tame type, we can choose a prime p and an integer $n < 2p - 3$ such that:

(1) for all $m > n$, $\pi_{r+m}(X)$ is p-divisible

(2) $\pi_{r+n}(X)$ contains a summand isomorphic to R_n or $\mathbf{Z}/p^s\mathbf{Z}$ for some s.

Let us denote $X\langle t\rangle$ the t-connected Postnikov fibre of X as usual. Since $\pi_*(X\langle r + n\rangle)$ is p-divisible,

$$H^+(X\langle r + m\rangle; \mathbf{Z}/p\mathbf{Z}) = 0,$$

and therefore

$$H^*(X\langle r + n - 1\rangle; \mathbf{Z}/p\mathbf{Z}) \cong H^*(K(\pi_{r+n}(X), r + n); \mathbf{Z}/p\mathbf{Z})$$

by the Serre spectral sequence, and the latter has infinite cup-length. Thus $\mathrm{cat}(X\langle r + n - 1\rangle) = \infty$ and by 1.1, $\mathrm{cat}(X) = \infty$.

We now introduce a new definition for the "tame" category for which we can prove a mapping theorem.

Let X be a space and Y be a subspace of X. Let

$$T^{n+1}(X, Y) = \{ (x_0, x_1, \ldots , x_n) \in X^{n+1}; \exists i \in \{0, \ldots , n\}, x_i \in Y\}.$$

Then $T^n(X) = T^n(X, *)$ is the usual fat wedge; let us recall that $\mathrm{cat}(X) < n$ if and only if the n-fold diagonal $X \to X^n$ factors through $T^n(X)$ up to homotopy. Let us also observe that while the r-taming commutes with products, it does not commute with wedges or fat wedges.

Definition 1.4. The *r-tame category* $\mathrm{Tcat}^{(r)}(X)$ of an $(r - 1)$-connected space X is the least integer n (or ∞ if there is no such integer) such that the r-taming of the diagonal map

$$X_T^{(r)} \to (X^{n+1})_T^{(r)} = (X_T^{(r)})^{n+1}$$

factors through the r-taming of the inclusion of the fat wedge

$$\gamma:(T^{n+1}X)_T^{(r)} \to (X^{n+1})_T^{(r)}.$$

One of course expects that r-tame category is invariant under r-tame equivalences; we first show that r-tame equivalences behave well with respect to cofibrations: actually, this essentially amounts to saying that the category of r-connected f.t.t. spaces, together with the usual cofibrations as "cofibrations" and the r-tame equivalences as "weak equivalences", make up a cofibration category in the sense of H. J. Baues (cf. [Berlin], Ch. I, or Baues's forthcoming book); since this is not the cofibration category explicitly considered in [**3**] or [Berlin], but the subcategory of fibrant objects (namely the r-tame spaces), we include a proof of this lemma.

LEMMA 1.5. *Let*

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Z & \xrightarrow{h'} & S \end{array}$$

be a pushout square, in which X, Y, Z are f.t.t. *spaces, h is a cofibration and f is an r-tame equivalence. Then q is an r-tame equivalence.*

Proof. If Y is r-tame the lemma follows from Proposition 1.1 (c) and diagram chasing in the cohomology exact sequences of the cofibrations h and h'. If Y is not r-tame, we consider the pushout of g and the r-taming α of Y:

$$\begin{array}{ccccc} X & \xrightarrow{h} & Y & \xrightarrow{\alpha} & Y_T^{(r)} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle s} \\ Z & \xrightarrow{h'} & S & \xrightarrow{\beta} & E \end{array}$$

Since $Y_T^{(r)}$ is r-tame and the composite of the two squares is a pushout, s is an r-tame equivalence; Proposition 1.1 (b) and diagram chasing in the homology exact sequences of α and β show that β is an r-tame equivalence, and so is g.

PROPOSITION 1.6. *If $f:X \to Y$ is an r-tame equivalence between $(r - 1)$-connected* f.t.t. *spaces, then so is the induced map $T^n(f)$: $T^n(X) \to T^n(Y)$ between the fat wedges.*

Proof. This should be a standard result in any cofibration category. The proposition is trivial for $n = 1$, and for $n \geqq 1$ one has

$$T^{n+1}(X) = (T^n(X) \times X) \cup (X^n \times *);$$

we may assume that f is a closed cofibration, and the induction step follows from Lemma 1.5 applied to the following pushout diagrams:

$$\begin{array}{ccc} T^n(X) \times X & \longrightarrow & T^n(Y) \times Y \\ \big\downarrow & & \big\downarrow \\ T^{n+1}(X) = T^n(X) \times X \cup X^n \times * & \longrightarrow & T^n(Y) \times Y \cup X^n \times * \end{array}$$

and

$$\begin{array}{ccc} X^n \times * & \longrightarrow & Y^n \times * \\ \big\downarrow & & \big\downarrow \\ T^n(Y) \times Y \cup X^n \times * & \longrightarrow & T^n(Y) \times Y \cup Y^n \times * = T^{n+1}(Y). \end{array}$$

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COROLLARY 1.7. *Within* f.t.t. *spaces, r-tame category is invariant under r-tame equivalences and one has*:

$$\mathrm{cat}(X_0) \leqq \mathrm{Tcat}^{(r-1)}(X) \leqq \mathrm{Tcat}^{(r)}(X) \leqq \mathrm{cat}(X)$$

where X_0 is the rationalization of X.

We can now state and prove a mapping theorem for r-tame category:

THEOREM 1.8. *If $f:X \to Y$ is a continuous map between r-connected* f.t.t. *spaces* ($r \geqq 3$) *such that $\pi_{r+k}(f) \otimes R_k$ is split injective for all $k \geqq 0$, then*

$$\mathrm{Tcat}^{(r)}(X) \leqq \mathrm{Tcat}^{(r)}(Y).$$

Proof. We may assume that f is a fibration and that the spaces X and Y are r-tame. Let F be the fibre of f. Let us consider the following diagram, in which the squares are pullbacks:

$$\begin{array}{ccccc}
F^{n+1} & = & F^{n+1} & = & F^{n+1} \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle i^{n+1}} \\
T^{n+1}(X, F) & \xrightarrow{\alpha'} & E & \longrightarrow & X^{n+1} \\
\downarrow & & \downarrow & & \downarrow \\
T^{n+1}(Y) & \xrightarrow{\alpha} & (T^{n+1}(Y))_T^{(r)} & \longrightarrow & Y^{n+1}
\end{array}$$

We first claim that α' is an r-tame equivalence. It is well-known that the inclusion map

$$j:T^{n+1}(Y) \to Y^{n+1}$$

is (split) surjective on homotopy groups, because Ωj admits a section. Since the exact homotopy sequence splits, the fibre of

$$(j)_T^{(r)}: (T^{n+1}(Y))_T^{(r)} \to Y^{n+1}$$

is the r-taming of the fibre of j, namely ΩX^{*n+1}, the join of $n + 1$ copies of ΩX. Thus the fibre of the inclusion of $T^{n+1}(X, F)$ into X^{n+1} (resp. of $E \to X^{n+1}$) is ΩX^{*n+1} (resp. $(\Omega X^{*n+1})_T^{(r)}$). So we have a map of fibrations:

$$\begin{array}{ccccc}
\Omega X^{*n+1} & \longrightarrow & T^{n+1}(X, F) & \longrightarrow & X^{n+1} \\
\downarrow & & \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle =} \\
(\Omega X^{*n+1})_T^{(r)} & \longrightarrow & E & \longrightarrow & X^{n+1}
\end{array}$$

in which the left and right vertical maps are r-tame equivalences. We conclude by the five-lemma applied to the homotopy exact ladder after tensoring the upper row by R_k.

Let us now assume that $\mathrm{Tcat}^{(r)}(Y) \leqq n$. Since Y is r-tame, the diagonal map $Y \to Y^{n+1}$ factors through $(T^{n+1}(Y))_T^{(r)}$, and therefore the diagonal map $X \to X^{n+1}$ factors through the pullback space E, which by the above argument has the r-tame homotopy type of $T^{n+1}(X, F)$. Thus we have got a factorization of the diagonal $X \to X^{n+1}$ through $(T^{n+1}(X, F))_T^{(r)}$. Now as in Theorem 1.2, f is Ω-split and so $i:F \to X$ is null-homotopic. Since we have assumed non-degenerate base-points, i is a cofibration by Strøm's lemma ([**10**], [**1**] (0.17) B), and there exists $\varphi: X \to X$ which is homotopic to the identity and such that $\varphi(F) = *$. Then

$$\Phi = \varphi^{n+1}: X^{n+1} \to X^{n+1}$$

is homotopic to the identity and satisfies

$$\Phi(T^{n+1}(X, F)) \subset T^{n+1}(X).$$

Applying the r-taming again yields a factorization of the diagonal $X \to X^{n+1}$ through $(T^{n+1}(X))_T^{(r)}$ and completes the proof.

We conclude this section with a question:

Question: Is the r-tame category of a space X always equal to the actual category of some space r-tame equivalent to X?

2. Non simply-connected spaces. We begin with a rather immediate generalization of Theorem 0.1 to non simply-connected spaces:

THEOREM 2.1. *Let $f:X \to Y$ be a continuous map between connected pointed spaces having the homotopy type of* CW-*complexes, such that*:
(1) *$\pi_1(f)$ is injective*
(2) *the universal cover $\tilde{f}:\tilde{X} \to \tilde{Y}$ of f is Ω-split*
then $\mathrm{cat}(X) \leqq \mathrm{cat}(Y)$.

Proof. If $\pi_1(f)$ is an isomorphism, one has the following commutative diagram:

$$\begin{array}{ccccccc}
\Omega\tilde{Y} & \xrightarrow{\tilde{\delta}} & F & \longrightarrow & \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \\
\downarrow & & \downarrow{=} & & \downarrow & & \downarrow \\
\Omega Y & \xrightarrow{\delta} & F & \xrightarrow{i} & X & \xrightarrow{f} & Y
\end{array}$$

Since $\tilde{f}$ is Ω-split, $\tilde{\delta}$ admits a section up to homotopy and so does δ: therefore i is null-homotopic and $\mathrm{cat}(X) \leqq \mathrm{cat}(Y)$ by Lemma 2 of [**6**]. If $\pi_1(f)$ is injective, let $p: Y' \to Y$ be the covering space which corresponds to the subgroup $\pi_1(f)(\pi_1(X))$ of $\pi_1(Y)$. The map f factors as $f = p \circ g$ where $g:X \to Y'$ induces an isomorphism on π_1; since p is a covering space one has $\mathrm{cat}(Y') \leqq \mathrm{cat}(Y)$, and since $\pi_1(g)$ is an isomorphism and $\tilde{f} = \tilde{g}$, one has $\mathrm{cat}(X) \leqq \mathrm{cat}(Y')$.

Following Henn [7] we shall say that a space is almost rational if its universal cover is a rational space; from Theorem 2 we readily deduce:

COROLLARY 2.2. *Let X and Y be almost rational spaces and $f:X \to Y$ be a map such that $\pi_*(f)$ is injective; then* $\mathrm{cat}(X) \leqq \mathrm{cat}(Y)$.

Remark 2.3. As in the 1-connected case, an almost rational associative H-space of finite type G has the homotopy type of a (weak) product of Eilenberg-MacLane spaces: indeed the Hurewicz map $\pi_*(G) \to H_*(G; \mathbf{Q})$ is injective in degrees $\geqq 2$ by the Milnor-Moore theorem. Therefore $H^*(G; \mathbf{Q}) \to H^*(\widetilde{G}; \mathbf{Q})$ is surjective, and since $\widetilde{G}$ is a product of Eilenberg-MacLane spaces, $p:\widetilde{G} \to G$ admits a homotopy retraction σ and

$$(\sigma, k_1):G \to \widetilde{G} \times K(\pi_1(G), 1)$$

is a homotopy equivalence. This result is dual to Henn's, who showed that an almost rational co-H-space has the homotopy type of a wedge of circles and rational spheres of dimension $\geqq 2$.

We now observe that the finiteness of the category of an almost rational space imposes a strong restriction on its fundamental group.

PROPOSITION 2.4. *The fundamental group of an almost rational space of finite category is torsion-free (i.e. contains no element $\neq 1$ of finite order).*

Proof. Let X be an almost rational space of finite category and let $\widetilde{X}$ be its universal cover, which is a rational space. If $\pi_1(X)$ is not torsion-free, it contains a cyclic group C_p of prime order p; let Y be the covering space of X such that $\pi_1(Y) = C_p$. The space Y is almost rational with universal cover $\widetilde{X}$. Since $H^+(\widetilde{X}; \mathbf{Z}/p\mathbf{Z}) = 0$, the spectral sequence of the fibration

$$\widetilde{X} \to Y \to K(C_p, 1)$$

shows that

$$H^*(Y, \mathbf{Z}/p\mathbf{Z}) \cong H^*(C_p; \mathbf{Z}/p\mathbf{Z}) \supset \mathbf{Z}/p\mathbf{Z}[\beta u_1].$$

Therefore $\mathrm{cat}(Y)$ is infinite, and so is $\mathrm{cat}(X)$ since Y is a covering space of X.

In the particular case of $K(\pi, 1)$'s, this result is a consequence of the fact that

$$\mathrm{cat}(K(\pi, 1)) = \mathrm{coh.dim}(\pi) \quad [4].$$

As in the 1-connected case again, we can use the mapping theorem to obtain some results on the Gottlieb groups of an almost rational space of finite category. Recall that the Gottlieb subgroup $G_p(X)$ of $\pi_p(X)$ is the set of classes of maps $g:S^p \to X$ such that $g \vee X:S^p \vee X \to X$ extends to $S^p \times X$. In particular a Gottlieb element is central for the Whitehead product.

PROPOSITION 2.5. *Let X be an almost rational space of category $\leqq m$. Then the Gottlieb subgroup $G_1(X)$ of $\pi_1(X)$ is a torsion free abelian group of finite rank r, and one has*

$$r + \sum_{k>0} \dim_{\mathbf{Q}} G_{2k+1}(X) \leqq m.$$

Moreover, one has $G_{2k}(X) = 0$ for all $k \geqq 1$.

Proof (cf [**5**, **8**]). Clearly $G_1(X)$ is abelian. Let $g_1, g_2, \ldots, g_s:S^1 \to X$ represent **Z**-independent elements in $G_1(X)$, and $g_{s+1}, \ldots, g_t$, with $g_j:S^{n_j} \to X$, represent **Q**-linearly independent elements in $G_{>1}(X)$. They define a map

$$g:S^1 \vee \ldots \vee S^1 \vee (S^{n_s} + 1)_0 \vee \ldots \vee (S^{n_t})_0 \to X$$

(where $S^{n_j})_0$ are rational odd spheres) which extends to the product of the s circles and the $t - s$ rational spheres, which is an almost rational space; this extension is injective on homotopy groups by construction, and since the category of this product is t, one must have $t \leqq m$ by Corollary 2.2. The second statement is proven as in [**5**].

Remark 2.6. For any connected cw-complex X of category m or less, since $G_*(X) \subset G_*(\tilde{X})$ and $\mathrm{cat}(\tilde{X}_0) \leqq \mathrm{cat}(\tilde{X}) \leqq \mathrm{cat}(X)$, one has

$$\sum_{k\geqq 1} \dim_{\mathbf{Q}} G_{2k+1}(X) \otimes \mathbf{Q} \leqq m \quad \text{and} \quad \forall k \geqq 1, \quad G_{2k}(X) \otimes \mathbf{Q} = 0.$$

However, in contrast with the simply connected case, the first inequality in Proposition 2.5 does not obviously extend to non almost rational spaces because category does not behave well under "almost-rationalization": the latter is obtained by fibrewise localization of the universal cover fibration

$$\tilde{X} \to X \to K(\pi_1(X), 1)$$

into

$$\tilde{X}_0 \to X_0 \to K(\pi_1(X), 1).$$

Unfortunately Proposition 2.4 above shows that one cannot have $\mathrm{cat}(X_0) \leqq \mathrm{cat}(X)$ in general: indeed for instance $\mathrm{cat}(\mathbf{RP}(2)) = 2$ while the argument in the proof of Proposition 1 shows that $\mathrm{cat}(\mathbf{RP}(2)_0) = \infty$.

Thus we do not know whether the full inequality

$$\sum_{k\geqq 0} \dim_{\mathbf{Q}} G_{2k+1}(X) \otimes \mathbf{Q} \leqq \mathrm{cat}(X)$$

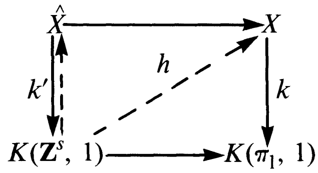
holds; we only can show:

PROPOSITION 2.7. *For any connected* cw-*complex of finite category m, one has*

$$\text{rank}(G_1(X)) \leqq m$$

(*and hence* $\sum_{k \geqq 0} \dim_{\mathbf{Q}} G_{2k+1}(X) \otimes \mathbf{Q} \leqq 2m$).

Proof. Let $g_1, g_2, \ldots, g_s : S^1 \to X$ represent $\mathbf{Z}$-independent elements in $G_1(X)$ as above, and let $h:(S^1)^n \to X$ be the extension of the map from the wedge of circles defined by the g_i's. Let $\hat{X}$ be the covering space of X corresponding to $\mathbf{Z}^s \subset G_1(X) \subset \pi_1(X)$. In the fibre square:



the projection k' has a section. Therefore $s = \text{cat}(K(\mathbf{Z}^s, 1)) \leqq \text{cat}(\hat{X}) \leqq \text{cat}(X) = m$.

Appendix. Let $T_*(r)$ be the ring system defined by $T_{r+k}(r) = R_k$ for all $k \geqq 0$. In [**3**], W. Dwyer has extended Quillen's theory to obtain an equivalence of homotopy categories between the homotopy category $\mathscr{S}(r, T_*(r))$ of r-reduced r-tame spaces and the homotopy category $\text{DGL}(r-1, T_*(r))$ of "tame" $(r-1)$-reduced differential graded Lie algebras: let us denote

$$\lambda : \mathscr{S}(r, T_*(r)) \to \text{DGL}(r-1, T_*(r))$$

this equivalence and μ its quasi-inverse.

PROPOSITION A.1. *Let X be an r-connected r-tame space of finite type; then ΩX has the homotopy type of a weak product of Eilenberg-MacLane spaces.*

Proof. Let $(L, d) = \lambda(X)$. L is a chain Lie $\mathbf{Z}$-algebra whose homology is of finite type over the ring system $T_*(r)$. Let us consider the short exact sequence of chain Lie algebras:

$$0 \to (s^{-1}L, D) \to (s^{-1}L \oplus L, D) \to (L, d) \to 0$$

with $(s^{-1}L)_n = L_{n+1}$

$$\forall l \in L, \quad Dl = dl + s^{-1}l \quad [s^{-1}l, l'] = 2^{-1}s^{-1}[l, l']$$

$$Ds^{-1}l = -s^{-1}dl \quad [s^{-1}l, s^{-1}l'] = 0.$$

One easily checks that $(s^{-1}L \oplus L, D)$ is acyclic. The above exact sequence is a fibration in $\text{DGL}(r-1, T_*(r))$ whose image under μ is a fibration in $\mathscr{S}(r, T_*(r))$ the total space of which is contractible. Therefore $(s^{-1}L, D)$ is equivalent in $\text{Ho DGL}(r-1, T_*(r))$ to $\lambda(\Omega X)$.

Since $s^{-1}L$ is an abelian Lie algebra, i.e., a $\mathbf{Z}$-complex, whose homology is of finite type over the principal ring system $s^{-1}T_*(r)$, there exist quasi-isomorphisms:

$$(s^{-1}L, D) \overset{\simeq}{\leftarrow} C_* \overset{\simeq}{\rightarrow} H_*(s^{-1}L, D) \cong \bigoplus_{i \geqq r} A_i$$

where C_* is $s^{-1}T_*(r)$-free and A_i is the abelian Lie algebra which consists of the group A_i concentrated in degree i, endowed with the zero differential. Clearly enough

$$\mu(A_i) \simeq K(A_i, i) \quad \text{and} \quad \mu(\bigoplus A_i) \simeq \varinjlim_n \prod_{i=r}^{n} K(A_i, i) \overset{\simeq}{\rightarrow} \Omega X.$$

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