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Continuity of Convolution and SIN Groups

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Abstract. Let the measure algebra of a topological group G be equipped with the topology of uniform convergence on bounded right uniformly equicontinuous sets of functions. Convolution is separately continuous on the measure algebra, and it is jointly continuous if and only if G has the SIN property. On the larger space $\mathsf{LUC}(G)^*$, which includes the measure algebra, convolution is also jointly continuous if and only if the group has the SIN property, but not separately continuous for many non-SIN groups.

1 Introduction

Throughout the paper we assume that topological groups are Hausdorff, linear spaces are over the field $\mathbb{R}$ of real numbers, and functions are real-valued. Our results hold also when scalars are the complex numbers, with essentially the same proofs.

When G is a topological group, the set of all continuous right-invariant pseudometrics on G induces the topology of G and its right uniformity [10, §3.2] [13, 7.4]. In what follows, we denote by G not only G with its topology but also G with its right uniformity. Since we do not consider other uniform structures on G, this convention will not lead to any confusion.

A pseudometric on G is *bi-invariant* if and only if it is both left and right-invariant. A topological group G is a *SIN group*, or has the *SIN property*, if and only if its topology (equivalently, its right uniformity) is induced by the set of all continuous bi-invariant pseudometrics [13, 7.12].

The space $\mathsf{LUC}(G) = \mathsf{U_b}(G)$ of bounded uniformly continuous functions on G has a prominent role in abstract harmonic analysis. It is a Banach space with the sup norm. Its dual $\mathsf{LUC}(G)^*$ is a Banach algebra in which the multiplication is the *convolution operation* $\star$, defined as follows. When φ is an expression with several parameters, $\smallsetminus_x \varphi$ denotes φ as a function of x. Define

$$\mathfrak{n}\bullet f(x) := \mathfrak{n}(\smallsetminus_y f(xy)) \qquad \text{for } \mathfrak{n} \in \mathsf{LUC}(G)^*, f \in \mathsf{LUC}(G), x \in G.$$
$$\mathfrak{m} \star \mathfrak{n}(f) := \mathfrak{m}(\mathfrak{n}\bullet f) \qquad \text{for } \mathfrak{m}, \mathfrak{n} \in \mathsf{LUC}(G)^*, f \in \mathsf{LUC}(G).$$

Here $(\mathfrak{n}, f) \mapsto \mathfrak{n}\bullet f$ is the canonical left action of $\mathsf{LUC}(G)^*$ on $\mathsf{LUC}(G)$.

We identify every finite Radon measure μ on G with the functional $\mathfrak{m} \in \mathsf{LUC}(G)^*$ for which $\mathfrak{m}(f) = \int f \, d\mu$, $f \in \mathsf{LUC}(G)$. That way the space $\mathsf{M_t}(G)$ of finite Radon (a.k.a. tight) measures on G is identified with a subspace of $\mathsf{LUC}(G)^*$. With convolution, this is the *measure algebra* of G, often denoted simply $M(G)$.

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Along with the norm topology, another topology on $\mathsf{LUC}(G)^*$ and $\mathsf{M}_t(G)$ commonly considered is the weak* topology $w(\mathsf{LUC}(G)^*, \mathsf{LUC}(G))$. Questions about separate weak* continuity of convolution on $\mathsf{LUC}(G)^*$ lead to the problem of characterizing the weak* topological centre of $\mathsf{LUC}(G)^*$ and of the LUC compactification of G (see [4, 6, 7] and [10, Chapter 9]). Joint weak* continuity of convolution on $\mathsf{LUC}(G)^*$ was studied by Salmi [14], who showed that convolution need not be jointly weak* continuous even on bounded subsets of $\mathsf{M}_t(G)$.

Here we consider the UEB topology on the space $\mathsf{LUC}(G)^*$. This topology, finer than the weak* topology, arises naturally in the study of continuity properties of convolution. When restricted to the space $\mathsf{M}_t(G)$, the UEB topology and the weak* topology $w(\mathsf{M}_t(G), \mathsf{LUC}(G))$ are closely related. It follows from general results in [10, Chapter 6] that these two topologies on $\mathsf{M}_t(G)$ have the same dual $\mathsf{LUC}(G)$ and the same compact sets (hence the same convergent sequences), and they coincide on the positive cone of $\mathsf{M}_t(G)$.

The UEB topology can be defined independently of the group structure of G for a general uniform space; for the details of the general theory we refer the reader to [10]. In our current setting of the right uniformity on a topological group G, the UEB topology is defined as follows. As in [10], for a continuous right-invariant pseudometric Δ on G and $\mathfrak{m} \in \mathsf{LUC}(G)^*$ let

$$\mathsf{BLip_b}(\Delta) := \left\{ f : G \to [-1,1] \;\middle|\; |f(x) - f(y)| \le \Delta(x,y) \text{ for all } x, y \in G \right\},$$
$$\|\mathfrak{m}\|_\Delta := \sup\left\{ \mathfrak{m}(f) \mid f \in \mathsf{BLip_b}(\Delta) \right\}.$$

The UEB topology on $\mathsf{LUC}(G)^*$ is the locally convex topology defined by the seminorms $\|\cdot\|_\Delta$, where Δ runs through continuous right-invariant pseudometrics on G.

In [9] the UEB topology is defined as the topology of uniform convergence on equi-LUC subsets of $\mathsf{LUC}(G)$. That definition is equivalent to the one given here, since by [10, Lemma 3.3] for every equi-LUC set $\mathcal{F} \subseteq \mathsf{LUC}(G)$, there are $r \in \mathbb{R}$ and a continuous right-invariant pseudometric Δ on G such that $\mathcal{F} \subseteq r\mathsf{BLip_b}(\Delta)$.

When the group G is locally compact and $\mathsf{M}_t(G)$ is identified with the algebra of right multipliers of $\mathsf{L}_1(G)$, the UEB topology on $\mathsf{M}_t(G)$ coincides with the right multiplier topology [9, Th. 3.3]. If G is discrete, then $\mathsf{LUC}(G) = \ell_\infty(G)$ and the UEB topology on $\mathsf{LUC}(G)^*$ is simply its norm topology. If G is compact then $\mathsf{LUC}(G)$ is the space of continuous functions on G and the UEB topology is the topology of uniform convergence on norm-compact subsets of $\mathsf{LUC}(G)$.

When the group G is metrizable by a right-invariant metric Δ, the seminorm $\|\cdot\|_\Delta$ on $\mathsf{LUC}(G)^*$ is a particular case of the Kantorovich–Rubinshteĭn norm, which has many uses in topological measure theory and in the theory of optimal transport [2, 8.3] [15, 6.2]. In this case the topology of $\|\cdot\|_\Delta$ coincides with the UEB topology on bounded subsets of $\mathsf{LUC}(G)^*$ [10, §5.4] but typically not on the whole space $\mathsf{LUC}(G)^*$. As we show in Section 3, when considered on the whole space $\mathsf{LUC}(G)^*$ or even $\mathsf{M}_t(G)$, convolution behaves better in the UEB topology than in the $\|\cdot\|_\Delta$ topology.

Our results in this paper complement those in [9]. By [9, Corollary 4.6 and Theorem 4.8], convolution is jointly UEB continuous on bounded subsets $\mathsf{LUC}(G)^*$ when G is a SIN group, and jointly UEB continuous on the whole space $\mathsf{LUC}(G)^*$ when G is a locally compact SIN group. Our main result (Theorem 3.2) states that convolution is jointly UEB continuous on $\mathsf{LUC}(G)^*$ if and only if it is jointly UEB continuous on $\mathsf{M_t}(G)$ if and only if G is a SIN group. In Section 4 we prove that convolution is separately UEB continuous on $\mathsf{M_t}(G)$ for every topological group G, but not separately continuous on $\mathsf{LUC}(G)^*$ for many non-SIN groups.

For locally compact groups, Lau and Pym [7] established the connection between the SIN property and the weak* continuity of multiplication in the LUC compactification. Corollary 4.5 extends one of their results to a larger class of topological groups.

2 Preliminaries

In this section we establish several properties of SIN groups that are needed in the proof of the main theorem in Section 3.

We specialize the notation of [10], where it is used for functions and measures on general uniform spaces, to the case of a topological group G. For every $x \in G$ we denote by $\partial(x)$ the *point mass* at x, the functional in $\mathsf{LUC}(G)^*$ defined by $\partial(x)(f) = f(x)$ for $f \in \mathsf{LUC}(G)$. $\mathsf{Mol}(G) \subseteq \mathsf{LUC}(G)^*$ is the space of *molecular measures*, that is, finite linear combinations of point masses. Obviously, $\mathsf{Mol}(G) \subseteq \mathsf{M_t}(G)$. For the molecular measure of the special form $\mathfrak{m} = \partial(x) - \partial(y)$, $x, y \in G$, and for any continuous right-invariant pseudometric Δ on G we have $\|\mathfrak{m}\|_\Delta = \min(2, \Delta(x,y))$, by Lemma 5.12 in [10].

The UEB closure of $\mathsf{Mol}(G)$ in $\mathsf{LUC}(G)^*$ is the space $\mathsf{M_u}(G) \supseteq \mathsf{M_t}(G)$ of *uniform measures* on the uniform space G. In this paper we do not deal with the space $\mathsf{M_u}(G)$; we only point out where a result that we prove for $\mathsf{M_t}(G)$ holds more generally for $\mathsf{M_u}(G)$. The reader is referred to [10] for the theory of uniform measures.

We start with a characterization of SIN groups, which is one part of [13, 2.17].

Lemma 2.1 *A topological group G with identity element e is a SIN group if and only if for every neighbourhood U of e there exists a neighbourhood V of e such that $xVx^{-1} \subseteq U$ for all $x \in G$.*

Lemma 2.2 *Let G be a SIN group and Δ a bounded continuous right-invariant pseudometric on G. Then there is a continuous bi-invariant pseudometric Θ on G such that $\Theta \geq \Delta$.*

Proof The proof mimics that of [10, Lemma 3.3]. It is enough to consider the case $\Delta \leq 1$. As G is a SIN group, there are continuous bi-invariant pseudometrics Θ_j for $j = 0, 1, \dots,$ such that

$$\forall x, y \in S \Big[\Theta_j(x,y) < 1 \Longrightarrow \Delta(x,y) < \frac{1}{2^{j+1}} \Big].$$

Define Θ by

$$\Theta(x,y) := \sum_{j=0}^{\infty} \frac{1}{2^j} \min\big(\Theta_j(x,y), 1\big).$$

If $x, y \in X$ and j are such that $\Theta(x,y) < 1/2^j$, then $\Theta_j(x,y) < 1$, whence $\Delta(x,y) < 1/2^{j+1}$. It follows that $\Theta \geq \Delta$. ∎

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Corollary 2.3 *Let G be a SIN group. Then the UEB topology on $\mathsf{LUC}(G)^*$ is defined by the seminorms $\|\cdot\|_\Delta$ where Δ runs through continuous bi-invariant pseudometrics on G.*

If Δ is a continuous or left or right-invariant pseudometric on G, then so is the pseudometric $\sqrt{\Delta}$ defined by $\sqrt{\Delta}(x,y) := \sqrt{\Delta(x,y)}$ for $x,y \in G$.

In the sequel we deal with functions of the form $f/\sqrt{\|f\|}$, where $f \in \mathsf{LUC}(G)$. To simplify the notation, we adopt the convention that $f/\sqrt{\|f\|} = f$ when f is identically 0.

Lemma 2.4 *Let Δ be a pseudometric on a set G and let $f \in \mathsf{BLip_b}(\Delta)$. Then*

$$f/\sqrt{\|f\|} \in \mathsf{BLip_b}(2\sqrt{\Delta}).$$

Proof Take any $x,y \in G$, and consider two cases:

(a) If $\|f\| \le \Delta(x,y)$, then $|f(x)/\sqrt{\|f\|}|, |f(y)/\sqrt{\|f\|}| \le \sqrt{\Delta(x,y)}$, hence

$$\left|\frac{f(x)}{\sqrt{\|f\|}} - \frac{f(y)}{\sqrt{\|f\|}}\right| \le 2\sqrt{\Delta(x,y)}.$$

(b) If $\|f\| > \Delta(x,y) > 0$, then

$$\left|\frac{f(x)}{\sqrt{\|f\|}} - \frac{f(y)}{\sqrt{\|f\|}}\right| \le \frac{|f(x)-f(y)|}{\sqrt{\Delta(x,y)}} \le \sqrt{\Delta(x,y)}. \qquad \blacksquare$$

The following lemma is a key ingredient in the proof of Theorem 3.2.

Lemma 2.5 *Let G be a topological group, $\mathfrak{m},\mathfrak{n} \in \mathsf{LUC}(G)^*$, and let Δ be a continuous bi-invariant pseudometric on G. Then*

$$\|\mathfrak{m} \star \mathfrak{n}\|_\Delta \le \sqrt{2}\|\mathfrak{m}\|_{\sqrt{\Delta}}\|\mathfrak{n}\|_{2\sqrt{\Delta}}.$$

Proof Take any $f \in \mathsf{BLip_b}(\Delta)$. As Δ is left-invariant, we have $\searrow_z f(xz) \in \mathsf{BLip_b}(\Delta)$ for every $x \in G$, and $\|\mathfrak{n}\bullet f\| \le \|\mathfrak{n}\|_\Delta$. Now $\mathsf{BLip_b}(\Delta) \subseteq \mathsf{BLip_b}(\sqrt{2\Delta}) \subseteq \mathsf{BLip_b}(2\sqrt{\Delta})$, because $\sqrt{2t} \ge t$ for $0 \le t \le 2$, and thus $\|\cdot\|_\Delta \le \|\cdot\|_{\sqrt{2\Delta}} \le \|\cdot\|_{2\sqrt{\Delta}}$. It follows that

$$\|\mathfrak{n}\bullet f\| \le \|\mathfrak{n}\|_\Delta \le \|\mathfrak{n}\|_{2\sqrt{\Delta}}. \tag{2.1}$$

For $x,y \in G$ we have $g := \frac{1}{2} \searrow_z (f(xz) - f(yz)) \in \mathsf{BLip_b}(\Delta)$, hence

$$g/\sqrt{\|g\|} \in \mathsf{BLip_b}(2\sqrt{\Delta})$$

by Lemma 2.4. Moreover, $2\|g\| \le \Delta(x,y)$, because Δ is right-invariant, so that

$$\begin{aligned} \big|\mathfrak{n}\bullet f(x) - \mathfrak{n}\bullet f(y)\big| &= 2|\mathfrak{n}(g)| = 2\sqrt{\|g\|}\left|\mathfrak{n}\Big(\frac{g}{\sqrt{\|g\|}}\Big)\right| \\ &\le \sqrt{2}\sqrt{\Delta(x,y)}\|\mathfrak{n}\|_{2\sqrt{\Delta}}. \end{aligned} \tag{2.2}$$

Putting (2.1) and (2.2) together, we get $\mathfrak{n}\bullet f \in \sqrt{2}\|\mathfrak{n}\|_{2\sqrt{\Delta}}\mathsf{BLip_b}(\sqrt{\Delta})$. Hence,

$$|\mathfrak{m}\star\mathfrak{n}(f)| = |\mathfrak{m}(\mathfrak{n}\bullet f)| \le \sqrt{2}\|\mathfrak{m}\|_{\sqrt{\Delta}}\|\mathfrak{n}\|_{2\sqrt{\Delta}}. \qquad \blacksquare$$

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3 Joint UEB Continuity

For any topological group G, the operation $\star$ is jointly UEB continuous on bounded subsets of $\mathsf{M}_\mathsf{t}(G)$ [9, 4.5], in fact, even on bounded subsets of $\mathsf{M}_\mathsf{u}(G)$ [10, Cor. 9.36]. However, as we shall see in this section, convolution need not be jointly UEB continuous on the whole space $\mathsf{M}_\mathsf{t}(G)$.

The UEB topology is defined by certain seminorms $\|\cdot\|_\Delta$. As a warm-up exercise, consider the continuity with respect to a single such seminorm. Let G be a metrizable topological group whose topology is defined by a right-invariant metric Δ. As we pointed out in the introduction, the topology of the norm $\|\cdot\|_\Delta$ coincides with the UEB topology on bounded subsets of $\mathsf{LUC}(G)^*$. Hence, $\star$ is jointly $\|\cdot\|_\Delta$ continuous on bounded subsets of $\mathsf{M}_\mathsf{t}(G)$. However, $\star$ is not jointly $\|\cdot\|_\Delta$ continuous on the whole space $\mathsf{M}_\mathsf{t}(G)$ or even $\mathsf{Mol}(G)$ for $G = \mathbb{R}$.

Example 3.1 Let G be the additive group $\mathbb{R}$ with the usual metric Δ defined by $\Delta(x,y) = |x-y|$. For $j = 1,2,\dots$, let $\mathfrak{m}_j := \mathfrak{n}_j := j(\partial(1/j^2) - \partial(0))$ and $f_j(x) := \min(1, |x - (1/j^2)|)$ for $x \in \mathbb{R}$. Then $f_j \in \mathsf{BLip_b}(\Delta)$ and

$$\mathfrak{m}_j \star \mathfrak{n}_j = j^2\Big(\partial(2/j^2) - 2\partial(1/j^2) + \partial(0)\Big),$$
$$\|\mathfrak{m}_j \star \mathfrak{n}_j\|_\Delta \geq \mathfrak{m}_j \star \mathfrak{n}_j(f_j) = 2$$

but $\lim_j \|\mathfrak{m}_j\|_\Delta = \lim_j \|\mathfrak{n}_j\|_\Delta = 0$.

Note that although the sequence $\{\mathfrak{m}_j\}_j$ converges in the norm $\|\cdot\|_\Delta$, it does not converge in the UEB topology; in fact, $\|\mathfrak{m}_j\|_{\sqrt{\Delta}} = 1$ for all j.

Next we shall see that the situation changes when we move from the topology defined by a single seminorm $\|\cdot\|_\Delta$ to the topology defined by all such seminorms, *i.e.,* the UEB topology.

Theorem 3.2 *The following properties of a topological group G are equivalent:*

(i) *Convolution is jointly UEB continuous on* $\mathsf{LUC}(G)^*$.
(ii) *Convolution is jointly UEB continuous on* $\mathsf{M}_\mathsf{t}(G)$.
(iii) *Convolution is jointly UEB continuous on* $\mathsf{Mol}(G)$.
(iv) *G is a SIN group.*

Proof Obviously, (i)⇒(ii)⇒(iii).

To prove (iii)⇒(iv), assume that convolution is jointly UEB continuous on $\mathsf{Mol}(G)$. Take any neighbourhood U of the identity element e. There is a continuous right-invariant pseudometric Θ such that $\{z \in G \mid \Theta(z,e) < 1\} \subseteq U$. By the UEB continuity there are a continuous right-invariant pseudometric Δ and $\varepsilon > 0$ such that if $\mathfrak{m}, \mathfrak{n} \in \mathsf{Mol}(G)$, $\|\mathfrak{m}\|_\Delta, \|\mathfrak{n}\|_\Delta \leq \varepsilon$, then $\|\mathfrak{m} \star \mathfrak{n}\|_\Theta < 1$. To conclude that G is a SIN group, in view of Lemma 2.1, it is enough to show that $xVx^{-1} \subseteq U$ for all $x \in G$, where $V := \{v \in G \mid \Delta(v,e) < \varepsilon^2\}$. To that end, take any $x \in G$ and $v \in V$ and define

$$\mathfrak{m} := \varepsilon\partial(x), \quad \mathfrak{n} := (\partial(v) - \partial(e))/\varepsilon$$

Then $\|\mathfrak{m}\|_\Delta = \varepsilon$ and $\|\mathfrak{n}\|_\Delta = \min(2, \Delta(v,e))/\varepsilon < \varepsilon$, hence

$$\min(2, \Theta(xv, x)) = \|\partial(xv) - \partial(x)\|_\Theta = \|\mathfrak{m} \star \mathfrak{n}\|_\Theta < 1,$$

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and therefore $\Theta(xvx^{-1}, e) = \Theta(xv, x) < 1$ and $xvx^{-1} \in U$. That completes the proof of (iii)⇒(iv).

To prove (iv)⇒(i), assume that G is a SIN group. Take any continuous bi-invariant pseudometric Δ on G. By Lemma 2.5, if $\mathfrak{m}, \mathfrak{m}_0, \mathfrak{n}, \mathfrak{n}_0 \in \mathsf{LUC}(G)^*$ are such that $\|\mathfrak{m} - \mathfrak{m}_0\|_{\sqrt{\Delta}} < \varepsilon$ and $\|\mathfrak{n} - \mathfrak{n}_0\|_{2\sqrt{\Delta}} < \varepsilon$, then

$$\begin{aligned}\|\mathfrak{m} \star \mathfrak{n} - \mathfrak{m}_0 \star \mathfrak{n}_0\|_\Delta &\le \|(\mathfrak{m} - \mathfrak{m}_0) \star \mathfrak{n}\|_\Delta + \|\mathfrak{m}_0 \star (\mathfrak{n} - \mathfrak{n}_0)\|_\Delta \\ &\le \sqrt{2}\varepsilon\|\mathfrak{n}\|_{2\sqrt{\Delta}} + \sqrt{2}\|\mathfrak{m}_0\|_{\sqrt{\Delta}}\varepsilon \\ &\le \sqrt{2}\varepsilon\big(\varepsilon + \|\mathfrak{n}_0\|_{2\sqrt{\Delta}} + \|\mathfrak{m}_0\|_{\sqrt{\Delta}}\big),\end{aligned}$$

which along with Corollary 2.3 proves that $\star$ is jointly UEB continuous at $(\mathfrak{m}_0, \mathfrak{n}_0)$. ∎

4 Separate UEB Continuity

By Theorem 3.2, convolution is jointly UEB continuous on $\mathsf{LUC}(G)^*$, and therefore also separately UEB continuous whenever G is a SIN group. On the other hand, as we explain at the end of this section, there are topological groups G for which convolution is not separately UEB continuous on $\mathsf{LUC}(G)^*$. Nevertheless, we now prove that convolution is separately UEB continuous on $\mathsf{M}_\mathsf{t}(G)$ for every topological group G. The same proof can be used to show that convolution is separately UEB continuous even on $\mathsf{M}_\mathsf{u}(G)$.

Lemma 4.1 *Let G be a topological group, $\mathfrak{m} \in \mathsf{M}_\mathsf{t}(G)$, and let Δ be a continuous right-invariant pseudometric on G. Then there exists a continuous right-invariant pseudometric $\Delta_\mathfrak{m}$ such that $\diagdown_y \mathfrak{m}(\diagdown_x f(xy)) \in \|\mathfrak{m}\|\mathsf{BLip}_\mathsf{b}(\Delta_\mathfrak{m})$ for every $f \in \mathsf{BLip}_\mathsf{b}(\Delta)$.*

Proof Evidently, $\|\diagdown_y \mathfrak{m}(\diagdown_x f(xy))\| \le \|\mathfrak{m}\|$ for every $f \in \mathsf{BLip}_\mathsf{b}(\Delta)$. To prove that the function $\diagdown_y \mathfrak{m}(\diagdown_x f(xy))$ is Lipschitz for a suitable $\Delta_\mathfrak{m}$, first note that if $\mathfrak{m} = \sum_j c_j \mathfrak{m}_j$, $\mathfrak{m}_j \in \mathsf{LUC}(G)^*$, is a finite linear combination such that

$$\big|\mathfrak{m}_j(\diagdown_x f(xy)) - \mathfrak{m}_j(\diagdown_x f(xz))\big| \le \Delta_j(y, z)$$

for every j and $y, z \in G$, then

$$\big|\mathfrak{m}(\diagdown_x f(xy)) - \mathfrak{m}(\diagdown_x f(xz))\big| \le \Delta'(y, z),$$

where $\Delta' = \sum_j |c_j|\Delta_j$. Thus, it is enough to prove the lemma assuming that $\mathfrak{m} \ge 0$.

We may also assume that $\Delta \le 2$, as replacing Δ by $\min(\Delta, 2)$ does not change $\mathsf{BLip}_\mathsf{b}(\Delta)$. For $\mathfrak{m} \ge 0$, $\mathfrak{m} \ne 0$, and $\Delta \le 2$, define $\Delta_\mathfrak{m}$ by

$$\Delta_\mathfrak{m}(y, z) := \mathfrak{m}\big(\diagdown_x \Delta(xy, xz)\big) / \|\mathfrak{m}\| \text{ for } y, z \in G.$$

Clearly, $\Delta_\mathfrak{m}$ is a right-invariant pseudometric. To see that it is continuous, first apply the estimate

$$\big|\Delta(xy, x) - \Delta(wy, w)\big| \le \Delta(xy, wy) + \Delta(x, w) = 2\Delta(x, w),$$

which shows that $\diagdown_x \Delta(xy, x) \in 2\mathsf{BLip}_\mathsf{b}(\Delta)$ for every $y \in G$. Since $\mathfrak{m}$ is a Radon measure, it is continuous on $2\mathsf{BLip}_\mathsf{b}(\Delta)$ in the compact-open topology. However, that

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topology on $2\mathsf{BLip_b}(\Delta)$ coincides with the topology of pointwise convergence. It follows that $\Delta_{\mathfrak{m}}(y,e)$, where e is the unit element of G, is a continuous function of y on G.

For any $f \in \mathsf{BLip_b}(\Delta)$, we have

$$\begin{aligned}\big|\mathfrak{m}(\diagdown_x f(xy)) - \mathfrak{m}(\diagdown_x f(xz))\big| &\leq \mathfrak{m}\big(\diagdown_x |f(xy) - f(xz)|\big)\\ &\leq \mathfrak{m}\big(\diagdown_x \Delta(xy,xz)\big) = \|\mathfrak{m}\|\Delta_{\mathfrak{m}}(y,z)\end{aligned}$$

for $y,z \in G$. ∎

Theorem 4.2 *For every topological group G, convolution is separately UEB continuous on $\mathsf{M_t}(G)$.*

Proof For every $\mathfrak{n} \in \mathsf{LUC}(G)^*$ the mapping $\mathfrak{m} \mapsto \mathfrak{m} \star \mathfrak{n}$ is UEB continuous; this is a special case of [10, Cor. 9.21].

For $\mathfrak{m} \in \mathsf{M_t}(G)$ and $\mathfrak{n} \in \mathsf{LUC}(G)^*$, we can reverse the order of applying $\mathfrak{m}$ and $\mathfrak{n}$ in the definition of convolution:

$$\mathfrak{m} \star \mathfrak{n}(f) = \mathfrak{n}\big(\diagdown_y \mathfrak{m}(\diagdown_x f(xy))\big) \text{ for } f \in \mathsf{LUC}(G).$$

This is a consequence of a variant of Fubini's theorem; see [10, §9.4] for a proof and discussion.

The UEB continuity of the mapping $\mathfrak{n} \mapsto \mathfrak{m} \star \mathfrak{n}$ for every $\mathfrak{m} \in \mathsf{M_t}(G)$ now follows from Lemma 4.1. ∎

In analogy with the commonly studied weak* topological centre of $\mathsf{LUC}(G)^*$, we can also consider its UEB topological centre Λ_{UEB}, the set of those $\mathfrak{m} \in \mathsf{LUC}(G)^*$ for which the mapping $\mathfrak{n} \mapsto \mathfrak{m} \star \mathfrak{n}$ is UEB continuous on $\mathsf{LUC}(G)^*$. Then $\Lambda_{UEB} = \mathsf{LUC}(G)^*$ for every SIN group G by Theorem 3.2. Example 4.7 in [9] (which is also [10, Example 9.39]) shows that $\Lambda_{UEB} \neq \mathsf{LUC}(G)^*$ when G is the group of homeomorphisms of the interval $[0,1]$ onto itself with the topology of uniform convergence. Next we will show that in fact $\Lambda_{UEB} \neq \mathsf{LUC}(G)^*$ for every topological group G that contains a non-SIN subgroup that is locally compact or metrizable.

For any topological group G denote by $\mathsf{RUC}(G)$ the space of those bounded continuous functions f on G for which the mapping $x \mapsto \diagdown_y f(yx)$ is continuous from G to the space $\ell_\infty(G)$ with the sup norm. In other words, $\mathsf{RUC}(G)$ is the space of bounded left uniformly continuous functions on G.

Note that $g \in \mathsf{LUC}(G)$ if and only if $\diagdown_x g(x^{-1}) \in \mathsf{RUC}(G)$. Thus, $\mathsf{LUC}(G) = \mathsf{RUC}(G)$ if and only if $\mathsf{LUC}(G) \subseteq \mathsf{RUC}(G)$. It is a long-standing open problem [3] whether every topological group G such that $\mathsf{LUC}(G) = \mathsf{RUC}(G)$ is a SIN group. The following partial answer was proved by Itzkowitz et al [5] and Milnes [8] for locally compact groups, and by Protasov [12] for almost metrizable (in particular locally compact or metrizable) groups.

Lemma 4.3 *Let G be a topological group that is locally compact or metrizable and such that $\mathsf{LUC}(G) \subseteq \mathsf{RUC}(G)$. Then G is a SIN group.*

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As in [10, §6.5], when each element x of a topological group G is identified with the point mass $\partial(x) \in \mathsf{LUC}(G)^*$ and $\mathsf{LUC}(G)^*$ is equipped with its weak* topology, we obtain topological embeddings $G \subseteq \widehat{G} \subseteq G^{\mathsf{LUC}} \subseteq \mathsf{LUC}(G)^*$. Here, $\widehat{G}$ is the completion of G (with its right uniformity) and $G^{\mathsf{LUC}} = \widehat{\mathfrak{p}}G$ is its uniform compactification. The embedding $G \subseteq \widehat{G}$ is not only topological but uniform as well. Both $\widehat{G}$ and G^{LUC} are subsemigroups of $\mathsf{LUC}(G)^*$ with the convolution operation.

The following theorem will be applied in two cases: When G is locally compact or completely metrizable, we let $S = G$. When G is merely metrizable, we let $S = \widehat{G}$.

Theorem 4.4 *Let G be a topological group that is locally compact or metrizable. Let S be a subsemigroup of G^{LUC} such that the following hold:*

(i) $G \subseteq S$;

(ii) *the topology of S is locally compact or completely metrizable;*

(iii) *for every $\mathfrak{m} \in G^{\mathsf{LUC}}$ the mapping $x \mapsto \mathfrak{m} \star x$ from S to G^{LUC} is continuous.*

Then G is a SIN group.

The main argument in the following proof is used in the proof of [1, 4.4.5].

Proof Take any $f \in \mathsf{LUC}(G)$. Define $\varphi: G^{\mathsf{LUC}} \times G \to \mathbb{R}$ by $\varphi(\mathfrak{m}, x) := \mathfrak{m} \star x(f)$ for $\mathfrak{m} \in G^{\mathsf{LUC}}$, $x \in G$.

From the definition of $\star$, for every $\mathfrak{n} \in \mathsf{LUC}(G)^*$ the mapping $\mathfrak{m} \mapsto \mathfrak{m} \star \mathfrak{n}$ is weak* continuous on $\mathsf{LUC}(G)^*$. That along with (iii) implies that the convolution operation is separately continuous on the product $G^{\mathsf{LUC}} \times S$, therefore jointly continuous on $G^{\mathsf{LUC}} \times G$ by [1, 1.4.2].

It follows that φ is jointly continuous on $G^{\mathsf{LUC}} \times G$. Then by [1, B.3] the mapping $x \mapsto \searrow_{\mathfrak{m}} \varphi(\mathfrak{m}, x)$ is continuous from G to $\ell_\infty(G^{\mathsf{LUC}})$ with the sup norm. Hence, the mapping $x \mapsto \searrow_{\mathfrak{m}} \varphi(\mathfrak{m}, x) \upharpoonright G$ is continuous from G to $\ell_\infty(G)$ with the sup norm. But $\varphi(y, x) = f(yx)$ for $x, y \in G$, and we get $f \in \mathsf{RUC}(G)$ by the definition of $\mathsf{RUC}(G)$. That proves $\mathsf{LUC}(G) \subseteq \mathsf{RUC}(G)$. Using Lemma 4.3, we conclude that G is a SIN group. ∎

For locally compact non-SIN groups the following corollary was proved by Lau and Pym [7, 3.1].

Corollary 4.5 *Let G be a non-SIN group whose topology is locally compact or completely metrizable. Then there exists $\mathfrak{m} \in G^{\mathsf{LUC}}$ for which the mapping $x \mapsto \mathfrak{m} \star x$ from G to G^{LUC} is not continuous.* ∎

Many infinite-dimensional groups of automorphisms, such as those discussed by Pestov [11], are metrizable by a complete metric and not SIN. This includes the groups of autohomeomorphisms of the interval $[0,1]$ and of the Cantor set $2^{\aleph_0}$ with the topology of uniform convergence, groups of automorphisms of many Fraïssé structures with the topology of pointwise convergence, and the unitary group of an infinite-dimensional Hilbert space.

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Corollary 4.6 *Let G be a metrizable non-SIN group. Then there exists $\mathfrak{m} \in G^{\mathsf{LUC}}$ for which the mapping $x \mapsto \mathfrak{m} \star x$ from $\widehat{G}$ to G^{LUC} is not continuous.*

Proof Apply Theorem 4.4 with $S = \widehat{G}$, which of course is completely metrizable. ∎

By [10, Cor. 6.13] the UEB and weak* topologies coincide on $\widehat{G}$. That together with the two corollaries shows that for any non-SIN group G that is locally compact or metrizable there exists $\mathfrak{m} \in G^{\mathsf{LUC}}$ for which the mapping $x \mapsto \mathfrak{m} \star x$ from $\widehat{G}$ to G^{LUC} is not UEB continuous, and thus convolution is not separately UEB continuous on G^{LUC}.

More generally, to exhibit such a discontinuity it is enough to show that one of the two corollaries applies to a subgroup H of G. Indeed, if H is a topological subgroup of G, then H is a uniform subspace of G when both are considered with their right uniformities [13, 3.24]. Hence, H^{LUC} is embedded in G^{LUC}, both topologically and algebraically (with the convolution operation). It follows that convolution is not separately UEB continuous on G^{LUC} whenever G contains a locally compact or metrizable subgroup that is not SIN.

Thus, Corollary 4.6 holds for a large class of not necessarily metrizable non-SIN groups. We do not know whether it holds for every non-SIN group.

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References

[1] J. F. Berglund, H. D. Junghenn, and P. Milnes, *Analysis on semigroups*. Canadian Mathematical Society Series of Monographs and Advanced Texts, John Wiley & Sons Inc., New York, 1989.

[2] V. I. Bogachev, *Measure theory. Vol. I, II*. Springer-Verlag, Berlin, 2007. http://dx.doi.org/10.1007/978-3-540-34514-5

[3] A. Bouziad and J.-P. Troallic, *Problems about the uniform structures of topological groups*. In: Open Problems in Topology II, Elsevier, Oxford, 2007, pp. 359–366.

[4] S. Ferri and M. Neufang, *On the topological centre of the algebra* LUC$(G)^*$ *for general topological groups*. J. Funct. Anal. **244**(2007), no. 1, 154–171. http://dx.doi.org/10.1016/j.jfa.2006.11.011

[5] G. Itzkowitz, S. Rothman, H. Strassberg, and T. S. Wu, *Characterization of equivalent uniformities in topological groups*. Topology Appl. **47**(1992), no. 1, 9–34. http://dx.doi.org/10.1016/0166-8641(92)90112-D

[6] A. T.-M. Lau, *Continuity of Arens multiplication on the dual space of bounded uniformly continuous functions on locally compact groups and topological semigroups*. Math. Proc. Cambridge Philos. Soc. **99**(1986), no. 2, 273–283. http://dx.doi.org/10.1017/S0305004100064197

[7] A. T.-M. Lau and J. Pym, *The topological centre of a compactification of a locally compact group*. Math. Z. **219**(1995), no. 4, 567–579. http://dx.doi.org/10.1007/BF02572381

[8] P. Milnes, *Uniformity and uniformly continuous functions for locally compact groups*. Proc. Amer. Math. Soc. **109**(1990), no. 2, 567–570. http://dx.doi.org/10.1090/S0002-9939-1990-1023345-7

[9] M. Neufang, J. Pachl, and P. Salmi, *Uniform equicontinuity, multiplier topology and continuity of convolution*. Arch. Math. (Basel) **104**(2015), no. 4, 367–376. http://dx.doi.org/10.1007/s00013-015-0726-9

[10] J. Pachl, *Uniform spaces and measures*. Fields Institute Monographs, 30, Springer, New York, 2013 (Corrections and supplements at http://www.fields.utoronto.ca/publications/supplements). http://dx.doi.org/10.1007/978-1-4614-5058-0

[11] V. Pestov, *Dynamics of infinite-dimensional groups*. University Lecture Series, 40, American Mathematical Society, Providence, RI, 2006. http://dx.doi.org/10.1090/ulect/040

[12] I. Protasov, *Functionally balanced groups*. (Russian) Mat. Zametki **49**(1991), no. 6, 87–91; translation in: Math. Notes **49**(1991), 614–616. http://dx.doi.org/10.1007/BF01156586

[13] W. Roelcke and S. Dierolf, *Uniform structures on topological groups and their quotients*. McGraw-Hill, New York, 1981.

[14] P. Salmi, *Joint continuity of multiplication on the dual of the left uniformly continuous functions*. Semigroup Forum **80**(2010), no. 1, 155–163. http://dx.doi.org/10.1007/s00233-009-9189-2

[15] C. Villani, *Optimal transport*. Grundlehren der Mathematischen Wissenschaften, 338, Springer-Verlag, Berlin, 2009. http://dx.doi.org/10.1007/978-3-540-71050-9

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