Can. J. Math., Vol. XXXIII, No. 5, 1981, pp. 1271–1278

CLOSED LIE IDEALS IN OPERATOR ALGEBRAS

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1. Introduction. If M is an associative algebra with product xy, M can be made into a Lie algebra by endowing M with a new multiplication $[x, y] = xy - yx$. The Poincare–Birkoff–Witt Theorem, in part, shows that every Lie algebra is (Lie) isomorphic to a Lie subalgebra of such an associative algebra M. A Lie ideal in M is a linear subspace $U \subseteq M$ such that $[x, u] \in U$ for all $x \in M$, $u \in U$. In [**9**], as a step in characterizing Lie mappings between von Neumann algebras, Lie ideals which are closed in the ultra-weak topology, and closed under the adjoint operation are characterized when M is a von Neumann algebra. However the restrictions of ultra-weak closure and adjoint closure seemed unnatural, and in this paper we characterize those uniformly closed linear subspaces which can occur as Lie ideals in von Neumann algebras. We follow, and use results from, the programme of Herstein [**8**] who characterized Lie ideals in simple rings. We expect, therefore, that the Lie ideal structure of M will be closely related to its associative ideal structure, and, indeed, this is the case. Using our characterization and a Herstein–Amitsur-like result, other characterizations of Lie ideals in terms of invariance properties are given. Finally, finite dimensional Lie ideals, and solvable Lie ideals are characterized in von Neumann algebras.

2. Uniformly closed Lie ideals. In what follows, if M is an algebra over the complex field, and S, T subsets of M then

$$[S, T] = \left\{ \sum_{i=1}^{n} \alpha_i[s_i, t_i] | \alpha_i \in \mathbf{C}, s_i \in S, t_i \in T \right\},$$

$\bar{S}$ will denote closure in the uniform topology, and $\bar{S}^{\text{uw}}$ closure in the ultraweak topology.

We shall need the following facts:

(1) If M is a properly infinite von Neumann algebra then $M = [M, M]$ [**15**] and M is the (non-closed) linear span of its projections [**11**].

(2) If M is a finite von Neumann algebra, $[M, M] = \{m \in M | m^\# = 0\}$ where $m^\#$ is the centre-valued trace ([**6**], Thèoréme 3.2, and [**10**], Theorem 1). This implies that if M is a II_1-factor then it is the (non-closed) linear span of its projections ([**10**], Theorem 3).

Received June 24, 1980. This research was partially supported by NSERC Grant A-7682.

https://doi.org/10.4153/CJM-1981-096-0 Published online by Cambridge University Press

(3) Any von Neumann algebra is the uniform closure of the linear span of its projections by the spectral theorem.

(4) If M is a ring with no 2-torsion, and no non-zero nilpotent ideals, and if W is a Lie ideal and subring of M then $W \subseteq Z_M$, the centre of M, or there exists a non-zero, two-sided ideal I of M such that $I \subseteq W$ ([**8**], Lemma 1.3).

The following lemma is due to John Bunce.

LEMMA 1. *If A is a C^*-algebra and J a uniformly closed, two-sided ideal in A, then* $\overline{[A, J]} = J \cap \overline{[A, A]}$.

Proof. $[A, J] \subseteq J \cap [A, A]$ so

$$\overline{[A, J]} \subseteq \overline{J \cap [A, A]} \subseteq \bar{J} \cap \overline{[A, A]} = J \cap \overline{[A, A]}.$$

Let $\{u_\lambda\}$ be a quasi-central approximate identity for J. That is, $\|u_\lambda\| \leqq 1$, $\{u_\lambda\} \subseteq J$, $u_\lambda \geqq 0$, $u_\lambda \nearrow$, $\|j - u_\lambda j\| \to 0$ for all $j \in J$, and $\|[u_\lambda, a]\| \to 0$ for all $a \in A$. (See [**1**] for a discussion of this idea.) Given $\epsilon > 0$, suppose $\|j - [x, y]\| < \epsilon$ for $j \in J$, $x, y \in A$. Then

$$\begin{aligned}\|j - [x, u_\lambda y]\| &= \|j - u_\lambda j + u_\lambda j - [x, u_\lambda y]\| \\ &\leqq \|j - u_\lambda j\| + \|u_\lambda j - [x, u_\lambda y]\| \\ &\leqq \|j - u_\lambda j\| + \|u_\lambda j + [x, u_\lambda]y - [x, u_\lambda y]\| + \|[x, u_\lambda]y\|.\end{aligned}$$

But

$$[x, u_\lambda]y - [x, u_\lambda y] = xu_\lambda y - u_\lambda xy - xu_\lambda y + u_\lambda yx = u_\lambda[y, x].$$

Hence

$$\|u_\lambda j + [x, u_\lambda y] - [x, u_\lambda y]\| = \|u_\lambda j - u_\lambda[x, y]\| \leqq \|j - [x, y]\| < \epsilon.$$

This implies that if $j \in J \cap \overline{[A, A]}$ then $j \in \overline{[A, J]}$.

COROLLARY. *If J is any two-sided ideal, then* $\overline{[A, J]} = \bar{J} \cap \overline{[A, A]}$.

LEMMA 2. *If M is a von Neumann algebra and J a uniformly closed, two-sided ideal in M then $Z_{(M/J)} = J + Z_M$ where Z_M is the centre of M.*

Proof. Let $\bar{x} = x + J$ be the coset in M/J containing x and suppose $\bar{x} \in Z_{(M/J)}$. Then $[\bar{x}, \bar{y}] = 0$ for all $y \in M$. That is $[x, y] \in J$ for all $y \in M$. In particular, if $u \in M$ is unitary, $[x, u] = xu - ux \in J$ so that $uxu^{-1} - x \in J$ for all unitary $u \in M$. Let

$$K_x = \{uxu^{-1} | \ u \text{ unitary in } M\}.$$

Then (by [**5**], Théorème 1, p. 253) $\overline{\text{co } K_x} \cap Z_M \neq \emptyset$ where co K_x is the convex hull of K_x. Let $t \in \overline{\text{co } K_x} \cap Z_M$ and $t = \lim t_n$ where

$$t_n = \sum \alpha_i^{(n)} u_i^{(n)} x (u_i^{(n)})^{-1}, \ \sum \alpha_i^{(n)} = 1.$$

https://doi.org/10.4153/CJM-1981-096-0 Published online by Cambridge University Press

Then

$$t_n - x = \left(\sum \alpha_i^{(n)} u_i^{(n)} x (u_i^{(n)})^{-1}\right) - x$$
$$= \sum \alpha_i^{(n)} (u_i^{(n)} x (u_i^{(n)})^{-1} - x) \in J.$$

Hence $t - x = j \in J$ or $x = j + t$ and $t \in Z_M$.

THEOREM 1. *Let M be a von Neumann algebra and U a uniformly closed Lie ideal in M. If M is properly infinite, there exists a closed two-sided ideal $J \subseteq M$ such that $J \subseteq U \subseteq J + Z_M$. If M is finite there exists a closed, two-sided ideal $J \subseteq M$ such that $J \subseteq U + Z_M \subseteq J + Z_M$.*

Proof. If $T(U) = \{x \in M | \, [x, M] \subseteq U\}$ then $T(U)$ is a Lie ideal and subring of M, containing U, and is uniformly closed since U is uniformly closed. Hence either $T(U) \subseteq Z_M$ or there exists a non-zero, two-sided ideal $J \subseteq T(U)$. If $T(U) \subseteq Z_M$ then $U \subseteq Z_M$ and $J = \{0\}$ in the theorem.

Otherwise let J be a maximal non-zero, two-sided ideal in $T(U)$. Since $T(U)$ is uniformly closed, so is J. We claim that $J \subseteq T(U) \subseteq J + Z_M$. For, if $T(U)/J \neq \{0\}$ in M/J then $T(U)/J$ is a Lie ideal and subring in M/J and is either contained in the centre of M/J or contains a non-zero, two-sided ideal K of M/J. In the first case, by Lemma 2, $T(U)/J \subseteq Z_{(M/J)} = J + Z_M$ or $T(U) \subseteq J + Z_M$. In the second case let $K_0 = \pi^{-1}(K)$ where π: $M \rightarrow M/J$ is the canonical map and notice that $J \subsetneq K_0 \subseteq T(U) + J = T(U)$ which is impossible since K_0 is a two-sided ideal and J is maximal. Hence we must have $J \subseteq T(U) \subseteq J + Z_M$. By Lemma 1,

$$J \cap \overline{[M, M]} = \overline{[M, J]} \subseteq U \subseteq T(U) \subseteq J + Z_M.$$

If M is infinite, then $[M, M] = M$ so that $J = J \cap [M, M] \subseteq U \subseteq J + Z_M$. If M is finite, $[M, M] = \{x \in M | x^\# = 0\} = M_0$. Moreover, if $j \in J$, $j - j^\# \in J \cap M_0$ ([**5**], Proposition 2, p. 256). Hence, in the finite case,

$$j - j^\# \in J \cap M_0 = J \cap [M, M] \subseteq U \subseteq J + Z_M.$$

Finally this implies $J \subseteq U + Z_M \subseteq J + Z_M$.

3. Characterizations of Lie ideals.

LEMMA 3. *If M is a von Neumann algebra and U a linear subspace such that $[U, [M, M]] \subseteq U$ then U is a Lie ideal in M.*

Proof. If M is properly infinite $[M, M] = M$. If M is finite, $[M, M] = \{m \in M | m^\# = 0\} = M_0$. If $x \in M$, $x - x^\# \in M_0$ and so $[x - x^\#, u] = [x, u] \in U$ when $u \in U$ so that U is a Lie ideal. In general $M = M_c \oplus M_{1-c}$ where c is a central projection, $M_c = \{cm | m \in M\}$, with M_c finite and M_{1-c} properly infinite. Moreover U_c and U_{1-c} satisfy the condition

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of the theorem in M_c and M_{1-c} respectively and are Lie ideals by the first part of the proof.

THEOREM 2 (See [**8**], Theorem 1.15, [**16**], p. 348). *Let M be a von Neumann algebra, W a subspace of M invariant under all special inner automorphisms of M. (That is $(1 + a)W(1 - a) \subseteq W$ for all a in M for which $a^2 = 0$.) If M is properly infinite, W is a Lie ideal of M. If M is finite, $\overline{W}$ is a Lie ideal of M.*

Proof. We follow ([**8**], Theorem 1.15). Given $a \in M$ with $a^2 = 0$ and $b \in W$ then

$$(1 + a)b(1 - a) = b + ab - ba - aba \in W.$$

Hence $ab - ba - aba \in W$. Let $\alpha \in \mathbf{C}$ with $\alpha^2 \neq \alpha$. Then $(\alpha a)^2 = 0$ implies

$$\alpha ab - \alpha ba - \alpha^2 aba \in W.$$

Since W is a subspace, $\alpha^2(ab - ba - aba) \in W$ so that

$$(\alpha^2 - \alpha)(ab - ba) \in W.$$

Hence $[a, b] \in W$ for all $b \in W$ and all a for which $a^2 = 0$.

If p is a projection in M then $a = xp - pxp$ is a nilpotent of index two for all $x \in M$. Hence

$$[a, b] = (xp - pxp)b - b(xp - pxp) \in W \text{ for all } b \in W, x \in M.$$

Similarly,

$$(px - pxp)b - b(px - pxp) \in W \text{ for all } b \in W, x \in M.$$

This implies $[[p, M], W] \subseteq W$ for all projections p in M. If P is the linear span of projections in M, then $[[P, M], W] \subseteq W$. If M is properly infinite, $P = M$ so that W is a Lie ideal of $[M, M] = M$. If M is finite, $\overline{P} = M$ so that $[[M, M], \overline{W}] \subseteq \overline{W}$. By Lemma 3, $\overline{W}$ is a Lie ideal in M.

COROLLARY. *A uniformly closed linear subspace W of a von Neumann algebra M is a Lie ideal if and only if it is invariant under all special inner automorphisms.*

Proof. Let W be a uniformly closed Lie ideal. If M is properly infinite there exists, by Theorem 1, a closed, two-sided ideal J such that $J \subseteq W \subseteq J + Z_M$. Hence, if $w \in W$, $w = j + z$ where $j \in J$, $z \in Z_M$. Let $a^2 = 0$ and $\phi_a(x) = (1 + a)x(1 - a)$. Then

$$\phi_a(w) = (1 + a)j(1 - a) + z \text{ and } \phi_a(w) - w \in J.$$

Thus $\phi_a(w) \in J + W \subseteq W$. A similar argument suffices for the finite case. Finally, ϕ_a preserves direct sums.

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Lie ideals have other invariance properties as shown in the following theorem.

THEOREM 3. *Let W be a linear subspace of an algebra A. If W is a Lie ideal then $u^*Wu \subseteq W$ for all $u = p + i(1 - p)$, p a projection in A. If A is generated as a linear space by its projections and if $u^*Wu \subseteq W$ for all $u = p + i(1 - p)$ then W is a Lie ideal. If A is a properly infinite von Neumann algebra and $u^*Wu \subseteq W$ for all $u = p + i(1 - p)$ then $u^*Wu \subseteq W$ for all unitaries $u \in A$.*

Proof. If W is a Lie ideal in A then for any unitary $u \in A$,

$$[[u, w], u^*] = uwu^* + u^*wu - 2w \in W$$

so that $uwu^* + u^*wu \in W$ for all $w \in W$. If p is a projection and $u = p + i(1 - p)$ then $2i[p, w] = u^*wu - uwu^* \in W$. Thus, if $u = p + i(1 - p)$ we have, by adding, that $u^*wu \in W$ for all $w \in W$.

If A is generated as a linear space by its projections and $u^*Wu \subseteq W$ for all $u = p + i(1 - p)$ then $2i[p, w] = u^*wu - uwu^* \in W$ for all p so $[A, W] \subseteq W$.

If A is a properly infinite von Neumann algebra and $u^*Wu \subseteq W$ for all $u = p + i(1 - p)$ then $(u^*)^2Wu^2 = vWv \in W$ where $v = u^2\,(u^*)^2 = 2p - 1$. Hence W is invariant under symmetries. But each unitary in A is the product of four symmetries ([**7**], Corollary, Theorem 3) so that for a general unitary $u \in A$, $u = v_1v_2v_3v_4$ and

$$u^*Wu = (v_1v_2v_3v_4)^*W(v_1v_2v_3v_4) = v_4v_3v_2v_1Wv_1v_2v_3v_4 \subseteq W$$

by the above.

Remark. For comparison to Theorem 3 we note that if A is an algebra with no 2-torsion generated as a linear space by its projections, then a linear subspace W is a Jordan ideal if and only if $pWp \subseteq W$ for all projections $p \in A$. For, if $pWp \subseteq W$ then $w \in W$ implies

$$\begin{aligned} w &= (p + (1 - p))w(p + (1 - p)) \\ &= pwp + (1 - p)w(1 - p) + (1 - p)wp + pw(1 - p) \\ &= (1 - p)w(1 - p) - pwp + wp + pw \in W. \end{aligned}$$

Hence $wp + pw \in W$ for all $p \in A$. On the other hand if W is a Jordan ideal then $wp + pw = w' \in W$ for all p. Hence

$$w'p + pw' = wp + pw + 2pwp \in W.$$

Hence $pwp \in W$.

Furthermore if A has no non-zero nilpotent ideals, a non-zero Jordan ideal contains a non-zero two-sided ideal ([**8**], Lemma 1.1) and any closed Jordan ideal in a C^*-algebra is a two-sided ideal ([**4**], Theorem 5.3).

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4. Finite dimensional and solvable Lie ideals.

THEOREM 4. *If U is a finite dimensional Lie ideal in a von Neumann algebra M and if c is the maximal finite central projection in M then $(1 - c)U \subseteq Z_{M_{1-c}}$ and $cU \cong \bigoplus_{i=1}^{k} U_{ni}$ where U_{ni} is $\{0\}$, M_{ni}, $\mathbf{C}1$, or $[M_{ni}, M_{ni}]$, and M_{ni} is an $n_i \times n_i$ matrix algebra over $\mathbf{C}$.*

Proof. $(1 - c)U$ is a finite dimensional and hence uniformly closed Lie ideal in M_{1-c}. By Theorem 1, there exists a closed two-sided ideal J in M_{1-c} such that

$$J \subseteq (1 - c)U \subseteq J + Z_{M_{1-c}}.$$

J is thus a finite dimensional two-sided ideal in the properly infinite algebra M_{1-c} and so $J = \{0\}$. Hence $(1 - c)U \subseteq Z_{M_{1-c}}$.

In M_c there exists a closed, two-sided ideal K such that

$$K \subseteq cU + Z_{M_c} \subseteq K + Z_{M_c}.$$

Thus $[K, K] = c[U, U]$ and $[K, K]$ is finite dimensional. We have $[K^{-\mathrm{uw}}, K^{-\mathrm{uw}}] \subseteq [K, K]^{-\mathrm{uw}}$, so that if $K^{-\mathrm{uw}} = M_d$, $d \leqq c$, d a central projection, then $[M_d, M_d]$ is finite dimensional. Notice that if p, q are projections in M_d with $p \sim q$ via the partial isometry $v \in M_d$ then $[v, v^*] = p - q \in [M_d, M_d]$.

If M_d has a non-zero continuous summand $M_{d'}$, and $0 \neq p$ a projection in $M_{d'}$ then p can be "halved" and written $p = p_1 + p_2$, $p_1 \perp p_2$, $p_1 \sim p_2$. Likewise p_1 and p_2 can be halved, $p_1 = p_{11} + p_{12}$, $p_2 = p_{21} + p_{22}$ where the p_{ij} are mutually $\perp$, $p_{11} \sim p_{12}$, $p_{21} \sim p_{22}$. Hence $p_{11} - p_{12}$ and $p_{21} - p_{22}$ are in $[M_d, M_d]$ and are independent vectors. Since the halving can be done infinitely many times, M_d can have no continuous part when $[M_d, M_d]$ is finite dimensional. Thus d is finite and discrete. A similar argument shows that M_d can have only a finite number of homogeneous summands and each of these type I_n summands has $n < \infty$. Thus

$$M_d = \prod_{i=1}^{k} A_{ni} \quad \text{where } A_{ni} \cong B_i \oplus \mathscr{L}(H_{ni}),$$

n_i a finite integer and B_i an abelian von Neumann algebra. If B_i has infinitely many orthogonal projections then $[A_{ni}, A_{ni}]$ and hence $[M_d, M_d]$ will contain infinitely many independent vectors. Hence B_i is an abelian von Neumann algebra with only finitely many projections. M_d is thus finite dimensional so

$$M_d = \bigoplus_{i=1}^{k} M_{ni}.$$

Let d be a maximal central projection such that $d \leqq c$, M_d is finite dimensional and

$$M_d \subseteq cU + Z_{M_c} \subseteq M_d + Z_{M_c}.$$

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If $(c - d)U \neq \{0\}$ the same argument will give

$$\{0\} \neq M_{d'} \subseteq (c - d)U + Z_{M_{c-d}} \subseteq M_{d'} + Z_{M_{c-d}}$$

where $M_{d'}$ is finite dimensional. Thus $M_d \oplus M_{d'} = M_{d+d'}$ will be a larger such ideal contradicting maximality. Hence $cU = dU$. The result now follows from ([**8**] Theorem 1.3) since the M_{n_i} are simple associative rings.

LEMMA 4. *If M is a C^*-algebra with $D^n M = \{0\}$ then M is abelian.* (*Here $D^0 M = M$, $D^1 M = [M, M]$, and $D^n M = D(D^{n-1}M)$.*)

Proof. Let n be the smallest positive integer for which $D^n M = \{0\}$. Then $U = D^{n-1}M$ is an abelian Lie ideal closed under the $*$-operation. Thus for $x \in U$, $y \in M$, $[x, [x, y]] = 0$. If $x = x^*$, $y = y^*$ this forces $[x, y] = 0$ by ([**14**], Theorem 1). Hence we have $U \subseteq Z_M$. The Kleinecke–Sirokov Theorem ([**12**], Theorem 1.3.1) implies $D^{n-1}M = \{0\}$. An induction argument now gives the result.

Recall that a Lie algebra L is solvable if $D^n L = \{0\}$ for some n.

THEOREM 5. *If M is a von Neumann algebra and U a solvable Lie ideal then $U \subseteq Z_M$.*

Proof. It suffices to assume U is uniformly closed. Let c be the maximal finite central projection in M. By Theorem 1 there exists a closed, two-sided ideal $J \subseteq M_{1-c}$ such that $J \subseteq (1 - c)U \subseteq J + Z_{M_{1-c}}$. Thus $[J, J] = (1 - c)[U, U]$ and $J^{-\mathrm{uw}} \subseteq (1 - c)U^{-\mathrm{uw}}$. Let $J^{-\mathrm{uw}} = M_d$, $d \leqq 1 - c$, d a central projection. We have

$$[M_d, M_d] \subseteq (1 - c)[U^{-\mathrm{uw}}, U^{-\mathrm{uw}}],$$

and in general,

$$D^n M_d \subseteq (1 - c) D^n U^{-\mathrm{uw}}.$$

For some $n > 0$, $D^n U = \{0\}$ which implies $D^n U^{-\mathrm{uw}} = \{0\}$. By Lemma 4, M_d is abelian. But $d \leqq 1 - c$ and M_{1-c} is properly infinite, so $d = 0$. That is, $J = \{0\}$ and $(1 - c)U \subseteq Z_{M_{1-c}}$.

On the other hand cU is a uniformly closed Lie ideal in the finite algebra M_c. There exists a uniformly closed, two-sided ideal $K \subseteq M_c$ such that

$$K \subseteq cU + Z_{M_c} \subseteq K + Z_{M_c}.$$

As before, $K^{-\mathrm{uw}} = M_e$ where $e \leqq c$ and e is a central projection. Now

$$K + Z_{M_c} \subseteq cU + Z_{M_c} \subseteq cU^{-\mathrm{uw}} + Z_{M_c}$$

so that

$$[M_e, M_e] = [K^{-\mathrm{uw}}, K^{-\mathrm{uw}}] \subseteq [K, K]^{-\mathrm{uw}} \subseteq c[U^{-\mathrm{uw}}, U^{-\mathrm{uw}}]^{-\mathrm{uw}}.$$

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In general $D^n M_e \subseteq c(D^n U^{-\mathrm{uw}})^{-\mathrm{uw}}$. $D^n U = 0$ implies $D^n U^{-\mathrm{uw}} = 0$ so that by Lemma 4, M_e is abelian. Hence $K \subseteq M_e \subseteq Z_{M_c}$. This implies

$$cU + Z_{M_c} \subseteq K + Z_{M_c} \subseteq Z_{M_c}$$

so that $cU \subseteq Z_{M_c}$.

Remark. It is possible that at least in some cases the restriction of uniform closure in Theorem 1 can be removed. The problem stems from the lack of Lemma 1 and Lemma 2 in the non-closed case. Lemma 2 holds for non-closed ideals when M is a type I factor on a separable Hilbert space ([**3**], Theorem 2.9) and, of course, when M is simple. Very little is known about $[A, J]$ even when J is closed. (See [**2**], [**13**]).

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