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Leafwise quasigeodesic foliations in dimension three and the funnel property

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Abstract. We construct one-dimensional foliations which are subfoliations of two-dimensional foliations in 3-manifolds. The subfoliation is by quasigeodesics in each two-dimensional leaf, but it is not funnel: not all quasigeodesics share a common ideal point in most leaves.

Key words: quasigeodesics, subfoliations, Anosov flows
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1. *Introduction*

The goal of this article is to analyze whether certain geometric conditions imply that a one-dimensional foliation in a 3-manifold is the foliation by flow lines of a topological Anosov flow. We do this analysis for one-dimensional foliations whose leaves lie inside leaves of two-dimensional foliations and whose leaves are quasigeodesics in these two-dimensional foliations. In other words, the goal of this article is to analyze whether some strictly geometric behavior implies strong dynamical systems behavior in this setting. This has important connections with partial hyperbolicity in dimension three.

A foliation $\mathcal{G}$ subfoliates a foliation $\mathcal{F}$ if each leaf of $\mathcal{F}$ has a foliation made up of leaves of $\mathcal{G}$. We call $\mathcal{G}$ the subfoliation and $\mathcal{F}$ the super foliation. This situation is very common, for example, if $\mathcal{F}_1$ and $\mathcal{F}_2$ are two foliations which are transverse to each other everywhere, then their intersection forms a subfoliation of each of them. This article aims to study geometric properties of leaves of subfoliations inside the leaves of the super foliation.

One very common and extremely important example is the following: let Φ be an Anosov flow and let $\mathcal{F}^{ws}, \mathcal{F}^{wu}$ be the weak stable and weak unstable foliations of Φ respectively [**Ano63**, **KH95**]. Then $\mathcal{F}^{ws}, \mathcal{F}^{wu}$ are transverse to each other—the intersection is the foliation by flow lines of Φ which is a subfoliation of each of them. This example has connections with geometry or large-scale geometry: the leaves of $\mathcal{F}^{ws}$, $\mathcal{F}^{wu}$ are Gromov hyperbolic. In rough terms, this means that they are negatively curved. The subfoliation by flow lines in, say, $\mathcal{F}^{ws}$ satisfies an additional strong geometric property:

in each leaf of $\mathcal{F}^{ws}$, the flow lines are quasigeodesics. This means that when lifted to the universal cover of the leaves, the flow lines are uniformly efficient up to a bounded multiplicative distortion in measuring length in the weak stable leaves. In other words, the flow lines are quasi-isometrically embedded in these weak stable leaves. The quasigeodesic property has many important consequences, for example, the flow lines are within a bounded distance from length minimizing geodesics when lifted to the universal cover of their respective weak stable leaves [**Gro87**, **Thu82**, **Thu97**]. Hence the flow lines have well-defined distinct ideal points in the Gromov boundary of the weak stable leaves in both directions. These properties and others are very strong and useful in many contexts. Obviously, this also works for the flow subfoliation of the weak unstable foliation.

A (one-dimensional) subfoliation made of quasigeodesics in the leaves of a super foliation by Gromov hyperbolic leaves is called a *leafwise quasigeodesic foliation*.

The Anosov case has an additional geometric property: in (say) a weak stable leaf, all flow lines are forward asymptotic, which is a defining property of the weak stable foliation. In particular, all flow lines in a given weak stable leaf have the same forward ideal point in the ideal boundary of the weak stable leaf (when lifted to the universal cover).

When all leaves of a leafwise quasigeodesic subfoliation in a leaf of the super foliation have a common ideal point, we call that leaf a *funnel leaf*. If all leaves of the super foliation are funnel leaves, then the leafwise quasigeodesic foliation is said to have the *funnel property*.

The motivation for this article is the following question: is the funnel property an additional property or is it a consequence of the leafwise quasigeodesic property? The importance of this is the following: in dimension three, we have a much stronger connection between some of these properties as follows. Suppose that $\mathcal{G}$ is a leafwise quasigeodesic foliation (which is a one-dimensional subfoliation of a two-dimensional foliation) which has the funnel property. The ambient manifold is three-dimensional. Suppose that the foliation $\mathcal{G}$ is orientable or, in other words, it is the foliation of a non-singular flow. Then one can prove that the flow in question is expansive. (We refer to [**BFP20**] for definitions of the terms used here and for detailed proofs.) This implies that the flow is a topological Anosov flow [**IM90**, Theorem 15], [**Pat93**, Lemma 7]. If the flow is transitive (the union of periodic orbits is dense), then the topological Anosov flow is in addition orbitally equivalent to a (smooth) Anosov flow [**Sha20**]. This means that if the leafwise quasigeodesic property implies the funnel property, then this purely geometric condition implies a very strong dynamical systems property: the foliation is the flow foliation of an Anosov flow, up to topological equivalence.

In this article, we prove that the funnel property is not a consequence of leafwise quasigeodesic behavior.

THEOREM 1.1. *There are examples of leafwise quasigeodesic foliations in dimension three which do not have the funnel property. In these examples, the two-dimensional foliations are C^0 with C^1 leaves and the subfoliation is by C^1 curves in the two-dimensional leaves.*

We now briefly explain one class of examples: start with the Franks–Williams example of a non-transitive Anosov flow Φ. This is obtained as follows: start with a suspension

Anosov flow and do a DA (derived from Anosov) blow up of a periodic orbit transforming it into (say) a repelling orbit α. Remove a tubular neighborhood of α so that the resulting semiflow is incoming in the complement of the removed tubular neighborhood of α. Glue this manifold with boundary with a copy of it which has a reversed flow. One fundamental result is that the ensuing flow Φ in the final manifold $\mathcal{M}$ is Anosov [**BBY17**, **FW80**]. This holds for certain isotopy classes of gluings and certain gluing maps satisfying transversality conditions. These were the first examples of non-transitive Anosov flows in dimension three. Our examples use this flow. There is a smooth torus T in $\mathcal{M}$ transverse to the flow. There is a single two-dimensional attractor and a single two-dimensional repeller of the flow Φ in M. Start with a one-dimensional foliation Z in T which is transverse to the intersections of both the weak stable and the weak unstable foliations of Φ with T. Saturate Z by the flow producing a collection of two-dimensional sets embedded in $\mathcal{M}$. The flow saturation of T is an open subset V of M, and the collection of the two-dimensional subsets described is a two-dimensional foliation in V. In addition, V is exactly the complement of the union of the attractor and the repeller of Φ. Complete the foliation in V to a foliation $\mathcal{F}$ in $\mathcal{M}$ which is the weak unstable foliation of Φ in the attractor of Φ and the weak stable foliation in the repeller of Φ. The proof that this is in fact a foliation of $\mathcal{M}$ depends on a careful choice of the one-dimensional foliation Z in T. There is a subtle point here in that if one chooses an arbitrary foliation Z in T, then when lifting to $\widetilde{\mathcal{M}}$, the lifted sets may not be properly embedded in $\widetilde{\mathcal{M}}$ and so $\mathcal{F}$ would not be a foliation in $\mathcal{M}$. This is carefully analyzed in §3 and there we prove that for appropriate choices of Z, the object $\mathcal{F}$ we construct is a foliation. The super foliation is this two-dimensional foliation $\mathcal{F}$. The subfoliation $\mathcal{G}$ of $\mathcal{F}$ is the foliation by flow lines of Φ. Each leaf of $\mathcal{F}$ is saturated by flow lines. We prove that $\mathcal{G}$ is a leafwise quasigeodesic subfoliation of $\mathcal{F}$, but $\mathcal{G}$ does not have the funnel property. There is an Anosov flow Φ in this example; however, notice that the super foliation $\mathcal{F}$ is neither the weak stable nor the weak unstable foliation of Φ, but rather a different foliation. In fact in the same way, one can construct an infinite number of inequivalent examples with the same starting flow Φ. The foliations are pairwise distinguished because of how they intersect the torus T in foliations which are not equivalent.

In this article, we consider more general examples. We prove that one can construct examples starting with any non-transitive Anosov flow Φ in dimension three so that all the basic sets have dimension two. As in the case of the Franks–Williams example, we construct super foliations which have Gromov hyperbolic leaves and whose leaves are saturated by flow lines of Φ. We show that the subfoliation $\mathcal{G}$ by flow lines of Φ is by quasigeodesics in each leaf of the super foliation $\mathcal{F}$. This is the hardest step to prove. This involves a very careful analysis of the geometry in these examples. The proof that $\mathcal{G}$ is not funnel is simpler than proving it is leafwise quasigeodesic as a subfoliation of $\mathcal{F}$.

In the course of the proof of Theorem 1.1, we prove another independent result which can be used in other contexts. In Definition 6.1, we define the notion of *continuity properties* for a pair of foliations $(\mathcal{F}, \mathcal{G})$ on $\mathcal{M}$ where $\mathcal{F}$ is a two-dimensional foliation sub-foliated by a one-dimensional foliation $\mathcal{G}$. We can show that the continuity property implies leafwise quasigeodisity.

THEOREM 1.2. *Suppose $\mathcal{F}$ is a two-dimensional foliation on a 3-manifold $\mathcal{M}$ and $\mathcal{F}$ is subfoliated by a one-dimensional foliation $\mathcal{G}$. If the pair $(\mathcal{F}, \mathcal{G})$ satisfies the continuity properties as defined in Definition 6.1, then $\mathcal{G}$ is leafwise quasigeodesic on $\mathcal{F}$.*

We finish this introduction mentioning another reason why we analyzed this question: this comes from partially hyperbolic dynamics. Let f be a partially hyperbolic diffeomorphism in a closed 3-manifold M (we refer to [**BFP20**] for definitions and properties of partially hyperbolic diffeomorphisms). Under very general orientability conditions, there is a pair of transverse two- dimensional branching foliations (center stable and center unstable foliations) associated with the partially hyperbolic diffeomorphism which intersect in an one-dimensional branching foliation, called the center foliation [**BI08**]. The center foliation subfoliates both the center stable and center unstable foliations. In some situations [**BFP20**], it is shown that the center foliation is a leafwise quasigeodesic subfoliation of both the center stable and center unstable foliations. However, in [**BFP20**], it is proved that in the partially hyperbolic setting the leafwise quasigeodesic property implies that the center foliation has the funnel property (as a subfoliation of both super foliations). The proof of this also uses dynamical system properties, namely partial hyperbolicity. An open question from the article [**BFP20**] was to whether the funnel property could be derived strictly from the leafwise quasigeodesic property in (say) the center stable foliation. In this article, we prove that this is not the case by constructing counterexamples for general foliations.

2. *Preliminaries*

A C^1-flow $\Phi_t : \mathcal{M} \to \mathcal{M}$ on a Riemannian manifold $\mathcal{M}$ is *Anosov* if the tangent bundle $T\mathcal{M}$ splits into three $D\Phi_t$-invariant sub-bundles $T\mathcal{M} = E^s \oplus E^0 \oplus E^u$ and there exists two constants $C, \lambda > 0$ such that:

- E^0 is generated by the non-zero vector field defined by the flow Φ_t;
- for any $v \in E^s$ and $t > 0$,

$$\|D\Phi_t(v)\| \leq Ce^{-\lambda t}\|v\|;$$

- for any $w \in E^u$ and $t > 0$,

$$\|D\Phi_t(w)\| \geq Ce^{\lambda t}\|w\|.$$

The definition is independent of the choice of the Riemannian metric $\|.\|$ as the underlying manifold $\mathcal{M}$ is compact. For a point $x \in \mathcal{M}$, the set $\gamma_x = \{\Phi_t(x) | t \in \mathbb{R}\}$ is called the *flow line* of x. The collection of all flow lines of a flow defines a one-dimensional foliation on $\mathcal{M}$. For an Anosov flow, there are several flow invariant foliations associated with the flow and these foliations play a key role in the study of Anosov flows.

Property 2.1. [**Ano63**] For an Anosov flow Φ_t on $\mathcal{M}$, the distributions E^u, E^s, $E^0 \oplus E^u$, and $E^0 \oplus E^s$ are uniquely integrable. The associated foliations are denoted by $\mathcal{F}^u$, $\mathcal{F}^s$, $\mathcal{F}^{wu}$, and $\mathcal{F}^{ws}$ respectively and they are called the strong unstable, strong stable, weak unstable, and weak stable foliation on $\mathcal{M}$.

For the remainder of this article, we will assume that $\mathcal{M}$ is a closed three-dimensional Riemannian manifold.

We also assume that $\mathcal{M}$ is equipped with an Anosov flow Φ_t and $\widetilde{\Phi}_t$ is the lift of the flow Φ_t in $\widetilde{\mathcal{M}}$, the universal cover of $\mathcal{M}$. The strong unstable, strong stable, weak unstable, and weak stable foliations of $\widetilde{\Phi}$ are the lifts of the foliations $\mathcal{F}^u$, $\mathcal{F}^s$, $\mathcal{F}^{wu}$, and $\mathcal{F}^{ws}$ in the universal cover $\widetilde{\mathcal{M}}$, and these foliations in $\widetilde{\mathcal{M}}$ are denoted by $\widetilde{\mathcal{F}}^u$, $\widetilde{\mathcal{F}}^s$, $\widetilde{\mathcal{F}}^{wu}$, and $\widetilde{\mathcal{F}}^{ws}$ respectively.

A map $f : (X_1, d_1) \to (X_2, d_2)$ is called a (K, s)-*quasi-isometric embedding* if there exits $K > 1$ and $s > 0$ such that for all $x, y \in X_1$,

$$\frac{1}{K} d_1(x, y) - s \le d_2(f(x), f(y)) \le K d_1(x, y) + s.$$

A (K, s)-*quasigeodesic* in X is the image of a (K, s)-*quasi-isometric embedding* $\gamma : [a, b] \to X$, where $[a, b]$ is a closed interval on $\mathbb{R}$ with the Euclidean metric. The interval could be infinite (that is, $a = -\infty$, $b = \infty$, or both), in which case the notation would be of a half open or open interval. If we have a map $\mathbb{R} \to X$ with rectifiable image, we consider the arclength metric in the domain $\mathbb{R}$.

LEMMA 2.2. *Flow lines on the leaves in $\widetilde{\mathcal{F}}^{ws}$ and $\widetilde{\mathcal{F}}^{wu}$ are quasigeodesics with respect to the induced path metric from $\widetilde{\mathcal{M}}$ in their respective leaves.*

Proof. Reparameterize the flow to have unit speed. The new flow is still Anosov with the same flow lines and the same weak stable and weak unstable foliations; however, the strong stable and strong unstable leaves may change [**Ano63**, **AS67**].

Any leaf L of $\widetilde{\mathcal{F}}^{wu}$ is subfoliated by $\widetilde{\mathcal{F}}^u$ and by the flow lines, these two foliations are transversal to each other. We can define a metric ds' on L by $ds' = dw + dy$, where dw measures length along flow lines and dy measures length along unstable curves. Suppose ds is the Riemannian metric induced on L^{wu} from $\widetilde{\mathcal{M}}$. The two path metrics induced in L from ds' and ds are uniformly quasi-isometric to each other [**Fen94**]. Moreover, each flow line in the leaf L is a length-minimizing curve in the ds' metric, and hence flow lines are uniform quasigeodesics with respect to the metric induced by ds. Similarly, it can be shown that flow lines on leaves in $\widetilde{\mathcal{F}}^{ws}$ are quasigeodesic with respect to the induced metric on their respective leaves. □

Definition 2.3. Suppose $\mathcal{F}$ is a two-dimensional foliation on $\mathcal{M}$ with Gromov hyperbolic leaves when lifted to the universal cover. Suppose that $\mathcal{G}$ is a one-dimensional foliation on $\mathcal{M}$ which subfoliates $\mathcal{F}$. In this situation, we say that leaves of $\mathcal{G}$ are leafwise quasigeodesic in $\mathcal{F}$ if every leaf of $\mathcal{G}$ is a quasigeodesic in the respective leaf of $\mathcal{F}$ containing it when lifted to the universal cover of the leaf. In that case, we say that $\mathcal{G}$ is a leafwise quasigeodesic subfoliation of $\mathcal{F}$.

In Lemma 2.2, the flow lines of Φ_t are shown to be *leafwise quasigeodesics* in the leaves of $\mathcal{F}^{ws}$ and $\mathcal{F}^{wu}$.

The leaves in $\widetilde{\mathcal{F}}^{ws}$ and $\widetilde{\mathcal{F}}^{wu}$ are Gromov hyperbolic with respect to the Riemannian metric on the leaves induced from the metric on $\widetilde{\mathcal{M}}$ [**Fen94**]. Suppose that L is a leaf either in $\widetilde{\mathcal{F}}^{ws}$ or in $\widetilde{\mathcal{F}}^{wu}$. As the leaves are Gromov hyperbolic, we can define the ideal boundary

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of L which is homeomorphic to the circle and we denote it as $S^1(L)$. The compactification $L \cup S^1(L)$ is homeomorphic to a closed disk. As the flow lines are quasigeodesics in L, they define two distinct ideal points on $S^1(L)$: if γ is a flow line in L, then the forward ray of γ defines an unique ideal point on $S^1(L)$ as γ is a quasigeodesic, which is called the *forward or positive ideal point* of γ. Similarly, we define the *backward or negative ideal point* as the limit of the ray in the backward direction. The following statement describes the equivalence between the forward and backward flow rays in the leaves of $\widetilde{\mathcal{F}}^{ws}$ and $\widetilde{\mathcal{F}}^{wu}$, and the points on their ideal boundaries.

Property 2.4. [Fen94] For a leaf L either in $\widetilde{\mathcal{F}}^{ws}$ or $\widetilde{\mathcal{F}}^{wu}$, all the points on $S^1(L)$ correspond to forward or backward flow rays on L. If $L \in \widetilde{\mathcal{F}}^{ws}$, then all the flow lines on L have a common forward ideal point and all the other ideal points are backward ideal points on $S^1(L)$ of the flow lines. No two different flow lines define a common negative or backward ideal point.

If $L \in \widetilde{\mathcal{F}}^{wu}$, then all the flow lines have a common backward ideal point and all the forward flow lines define all the other ideal points on $S^1(L)$. No two different flow lines define the same positive or forward ideal point.

The property for forward ideal points in $\widetilde{\mathcal{F}}^{ws}$ is immediate as these flow lines are forward asymptotic, a direct consequence of the definitions. The property for backward ideal points in leaves of $\widetilde{\mathcal{F}}^{ws}$ is not as immediate and is proved in [Fen94].

Definition 2.5. Suppose that $\mathcal{G}$ is a leafwise quasigeodesic subfoliation of $\mathcal{F}$. If a leaf L of $\widetilde{\mathcal{F}}$ has all leaves of $\widetilde{\mathcal{G}}$ in it sharing a common ideal point, then the projected leaf $\pi(L)$ of $\mathcal{F}$ in $\mathcal{M}$ is called a *funnel leaf*. In this case, the common ideal point shared by all the flow lines in L is called the *funnel point* of L.

COROLLARY 2.6. *By Property 2.4, for an Anosov flow Φ_t on a 3-manifold $\mathcal{M}$, with the flow foliation a leafwise quasigeodesic subfoliaton of both $\mathcal{F}^{ws}$ and $\mathcal{F}^{wu}$, the following happens: all the leaves in weak stable foliation $\mathcal{F}^{ws}$ and weak unstable foliation $\mathcal{F}^{wu}$ are funnel leaves, as shown in Figure 1.*

2.1. *Basic sets of Anosov flows on 3-manifolds.* The Anosov flow Φ is called *transitive* if there exists a flow line γ dense in $\mathcal{M}$, otherwise the flow is *non-transitive*. The first example of a non-transitive Anosov flow was constructed by John Franks and Bob Williams in their 1980's article [FW80].

A point $x \in \mathcal{M}$ is called *non-wandering* if for any open neighborhood U of x and any $t_0 > 0$, there exists $t > t_0$ such that $\Phi_t(U) \cap U \neq \emptyset$, the set of all non-wandering points is denoted by $\Omega(\Phi)$. For a non-transitive Anosov flow Φ_t, the non-wandering set $\Omega(\Phi)$ is not equal to the whole manifold $\mathcal{M}$ and according to *spectral decomposition theorem* [Sma67], $\Omega(\Phi)$ is decomposed into finitely many closed, disjoint, Φ_t-invariant, and transitive *basic* sets $\{\Lambda_i, i = 1, \ldots, n\}$, so $\Omega(\Phi) = \bigsqcup_{i=1}^n \Lambda_i$.

Suppose Λ is a basic set of a non-transitive Anosov flow Φ_t on a 3-manifold. Then Λ can be characterized into four different types [Bru93, Sma67]:

- $\dim(\Lambda) = 2$, and the basic set Λ is an *attractor*, i.e. there exists an open set U containing Λ such that $\bigcap_{t>0} \Phi_t(U) = \Lambda$;

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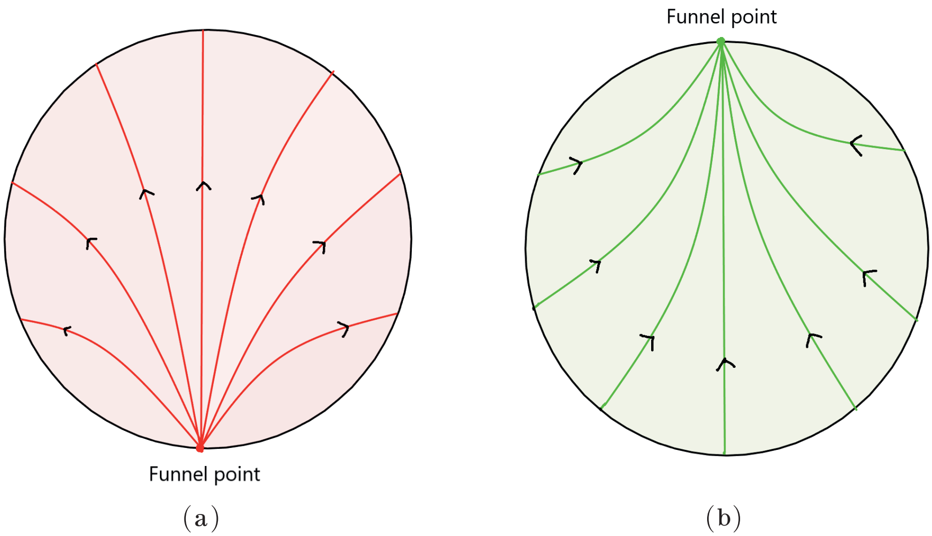

FIGURE 1. Geometry of flow lines on the leaves in $\widetilde{\mathcal{F}}^{wu}$ (a) and $\widetilde{\mathcal{F}}^{ws}$ (b).

- $\dim(\Lambda) = 2$, and the basic set Λ is a *repeller*, i.e Λ is an attractor for the reversed flow $\Psi_t = \Phi_{-t}$;
- $\dim(\Lambda) = 1$, and Λ is a saddle with local cross section a Cantor set;
- $\dim(\Lambda) = 1$, and λ is a hyperbolic periodic orbit.

Property 2.7. If Λ is an *attractor*, then Λ is saturated by weak unstable leaves. If Λ is a *repeller*, then Λ is saturated by weak stable leaves.

From now on we assume the following.

Assumption 2.8. We assume throughout that the Anosov flow Φ on $\mathcal{M}$ is non-transitive and its non-wandering set Ω consists of two-dimensional basic sets only.

In other words, we assume that Φ has no one-dimensional basic set. As $\mathcal{M}$ is compact, there exits at least one attracting basic set and one repelling basic set. Suppose $\mathcal{A}$ denotes the union of all attracting basic sets and $\mathcal{R}$ denotes the the union of all repelling basic sets. We will denote the collection of all lifts of $\mathcal{A}$ in $\widetilde{\mathcal{M}}$ by $\widetilde{\mathcal{A}}$. Here, $\widetilde{\mathcal{A}}$ is the the attracting set for $\widetilde{\Phi}_t$ defined on $\widetilde{\mathcal{M}}$. The union of all lifts of $\mathcal{R}$ is denoted by $\widetilde{\mathcal{R}}$ similarly.

Property 2.9. [**KH95**] Suppose γ is a flow line not contained in $\mathcal{A}$ or $\mathcal{R}$. Then there exists a flow line in $\mathcal{A}$, say α, such that the forward rays of γ and α are asymptotic in $\mathcal{M}$. Similarly, there exits a flow line β in $\mathcal{R}$ such that the backward rays of γ and β are asymptotic in $\mathcal{M}$.

Proof. This is classical [**KH95**], we explain briefly. Given the orbit γ, it gets closer and closer to the attractor $\mathcal{A}$ in future time. Fix x in γ. Every point in the attractor has a local product structure, see, for example, Proposition 6.4.21 of [**KH95**]. Hence for a t sufficiently

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big, $\Phi_t(x)$ is ϵ near the attractor where ϵ is smaller than the size of product boxes of the hyperbolic set $\mathcal{A}$. Hence, $\Phi_t(x)$ is ϵ near some point z in $\mathcal{A}$ and there is w in $\mathcal{A}$ near z so that $\Phi_t(x)$ is in the stable manifold of w because of the local product structure in sets of size ϵ. This proves the result. $\square$

The attractor is saturated by leaves of $\mathcal{F}^{wu}$ and the repeller saturated by leaves of $\mathcal{F}^{ws}$. In the property above, one can choose the flow line α in the attractor $\mathcal{A}$ to be contained in the boundary of the attractor. This means the following: let $x \in \alpha$ and L the $\mathcal{F}^{ws}$ leaf through x. Let D be a small disk in L with x in the interior. The local flow line of x cuts D into two components D_1, D_2 (which are also disks). The condition is that one of D_1 or D_2 does not intersect the attractor $\mathcal{A}$. Suppose it is D_1. The 'D_1 side' of α in L is the side so that γ is getting increasingly closer to α.

3. *The foliation $\mathcal{F}$*

Throughout the article, we will fix a non-transitive Anosov flow Φ as in the previous section, that is, Φ has only two-dimensional basic sets.

To prove our results, we will consider a two-dimensional foliation $\mathcal{F}$ in $\mathcal{M}$ such that:

- on the attractor $\mathcal{A}$, $\mathcal{F}|_{\mathcal{A}} = \mathcal{F}^{wu}|_{\mathcal{A}}$;
- on the repeller $\mathcal{R}$, $\mathcal{F}|_{\mathcal{R}} = \mathcal{F}^{ws}|_{\mathcal{R}}$;
- on $\mathcal{M} \setminus \{\mathcal{A} \cup \mathcal{R}\}$, $\mathcal{F}$ is transversal to both $\mathcal{F}^{ws}$ and $\mathcal{F}^{wu}$;
- every leaf $L \in \mathcal{F}$ is subfoliated by the flow lines of Φ, i.e. every leaf L is $\Phi_{\mathbb{R}}$-invariant.

We will denote the lift of $\mathcal{F}$ in the universal cover $\widetilde{\mathcal{M}}$ by $\widetilde{\mathcal{F}}$. Leaves of $\widetilde{\mathcal{F}}$ in $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{R}}$ look like the leaves in Figure 1. Leaves in $\widetilde{\mathcal{M}} \setminus (\widetilde{\mathcal{A}} \cup \widetilde{\mathcal{R}})$ are described in Figure 2. It is not immediate from the definition why the leaves not contained in $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{R}}$ are as described in Figure 2, but we will prove this later in this article.

THEOREM 3.1. *There exists foliations $\mathcal{F}$ with the properties described above.*

Start of the proof of Theorem 3.1. We start with an Anosov flow as described above. For simplicity, assume that M is orientable as well. There is a finite collection of disjoint tori $\{T_i\}$ transverse to the flow Φ which separate the basic sets [**Bru93**, **Sma67**]. We choose T_i to be smooth. The collection of tori is supposed to be minimal respective to the property that if an orbit is not in $\mathcal{R}$ or $\mathcal{A}$, then it intersects one of the $\{T_i\}$. Let γ be such an orbit intersecting a specific T_i, let x be a point in the intersection. Then the forward orbit of x is asymptotic to a component A of the attractor $\mathcal{A}$—this uses the important fact that there are no one-dimensional components of the non-wandering set of Φ by assumption. The set of such x so that the forward ray of x is asymptotic to A is open in T_i. This holds for any component A of the attractor $\mathcal{A}$. Since the union over such components of $\mathcal{A}$ is all of T_i and T_i is connected, it follows that all orbits in T_i are forward asymptotic to a single component A of $\mathcal{A}$.

In a similar way, one proves that if T_1, T_2 are tori contained in the complement of the union of the attractor and repeller, and T_1, T_2 intersect a common orbit of Φ, then T_1, T_2 intersect exactly the same set of orbits of Φ. In other words, if B is a component of $\mathcal{M} - (\mathcal{A} \cup \mathcal{R})$, then there is a torus T contained in B, transverse to Φ so that B is the flow saturation of T. Hence, we can choose a minimal collection $\{T_i\}$ of tori transverse to Φ and

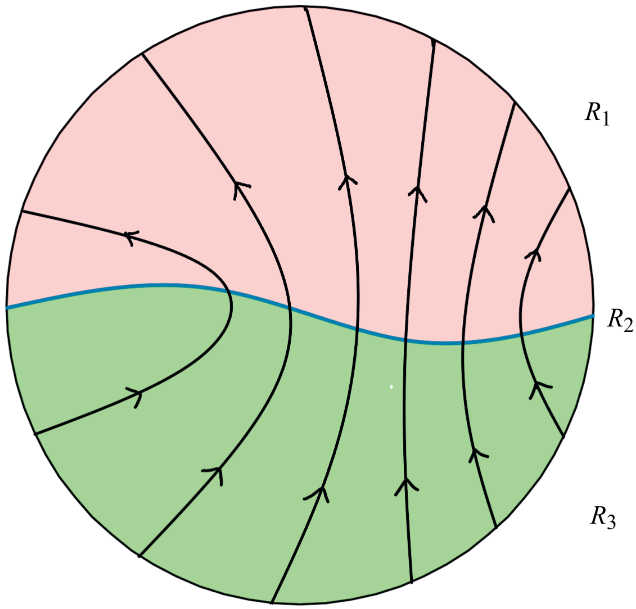

FIGURE 2. An example of a leaf $L \in \widetilde{\mathcal{F}}$ not contained in $\widetilde{\mathcal{A}}$ or $\widetilde{\mathcal{R}}$. In R_1, forward rays are asymptotic to $\widetilde{\mathcal{A}}$; in R_3, backward rays are asymptotic to $\widetilde{\mathcal{R}}$; R_2, the blue line, represents the intersection of L with some lift $\widetilde{T}_i$ of some torus T_i.

intersecting all orbits in the complement of $\mathcal{A} \cup \mathcal{R}$, and any such orbit intersects a unique T_i and only once.

3.1. *Construction of* $\mathcal{F}$. Now we construct $\mathcal{F}$. The foliations $\mathcal{F}^{ws}, \mathcal{F}^{wu}$ are C^0 with C^1 leaves [**KH95**], and so are the intersections with each T_i. On each T_i, choose a one-dimensional C^1 foliation F_i transverse to both

$$\mathcal{F}^{ws} \cap T_i, \quad \text{and} \quad \mathcal{F}^{wu} \cap T_i.$$

Saturate F_i by the flow to produce a two-dimensional foliation in the flow saturation of T_i. Note that the leaves in the flow saturation are either a plane or an infinite annulus. Let $\mathcal{F}$ be this foliation in the complement of the attractor union the repellor. Figure 2 describes a possible leaf in the lift $\widetilde{\mathcal{F}}$ of $\mathcal{F}$ to $\widetilde{\mathcal{M}}$, where R_2, the blue line, represents its intersection with some lift of T_i. The figure depicts the following several properties that we are going to prove later and that are essential to the results of this article: (1) we will show later that leaves of $\widetilde{\mathcal{F}}$ are Gromov hyperbolic; (2) we will also show that for $L \in \widetilde{\mathcal{F}}$ not in the lift of the attractor or repeller, each flow ray in L converges to a single point in $S^1(L)$ and distinct flow rays do not forward converge to the same ideal point in $S^1(L)$. Similarly for backward flow rays.

A flow line that does not intersect any T_i has to lie either in the attractor ($\mathcal{A}$) or in the repeller ($\mathcal{R}$). We define $\mathcal{F}$ to be $\mathcal{F}^{wu}$ in the attractor, $\mathcal{F}^{ws}$ in the repeller, and the saturation of the F_i everywhere else.

3.2. *Properties of $\mathcal{F}$.* At this point, $\mathcal{F}$ is just a collection of two-dimensional subsets of $\mathcal{M}$. We will prove that $\mathcal{F}$ is a foliation of $\mathcal{M}$. Clearly, $\mathcal{F}$ is a foliation in the complement of the union of the attractor and the repeller, because this is an open set and because of the definition of $\mathcal{F}$: each component $\mathcal{C}$ of $\mathcal{M} \setminus (\mathcal{A} \cup \mathcal{R})$ is equal to $\Phi_{\mathbb{R}}(T_i)$ for some T_i and this is homeomorphic to $T_i \times \mathbb{R}$ with the product topology (the topology in T_i is induced from $\mathcal{M}$). The foliation $\mathcal{F}$ in $\mathcal{C}$ is equivalent to the foliation $F_i \times \mathbb{R}$ in $T_i \times \mathbb{R}$.

The interaction between $\mathcal{F}$ in $\mathcal{M} - (\mathcal{A} \cup \mathcal{R})$ and $\mathcal{F}$ in $\mathcal{A} \cup \mathcal{R}$ is more complex. There is a subtle point here, which we now explain. Let $\widetilde{\mathcal{F}}$ be the lift of $\mathcal{F}$ to $\widetilde{\mathcal{M}}$. If $\mathcal{F}$ is a foliation, then it will follow that $\widetilde{\mathcal{F}}$ is a foliation of $\widetilde{\mathcal{M}}$ by properly embedded planes. By construction, the 'leaves' of $\mathcal{F}$ intersecting the attractor are contained in the attractor and similarly for the repeller. Therefore, the leaves of $\mathcal{F}$ in the complement of $\mathcal{A} \cup \mathcal{R}$ are entirely contained in the complement of $\mathcal{A} \cup \mathcal{R}$ as well. In particular, if L is a lift of a leaf of $\mathcal{F}$ in the complement of the attractor and repeller, then it should be properly embedded when lifted to $\widetilde{\mathcal{M}}$. As it turns out, this property is not true if one starts with an arbitrary foliation F_i in T_i. Let us review the construction: we start with a foliation F_i in T_i and saturate it by the flow to produce a foliation in an open set in $\mathcal{M}$. Then consider a lift L of a leaf of this foliation to the universal cover. Is L always properly embedded in $\widetilde{\mathcal{M}}$? In general, this is not true. For example, start with the Franks–Williams non-transitive flow [**FW80**], consider a smooth torus T which separates the attractor and repeller, and start with say the intersection of the stable foliation of Φ with T, which we call F. Then for some of the leaves of F, it follows that if L is a lift of the flow saturation to $\widetilde{\mathcal{M}}$, then L is not properly embedded in $\mathcal{M}$. For example, [**FW80**, Fig. 3, p. 164] depicts the foliations induced by the weak stable and unstable foliations in T for the Franks–Williams flow. Each has two Reeb components. Take α to be a leaf of the stable foliation which is not in the interior of a Reeb component, that is, a horizontal line in the figure, and also that α is a closed curve. Lift it to $\widetilde{\alpha}$ in $\widetilde{\mathcal{M}}$. If C is the flow saturation of $\widetilde{\alpha}$, then C is not properly embedded in $\widetilde{\mathcal{M}}$: there is an orbit γ of $\widetilde{\Phi}$ which is not in C but is contained in the closure of C. This orbit γ is the lift of a periodic orbit contained in the attractor of the Franks–Williams flow. The same would happen if we took F to be the intersection of the unstable foliation with T, α a closed leaf of F, and considering the repeller of Φ instead of the attractor.

The reason why our construction of $\mathcal{F}$ as above produces a foliation is because we start with a foliation F_i in T_i which is transverse to both the stable and unstable foliations in T_i. We first prove the following result.

LEMMA 3.2. *Let ℓ be a leaf of F_i and let E be the flow saturation of ℓ. Then, with the induced path metric from $\mathcal{M}$, it follows that E is complete.*

Proof. Let E be the flow saturation of ℓ. Since ℓ is smooth and the flow is C^1 it follows that E is C^1. For any x, y in ℓ if

$$\Phi_t(x) = \Phi_s(y),$$

then $x = y$ and $t = s$, since the component of $\mathcal{M} - (\mathcal{A} \cup \mathcal{R})$ containing ℓ is homeomorphic to $T \times \mathbb{R}$ and ℓ is injectively immersed in T. Hence, E is parameterized as $\ell \times \mathbb{R}$, that is, every point p in E can be represented as (x, t) where $x \in \ell$ and $t \in \mathbb{R}$.

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The Riemannian metric in $\mathcal{M}$ induces a Riemannian metric in E and a path metric in E. What we prove is the following.

CLAIM 1. *There is $a_0 > 0$ so that any point $p = (x, t)$ in E is the center of a metric disk of radius a_0 in E.*

Proof. This is obvious for any point p in ℓ or, in other words, if $t = 0$.

We now prove the claim for $t > 0$ using the unstable foliation. The analogous proof shows the result for $t < 0$ using the stable foliation. The foliation F_i is transverse to both the stable and unstable foliations induced in T_i, hence uniformly transverse to these foliations (which means the angles between F_i and $\mathcal{F}^{wu} \cap T_i$ or $\mathcal{F}^c s \cap T_i$ are uniformly bounded away from 0 on T_i). Given any smoothly embedded curve α in M, let $l_u(\alpha)$ be its unstable length: we integrate only the component of the tangent vector in the direction of the unstable bundle. For example, if α is contained in a weak stable leaf, then $l_u(\alpha)$ is zero, while if α is contained in a strong unstable leaf, then $l_u(\alpha)$ is the same as its length under the Riemannian metric of $\mathcal{M}$. In particular, if α is a curve not contained in a strong stable or unstable leaf, then the original length $l(\alpha)$ is always strictly greater than the unstable length $l_u(\alpha)$.

By the definition of an Anosov flow, there exist constants $C > 0$, $\lambda > 1$ such that if we flow forward a segment with t amount of time, the new unstable length is at least $C\lambda^t$ times the original unstable length. Hence if we let $a_1 = C$, then for any smooth segment, any flow forward of that segment has unstable length which is at least a_1 times the original unstable length.

Since F_i is uniformly transverse to $\mathcal{F}^{ws} \cap T_i$ by our construction, it follows that any point x in T_i is the midpoint of a segment β in its leaf of F_i of unstable length 2. For any $t \geq 0$, the unstable length of $\Phi_t(\beta)$ is at least $2a_1$. This constant a_1 is defined globally. In addition, if v is a non-zero vector tangent to β, then v makes a definite positive angle with the flow direction. Since flowing forward increases the size of unstable vectors more than the size of tangent vectors (where $t > t_0 > 0$ for some t_0 big enough), it follows that there is a global constant $\theta > 0$ so that $D\Phi_t v$ also makes an angle $> \theta$ with the tangent to the flow. Consider the infinitesimal arclengths dt, ds, du along the flow, stable, and unstable bundles. The (non-Riemannian) metric

$$|dt| + |ds| + |du|$$

is quasicomparable (this means Lipschitz equivalent) with the Riemannian metric in M: there is $a_2 > 0$ so that the Riemannian length is at least a_2 times the length in this metric. Consider the following set:

$$A = \Phi_{[t-1,t+1]}(\beta)$$

for $t \geq 0$. The segment β of F_i is contained in the leaf E of $\mathcal{F}$. From any point in the boundary of A to $\Phi_t(x)$ along E, one has to have at least a_1 unstable length and flow length of at least 1. It follows that there is a global constant a_0 (depending only on a_1) so that A contains a disk in the Riemannian metric of radius a_0 and centered at $\Phi_t(x)$.

For $t < 0$, we use the stable foliation and flow backward instead of forward. This finishes the proof of the claim. □

The claim shows that E is complete and finishes the proof of the lemma. □

Continuation of the proof of Theorem 3.1 We consider the collection $\mathcal{F}$ as constructed in the beginning of this section. This object $\mathcal{F}$ is a foliation restricted to $\mathcal{M} - (\mathcal{A} \cup \mathcal{R})$ and this is an open set.

The only remaining thing to prove is that if a sequence x_n in $\mathcal{M} - (\mathcal{A} \cup \mathcal{R})$ converges to x in $\mathcal{A} \cup \mathcal{R}$, then the leaves of $\mathcal{F}$ through x_n converge to the leaf of $\mathcal{F}$ through x. Without loss of generality, we may assume that x is in an attractor.

Let $p_n \in T_i$ so that x_n are in $\Phi_{\mathbb{R}}(p_n)$. There are $t_n \in \mathbb{R}$ with $x_n = \Phi_{t_n}(p_n)$. Since x is in the attractor, then t_n converges to positive ∞. The leaf of $\mathcal{F}$ through p_n is the Φ flow saturation of the leaf of F_i through p_n. The tangent to this two-dimensional set through p_n is generated by the Anosov vector field generating Φ and a tangent vector v to F_i at p_n. The leaf of $\mathcal{F}$ is Φ-flow invariant. Flowing forward, the flow vector remains invariant. The vector v is transverse to the weak stable foliation and hence it flows increasingly more (does not matter how fast) to the weak unstable direction. So flowing forward, these leaves become increasingly more tangent to the $E^0 \oplus E^u$ bundle and limit to leaves of $\mathcal{F}^{wu}$. Since flowing forward limits to the attractor, this shows that the leaves of $\mathcal{F}$ through x_n converge to the leaf of $\mathcal{F}$ through x.

In addition, the previous lemma shows that the leaves of $\mathcal{F}$ through x_n are complete in their path metrics. This shows that $\mathcal{F}$ defines a foliation. We stress that Lemma 3.2 is needed to ensure that $\mathcal{F}$ is a foliation. Otherwise, even if the tangent directions of $\mathcal{F}$ in $\mathcal{M} - (\mathcal{A} \cup \mathcal{R})$ and $\mathcal{F}$ in $\mathcal{A} \cup \mathcal{R}$ match continuously, one would have that leaves of the first set 'arrive' at leaves of the second set in finite distance. In other words, the union of a leaf in $\mathcal{M} - (\mathcal{A} \cup \mathcal{R})$ and a leaf in $(\mathcal{A} \cup \mathcal{R})$ would form a branched surface. This would produce a branching foliation, instead of a foliation.

This finishes the proof of Theorem 3.1. □

We remark that the construction of $\mathcal{F}$ highlights why our methods do not work when there are one-dimensional basic sets. For simplicity, suppose that there is a basic set which is a periodic orbit γ. There is a torus T so that negative saturation limits on γ. If we start with F in T transverse to both $\mathcal{F}^{ws} \cap T$ and $\mathcal{F}^{wu} \cap T$, then flowing backward will make it limit to the weak stable leaf of γ. So the weak stable leaf of γ is in the collection $\mathcal{F}$ so constructed. However, there is also a torus T' so that the forward flow saturation limits on γ. The similar argument shows that the weak unstable foliation of γ also has to lie in the collection $\mathcal{F}$. Hence the collection $\mathcal{F}$ has sets which intersect transversely and cannot be a foliation.

Remark 3.3. By construction, the foliation $\mathcal{F}$ does not have compact leaves: any leaf in $\mathcal{R} \cup \mathcal{A}$ is not compact as they are weak stable leaves of an Anosov flow. Each leaf in $\mathcal{M} - (\mathcal{R} \cup \mathcal{A})$ limits on $\mathcal{R}$ and hence cannot be compact. Since $\mathcal{F}$ does not have compact leaves, it follows from Novikov's theorem [**Cal01**] that leaves of $\widetilde{\mathcal{F}}$ are properly embedded planes in $\widetilde{\mathcal{M}}$.

To prove Theorem 1.1, we will prove the following properties for such a foliation $\mathcal{F}$:

(1) the flow lines are *leafwise quasigeodesics* in leaves of $\mathcal{F}$;

(2) every leaf of $\mathcal{F}$ not contained in $\mathcal{A}$ or $\mathcal{R}$ is a non-funnel leaf, as in Figure 2.

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4. *Gromov hyperbolicity of the leaves of $\mathcal{F}$*

We will consider a foliation $\mathcal{F}$ as constructed in the previous section.

In this section, we will show that there exists a metric g such that every leaf of the foliation $\mathcal{F}$ is Gromov hyperbolic. By *Candel's uniformization theorem*, this condition is equivalent to the fact that every holonomy invariant non-trivial measure μ on $\mathcal{M}$ has Euler characteristic $\chi_\mu(\mathcal{M}, \mathcal{F}) < 0$, which includes the case when there exists no invariant measure. For more details about the Euler characteristic, see [**Can93**] or [**CC00**]. In our context, we will prove that there is no holonomy invariant transverse measure to $\mathcal{F}$. Candel's theorem requires that the foliation has C^∞ leaves. To obtain that, we use Calegari's result [**Cal01b**] which implies that $\mathcal{F}$ is isotopic to a foliation with C^∞ leaves. This does not change the property that $\mathcal{F}$ has or does not have holonomy invariant transverse measures. Once this is obtained, Candel proved that there is a metric in $\mathcal{M}$ inducing a smooth Riemannian metric in the leaves so that curvature in each leaf of $\mathcal{F}$ is constant equal to -1. A precise statement can be found in [**Can93**], [**CC00**], or [**Cal01**]. We call such a metric a *Candel metric*. This Candel metric is not smooth in the transverse direction.

Here is the precise statement on the equivalence of Gromov hyperbolicity of leaves of a foliation and negative Euler characteristic of a positive invariant measure.

PROPOSITION 4.1. [**Can93**] *Let (M, F) be a compact oriented surface lamination with a Riemannian metric g. Then $\chi(M, \mu) < 0$ for every positive invariant transverse measure μ if and only if there is a metric in M which induces a metric in each leaf of F which makes it into a hyperbolic surface. In particular, this holds true if M has no invariant measure.*

To prove that all the leaves of $\mathcal{F}$ are Gromov hyperbolic, we will show that there does not exist any invariant measure. We will argue by contradiction, we assume that there exists a invariant measure μ, and we will attain a contradiction.

The *support* of μ on $\mathcal{M}$, denoted by $supp(\mu)$, is defined as the collection of all points $x \in \mathcal{M}$ such that if τ is a one-dimensional manifold transverse to $\mathcal{F}$ which contains x in its interior, then $\mu(\tau) > 0$. The support of a holonomy invariant transverse measure is a closed set and it is saturated by $\mathcal{F}$, which means $supp(\mu)$ is a union of leaves of $\mathcal{F}$. The orientation hypothesis is not essential as it can be achieved by a double cover. The double cover does not change the conformal type of any leaf.

LEMMA 4.2. *The support of μ on $\mathcal{M}$ contains at least one leaf from the attractor $\mathcal{A}$ or the repeller $\mathcal{R}$.*

Proof. Consider a point $x \in supp(\mu)$ and suppose L_x is the leaf in $\mathcal{F}$ which contains x, then $L_x \subset supp(\mu)$ as $supp(\mu)$ is $\mathcal{F}$-saturated. If $x \in \mathcal{A}$, then $L_x \subset \mathcal{A}$ and the claim is true. Similarly if x is in $\mathcal{R}$, then its leaf is contained in $supp(\mu)$. Finally suppose that $x \notin (\mathcal{A} \cup \mathcal{R})$. Then consider the sequence $\{\Phi_n(x)\}$ as $n \to \infty$. Let z be an accumulation point of $\{\Phi_n(x)\}$. As $supp(\mu)$ is closed, z is in $supp(\mu)$ and hence $L_z \subset supp(\mu)$. Since z is an accumulation point of $\Phi_n(x)$, it implies that z is a non-wandering point, and hence $z \in \mathcal{A} \cup \mathcal{R}$ and $L_z \subset (\mathcal{A} \cup \mathcal{R}) \cap supp(\mu)$. In fact, since $n \to \infty$, it follows that z is in the attractor, so $L_z \subset \mathcal{A}$. □

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Suppose L is a leaf in $supp(\mu)$ which is contained in $\mathcal{A}$ (assume in $\mathcal{A}$ without loss of generality). By [**Pla75**, Theorem 6.3], we know that if μ is a holonomy invariant transverse measure on a compact manifold foliated by a codimension-one foliation $\mathcal{F}$, then any leaf contained in $supp(\mu)$ has polynomial growth. Then the leaf L_z in the attractor $\mathcal{A}$, as obtained in the previous paragraph, has polynomial growth. Recall that the leaves of $\mathcal{F}$ are either planes or annuli. At the same time, L_z is contained in the attractor and each leaf in the attractor belongs to the weak unstable foliation of the Anosov flow Φ. However, weak stable and weak unstable leaves of Anosov flows have exponential growth, a contradiction.

As each leaf $L \in \widetilde{\mathcal{F}}$ is Gromov hyperbolic with respect to the path metric from the induced Riemannian metric from $\widetilde{\mathcal{M}}$, we can define the circle at infinity or the ideal boundary $S^1(L)$ of each leaf L.

Next we will describe the topology we will use on the spaces

$$S^1(\widetilde{\mathcal{M}}) = \bigcup_{L\in\widetilde{\mathcal{F}}} S^1(L) \quad \text{and}$$

$$\widetilde{\mathcal{M}} \cup S^1(\widetilde{\mathcal{M}}) = \bigcup_{L\in\widetilde{\mathcal{F}}} (L \cup S^1(L)).$$

For this, we will assume first that $\mathcal{M}$ has a Candel metric.

Suppose τ is an open segment homeomorphic to (0,1) and transversal to $\widetilde{\mathcal{F}}$. We define the the following sets

$$\mathcal{P}_\tau = \bigcup_{y\in\tau} S^1(L_y) \quad \text{and} \quad \mathcal{Q}_\tau = \bigcup_{y\in\tau} (L_y \cup S^1(L_y))$$

If $T^1(\tau)$ denotes the unit tangent bundle of $\widetilde{\mathcal{F}}$ restricted to τ, then $T^1(\tau)$ is naturally homeomorphic to the standard cylinder. The natural identification between $T^1(\tau)$ and $\mathcal{P}_\tau$ induces the topology on $\mathcal{P}_\tau$ homeomorphic to the standard annulus. In [**Fen02**], it is proved that this topology is independent of the particular transversal τ that is chosen intersecting the same sets of leaves of $\widetilde{\mathcal{F}}$. This is because the metrics induced in $S^1(L)$ from the visual metric in any point are Hölder equivalent.

Similarly, $\mathcal{Q}_\tau$ has a natural topology homeomorphic to the standard solid cylinder.

The collection of all $\mathcal{P}_\tau$ sets over a $\pi_1(\mathcal{M})$-invariant discrete collection of transversals defines a topology on $S^1(\widetilde{\mathcal{M}})$. Similarly, the collection of $\mathcal{Q}_\tau$ sets over the same collection of transversals defines a topology on $\widetilde{\mathcal{M}} \cup S^1(\widetilde{\mathcal{M}})$. Deck transformations act by homeomorphisms on both sets. For more details, see [**Fen02**], [**Cal00**], or [**Cal01**].

After the fact, it is easy to see that the topologies described are independent of the specific metric in $\mathcal{M}$ chosen and also work for any Riemannian metric in $\mathcal{M}$.

5. *Properties of flow lines*

This section describes the behavior of forward rays of flow lines, in particular their asymptotic behavior toward the the boundary at infinity $\bigcup\{S^1(L) | L \in \widetilde{\mathcal{F}}\}$. In particular, we will prove that the rays are quasigeodesics in their respective leaves of $\widetilde{\mathcal{F}}$. Notice that this is definitely much weaker than saying that full flow lines are quasigeodesics in their respective leaves. We will also show that in some leaves, the forward ideal points are

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pairwise distinct and the negative ideal points are also pairwise distinct. In particular, even if the flow foliation is a leafwise quasigeodesic subfoliation of $\mathcal{F}$, it will not have the funnel property.

We now introduce a family of sets in $\widetilde{\mathcal{M}}$ which will be extremely useful for us.

5.1. *The sets $\mathcal{U}$.* Consider an arbitrary point $x \in \widetilde{\mathcal{A}} \subset \widetilde{\mathcal{M}}$ and the forward ray from x which is denoted by

$$\gamma_x^+ = \widetilde{\Phi}_{[0,\infty)}(x)$$

starting at x, and let $L_x \subset \widetilde{\mathcal{A}}$ be the leaf containing γ_x^+. Recall that in the attractor $\mathcal{A}$, the foliation $\mathcal{F}$ is equal to $\mathcal{F}^{wu}$, and hence transverse to $\mathcal{F}^{ws}$. Therefore, the foliations $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{F}}^{ws}$ are transversal to each other near $\widetilde{\mathcal{A}}$.

Let U be a compact rectangle transverse to the flow and with x in the interior of U. We assume that U is contained in the foliation boxes of all foliations $\widetilde{\mathcal{F}}$, $\widetilde{\mathcal{F}}^s$, and $\widetilde{\mathcal{F}}^{ws}$ such that U is made up of a union of stable segments, every one of which intersects the local strong unstable segment through x. Consider the set

$$\mathcal{U} = \widetilde{\Phi}_{[0,\infty)}(U).$$

The set $\mathcal{U}$ is a neighborhood of the forward ray $\widetilde{\Phi}_{[0,\infty)}(x)$. We can assume that $\mathcal{U}$ is homeomorphic to $[-1, 1] \times [-1, 1] \times [0, \infty)$ with $x = (0, 0, 0)$ and we can define coordinates on $\mathcal{U}$ such that the following hold.

- U is identified with $[-1, 1] \times [-1, 1] \times \{0\}$ and points on U are represented as $(r, s, 0)$ for $r, s \in [-1, 1]$. In particular, $x = (0, 0, 0)$.
- For a point $y = (r, s, 0)$, $\widetilde{\Phi}_t(y)$ has coordinates (r, s, t), that is, the ray $\{(r, s, t) | t \in [0, \infty)\}$ represents the ray $\Phi_{[0,\infty)}(y)$.
- For a point $y' = (r', s, t') \in \mathcal{U}$, $P_{y'}$ denotes the horizontal infinite strip

$$P_{y'} = \{(r, s, t) | r \in [-1, 1], t \in [0, \infty)\}.$$

  The infinite strip $P_{y'}$ is contained in the leaf $L_{y'} \in \widetilde{\mathcal{F}}$ which contains y'.
- For a point $y' = (r, s', t') \in \mathcal{U}$, $Q_{y'}$ denotes the vertical infinite strip

$$Q_{y'} = \{(r, s, t) | s \in [-1, 1], t \in [0, \infty)\}.$$

  The infinite strip $Q_{y'}$ is contained in the leaf $E_{y'} \in \widetilde{\mathcal{F}}^{ws}$ which contains y'.

As $x = (0, 0, 0) \in \widetilde{\mathcal{A}}$, the leaf of $\widetilde{\mathcal{F}}$ through x is actually the weak unstable leaf of $\widetilde{\Phi}$ through x, and hence P_x is contained in the $\widetilde{\mathcal{F}}^{wu}$ leaf through x.

The sets $\mathcal{U}$ will be used throughout this section. Any such particular set $\mathcal{U}$ is completely determined by the rectangle U.

We can define a projection map $\Pi : \mathcal{U} \to P_x$ by the formula $\Pi(y) = S_y \cap P_x$, where S_y is the one-dimensional leaf of the strong stable foliation $\widetilde{\mathcal{F}}^s$ containing y. This is possible because one can do that in the original rectangle U as it is a union of strong stable segments, and then $\mathcal{U}$ is the flow forward saturation of U and the maps $\widetilde{\Phi}_t$ preserve the strong stable foliation in $\widetilde{\mathcal{M}}$. These projection maps are well defined and continuous because of the foliation structures on $\mathcal{U}$.

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Observation 5.1. Here we list out all of the important observations from the above construction of $\mathcal{U}$ which we need in the rest of the article.

(1) For any $y \in \mathcal{U}$, the rays $\widetilde{\Phi}_{[0,\infty)}(y)$ and $\widetilde{\Phi}_{[0,\infty)}(\Pi(y))$ are asymptotic as they lie on the same weak stable leaf.

(2) We can assume that lengths of all the line segments $\{(r, s, 0)|s \in [-1, 1]\}$ are less than a fixed $\epsilon > 0$ in $\widetilde{\mathcal{M}}$. Without loss of generality, we assume ϵ is small enough that for any point $p \in \mathcal{M}$, the $\epsilon-$neighborhood of p is contained in a covering neighborhood of all the foliations $\mathcal{F}, \mathcal{F}^u, \mathcal{F}^s, \mathcal{F}^{wu}, \mathcal{F}^{ws}$.

(3) We have considered a Candel metric on the leaves of $\widetilde{\mathcal{F}}$ in $\widetilde{\mathcal{M}}$ and the Candel metric varies continuously on the leaves transversally.

The line segment $\lambda = \{(0, s, 0)|s \in [-1, 1]\}$ is transversal to $\widetilde{\mathcal{F}}$. Consider the open sets $\mathcal{V} = \bigcup_{x' \in \lambda} S^1(L_{x'})$ and $\mathcal{W} = \bigcup_{x' \in \lambda}(L_{x'} \cup S^1(L_{x'}))$.

Definition 5.2. Let γ be a flow line of $\widetilde{\Phi}$ contained in a leaf L of $\widetilde{\mathcal{F}}$. Given z in γ, if the forward ray of γ converges to a single point of $S^1(L)$, we let this be $\eta^+(z)$. Similarly define $\eta^-(z)$. In addition, given a point a in $\widetilde{\mathcal{M}}$, let γ_a be the flow line of $\widetilde{\Phi}$ containing a.

At this point, we only know that flow lines on $\widetilde{\mathcal{A}} \cup \widetilde{R}$ are leafwise quasigeodesics (by Lemma 2.2). Hence for $x \in \widetilde{\mathcal{A}} \cup \widetilde{R}$, both $\eta^+(x)$ and $\eta^-(x)$ are well-defined points in $S^1(L)$. The next lemma shows that every flow ray on $\widetilde{\mathcal{M}}$ is leafwise quasigeodesic (in the respective leaf of $\widetilde{\mathcal{F}}$), and hence $\eta^+(x)$ and $\eta^-(x)$ are well defined for any arbitrary $x \in \widetilde{\mathcal{M}}$. The proof of this lemma is quite involved.

LEMMA 5.3. *For any $w \in \widetilde{\mathcal{M}}$, the forward and the backward rays of the flow line $\gamma_w = \widetilde{\Phi}_{\mathbb{R}}(w)$ are quasigeodesics on the leaf L_w in $\widetilde{\mathcal{F}}$ which contains the flow line.*

Proof. In §4, we proved that $\mathcal{F}$ does not admit any holonomy invariant transverse measure, so by Candel's theorem, $\mathcal{M}$ admits a Candel metric. To prove the current lemma, we assume a Candel metric in $\mathcal{M}$ so that leaves of $\mathcal{F}$ are hyperbolic surfaces. This metric is not Riemannian, but the result is independent of the metric.

By Lemma 2.2, every forward or backward ray in a leaf in $\widetilde{\mathcal{F}}^{wu}$ or $\widetilde{\mathcal{F}}^{ws}$ is quasigeodesic in its respective leaf. In particular, every flow line is a quasigeodesic in the respective leaf of $\mathcal{F}$ if contained in the attractor or repeller.

So we may assume that the ray is in a leaf not in the attractor or repeller. Property 2.9 shows that every forward (respectively backward) ray is asymptotic with a ray in the attractor (respectively repeller). We will prove the result for a forward ray and the backward case is similar. Suppose γ denotes a forward ray not contained in the attractor. By taking a subray, we may assume that the ray γ is in the weak stable leaf of a point x in the attractor and the initial point w of the ray γ is very near x and contained in the strong stable segment of x. Hence we may assume that the initial point is contained in a local cross section U to $\widetilde{\Phi}$ which is a rectangle centered at x, as described in the construction of the set $\mathcal{U}$ in the beginning of this section. Let L_x be the leaf of $\widetilde{\mathcal{F}}$ containing x, and similarly define L_w.

Recall that L_x is also the weak unstable leaf of $\widetilde{\Phi}$ containing x.

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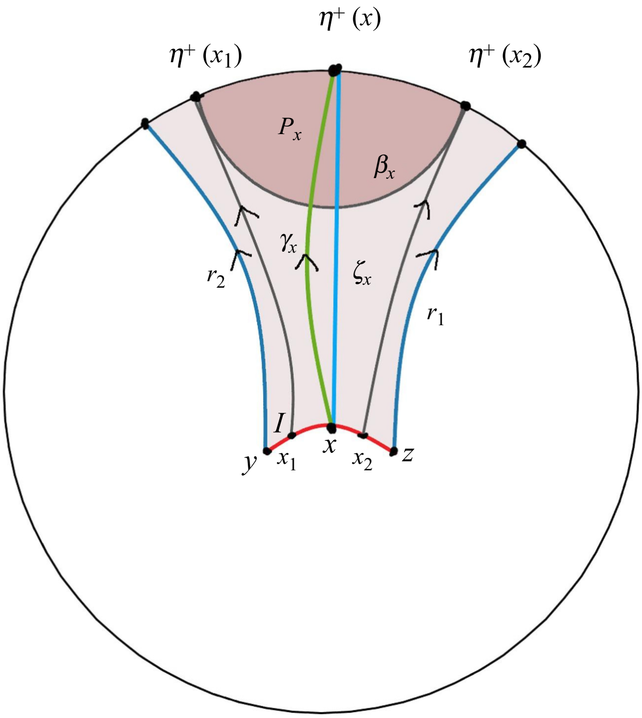

FIGURE 3. The region A_x in L_x and the half-space P_x. The region A_x is the region bounded by the curve $c = r_1 \cup I \cup r_2$.

Therefore, it is sufficient to show that every forward ray in the set $\mathcal{U}$ described above is quasigeodesic in its respective leaf of $\widetilde{\mathcal{F}}$. In the leaf L_x through x, we consider a curve c as follows.

Let I be the compact unstable segment $U \cap L_x$ which has endpoints z, y. Let $r_1 = \widetilde{\Phi}_{[0,\infty)}(z)$ and $r_2 = \widetilde{\Phi}_{[0,\infty)}(y)$ be the forward rays of $\widetilde{\Phi}$ through z and y. Then $c := r_1 \cup I \cup r_2$ is the bi-infinite curve on L_x, as shown in Figure 3.

The two rays r_1 and r_2 are quasigeodesics in L_x by Lemma 2.2 as L_x is a weak unstable leaf of $\widetilde{\Phi}$. Moreover, they converge to distinct ideal points in $S^1(L_x)$. Let ν be the interval in $S^1(L_x)$ bounded by these ideal points and containing the ideal point of $\widetilde{\Phi}_{[0,\infty)}(x)$.

The curve c bounds a region A_x in L_x (as in Figure 3) which is exactly $\widetilde{\Phi}_{[0,\infty)}(I)$. The region A_x is contained in $\mathcal{U}$, in fact,

$$A_x = \widetilde{\Phi}_{[0,\infty)}(I) = \mathcal{U} \cap L_x.$$

This region contains a half plane in L_x, which we denote as P_x, as shown in Figure 3.

Recall that we are considering w, a point in $U \cap \widetilde{\mathcal{F}}^s(x)$, where $\widetilde{\mathcal{F}}^s(x)$ is the strong stable leaf of x, in other words, $\Pi(w) = x$ in $\mathcal{U}$ according to the definition of the map Π above. Let J be the intersection of $L_w \cap U$, where L_w is the leaf of $\widetilde{\mathcal{F}}$ through w. Then $B_w := \widetilde{\Phi}_{[0,\infty)}(J)$ is contained in L_w and contained in $\mathcal{U}$. In addition, $\Pi(B_w) = A_x$.

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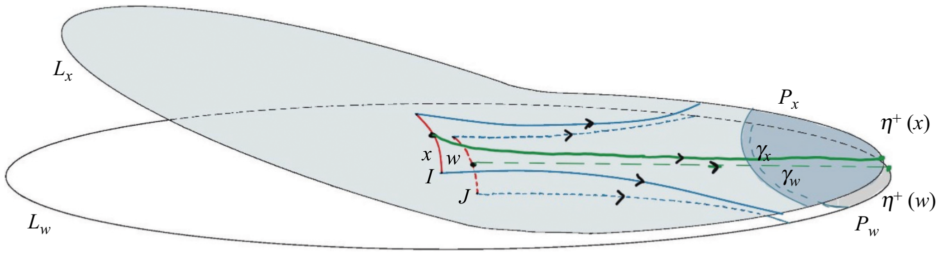

FIGURE 4. P_x in L_x is asymptotic to P_w in L_w.

Since every point in J is in the strong stable leaf of a point in I, it follows that every flow ray in B_w is asymptotic to a flow ray in A_x by Observation 5.1(2). In fact, as points leave compact sets in B_w, they become closer and closer to A_x.

The flow ray $r_x = \widetilde{\Phi}_{[0,\infty)}(x)$ is quasigeodesic on L_x by Lemma 2.2 as L_x is a weak unstable leaf of $\widetilde{\Phi}$. We want to show that $r_w = \widetilde{\Phi}_{[0,\infty)}(w)$ is also quasigeodesic with respect to the induced path metric on L_w. The key idea of the proof is as follows: the induced metrics on the leaves $\widetilde{\mathcal{F}}$ vary continuously and the region $A_x \subset L_x$ is very close to $B_w \subset L_w$. As r_x is quasigeodesic in its leaf and asymptotic to the ray $r_w = \widetilde{\Phi}_{[0,\infty)}(w)$, it follows that the other ray is also a quasigeodesic in its $\widetilde{\mathcal{F}}$ leaf.

Next we provide more specific details. In the leaf L_x, choose two points x_1, x_2 in I with x in between them so that the geodesic β_x in L_x with ideal points $\eta^+(x_1), \eta^+(x_2)$ is contained in the interior of A_x. This is possible since the flow lines in L_x are uniform quasigeodesics and they spread out in the forward direction. We stress that, in general, it is not possible to choose x_1, x_2 as the endpoints of ν as the flow lines are only quasigeodesics and not geodesics in L_x. Recall that ν is the interval of $S^1(L_x)$ defined previously. Let P_x be the half plane of L_x bounded by β_x and containing a forward ray from x. We also may assume that every point in P_x is ϵ_1 close to L_w with ϵ_1 very close to zero. Then β_x is ϵ_1 close to a curve β' in L_w which has geodesic curvature in L_w very close to zero. To obtain this property of β' with small geodesic curvature in L_w was one of the reasons to choose a Candel metric with hyperbolic leaves varying continuously, and hence uniformly continuously, since $\mathcal{M}$ is compact. Notice that using this continuity only gives us a curve β' with small geodesic curvature, but not necessarily one which has zero geodesic curvature. However, since the induced path metric in L_w is hyperbolic, it now follows that this curve β' with very small geodesic curvature is very close in L_w to an actual geodesic in L_w. This geodesic is denoted by β_w. Let P_w be the union of the size-1 neighborhood of β_w in L_w and the half plane of L_w which is very close to P_x, as shown in Figure 4.

Note that $\Pi^{-1}(r_x) = r_w$. We choose ϵ_1 small enough so that Π^{-1} is defined in P_x and $\Pi^{-1}(L_x) \subset P_w$. This is the reason to include a neighborhood of β_w in L_w.

We will now show that the map $\Pi^{-1} : P_x \to P_w$ is a quasi-isometry. Using the fact that P_w is very close to L_x, continuity of leafwise Riemannian metric on $\mathcal{M}$, and the compactness of $\mathcal{M}$, we obtain the following: given $\epsilon > 0$, we can consider $\epsilon_1 > 0$ small enough, such that

$$\text{if } d_{L_x}(a, b) \leq 1, \quad \text{for } a, b \in P_x, \quad \text{then } d_{L_w}(\Pi^{-1}(a), \Pi^{-1}(b)) \leq 1 + \epsilon.$$

Next consider a_0 and b_0 on P_x and let ρ be the geodesic connecting them on P_x. Partition ρ in n subintervals $a_0, a_1, a_2, \ldots, a_{n+1} = b_0$ such that $d_{L_x}(a_i, a_{i+1}) = 1$ for all $0 \leq i < n-1$ and $0 < d_{L_x}(a_n, b_0) \leq 1$. As $d(a_i, a_{i+1}) \leq 1$ for all i,

$$\begin{aligned} d_{L_w}(\Pi^{-1}(a_0), \Pi^{-1}(b_0)) &\leq \sum_{i=0}^{n} d_{L_w}(\Pi^{-1}(a_i), \Pi^{-1}(a_{i+1})) \\ &\leq (n+1)(1+\epsilon) = n(1+\epsilon) + (1+\epsilon). \end{aligned}$$

By construction, $n < d_{L_x}(a_0, b_0)$. Hence, we conclude

$$d_{L_w}(\Pi^{-1}(a_0), \Pi^{-1}(b_0)) \leq d_{L_x}(a_0, b_0)C_0 + C_0,$$

where $C_0 = (1+\epsilon)$ is a fixed constant. Similarly, if a_0, b_0 are in $\Pi^{-1}(P_x)$, we get that $d_{L_x}(\Pi(a_0), \Pi(b_0)) < C_1 d_{L_w}(a_0, b_0) + C_2$ for some globally fixed constants C_1, C_2. Since $\Pi^{-1}(P_x)$ is 2-dense in P_w, it follows that Π^{-1} is a quasi-isometry from P_x to P_w.

Finally, as we know that r_x is a quasigeodesic on P_x, then its image via the quasi-isometry Π^{-1}, $r_w = \Pi^{-1}(r_x)$ is a quasigeodesic on $P_w \subset L_w$ with respect to the metric d_{L_w}. Since P_w is a quasi-isometrically embedded in L_w, it now follows that r_w is a quasigeodesic in L_w.

If we reverse the flow, every backward ray becomes a forward ray, and hence leafwise quasigeodesic.

This finally finishes the proof of Lemma 5.3. □

By compactness and continuity, there is global $K_0, s_0 > 0$ so that given any flow line γ, there is a forward ray γ^+ and a backward ray γ^- of γ which are (K_0, s_0) quasigeodesics in leaf L_γ, the leaf of $\widetilde{\mathcal{F}}$ containing γ. Note that all the leaves $L \in \widetilde{\mathcal{F}}$ are hyperbolic and we can define their boundary at infinity $S^1(L)$. As the flow rays γ^+ and γ^- are quasigeodesics on F_γ, they define unique points on the ideal boundary $S^1(L_\gamma)$. Hence for all $a \in \gamma \subset \widetilde{\mathcal{M}}$, the forward subray γ_a^+ limits on a single point in $S^1(L_\gamma)$ and $\eta^+(a)$ is well defined as in Definition 5.2. Similarly, $\eta^-(a)$ is also well defined by the backward subray γ_a^-.

In the next proposition, we consider the sets P_y contained in $\mathcal{U}$.

PROPOSITION 5.4. *Suppose $a, b \in P_y \subset L_y$ but $\gamma_a \neq \gamma_b$, then $\eta^+(a) \neq \eta^+(b)$ in $S^1(L_y)$.*

Proof. By the previous Lemma 5.3, we already know that all rays are quasigeodesics in their respective leaves. We do the proof by contradiction and assume that $\eta^+(a) = \eta^+(b)$ on $S^1(L_y)$. Since the rays $\widetilde{\Phi}_{[0,\infty)}(a), \widetilde{\Phi}_{[0,\infty)}(b)$ are quasigeodesics in L_y and by assumption they have the same ideal point in $S^1(L_y)$, the following happens: there is $d_0 > 0$ and points p_i, q_i in $\widetilde{\Phi}_{[0,\infty)}(a), \widetilde{\Phi}_{[0,\infty)}(b)$ respectively, escaping in the rays so that $d_{L_y}(p_i, q_i) < d_0$. Consider the points $\Pi(a)$ and $\Pi(b)$ on P_x. Since

$$\widetilde{\Phi}_{[0,\infty)}(a), \widetilde{\Phi}_{[0,\infty)}(\Pi(a))$$

are asymptotic in the weak stable leaf of $\widetilde{\Phi}$ in $\widetilde{\mathcal{M}}$, there are p_i' in $\widetilde{\Phi}_{[0,\infty)}(\Pi(a))$ with $d(p_i, p_i') \to 0$. Here d is the ambient distance in $\widetilde{\mathcal{M}}$. Similarly, there are q_i' in $\widetilde{\Phi}_{[0,\infty)}(\Pi(b))$ with $d(q_i, q_i') \to 0$. By the local product structure of the foliation $\mathcal{F}$, it follows that $d_{L_x}(p_i', q_i') < d_0 + 1$ for i sufficiently big.

We explain this in more detail. We choose a finite cover of M by foliated boxes of $\mathcal{F}$, each of which contains a ball of radius $2/m$, where m is a fixed integer. Any disk in a leaf of $\mathcal{F}$ which has diameter less than $1/m$ which is product foliated and any path in the disk is approximated by a path in another leaf with length very close to the length of the original path. Let n be the smallest positive integer bigger than d_0. Using compactness of $\mathcal{M}$, it follows that any connected union of at most nm such disks in a leaf has a transversal neighborhood of fixed size which is product foliated and has the property above on lengths of paths. Therefore, the paths from p_i to q_i in L_y can be approximated by paths from p_i' to q_i' in L_x with length very close, resulting in $d_{L_x}(p_i', q_i') < d_0 + 1$ for i sufficiently big.

Therefore, the rays $\widetilde{\Phi}_{[0,\infty)}(\Pi(a))$, $\widetilde{\Phi}_{[0,\infty)}(\Pi(b))$ converge to the same ideal point in $S^1(L_x)$. However, L_x is also a weak unstable leaf of $\widetilde{\Phi}$ and as the flow lines $\widetilde{\Phi}_{\mathbb{R}}(\Pi(a))$ and $\widetilde{\Phi}_{\mathbb{R}}(\Pi(b))$ are distinct flow lines in L_x, by the description of ideal points of flow lines in weak unstable leaves as in Property 2.4, the forward limit points are distinct, that is,

$$\eta^+(\Pi(a)) \neq \eta^+(\Pi(b)) \text{ in } S^1(L_x).$$

This is a contradiction and shows that $\eta^+(a) \neq \eta^+(b)$ in $S^1(L_y)$. □

LEMMA 5.5. *In each leaf L of $\widetilde{\mathcal{F}}$, the leaf space of the flow foliation is Hausdorff and homeomorphic to the real line $\mathbb{R}$.*

Proof. For the leaves of $\widetilde{\mathcal{F}}$ in lifts $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{R}}$ of the attractor and repeller, the result is obvious, since the foliation by flow lines satisfies this property in weak stable and weak unstable leaves of Anosov flows [**Fen94**]. Any other leaf L of $\widetilde{\mathcal{F}}$ is the lift of a leaf of $\mathcal{F}$ which intersects a torus T from the collection of tori $\{T_i\}$ which separates $\mathcal{A}$ and $\mathcal{R}$. Hence L intersects a lift $\widetilde{T}$ of T in a curve β. The flow saturation of β is exactly L, since every flow line in M is either in the attractor or repeller; or intersects a torus in $\{T_i\}$. The curve β is transverse to the weak stable and weak unstable foliations, and hence intersects a flow line exactly once. Hence β parameterizes the flowlines in L. This proves the result. □

For each L of $\widetilde{\mathcal{F}}$, the map η^+ induces a map from the flow foliation leaf space in L (which is $\cong \mathbb{R}$) to $S^1(L)$. Since flow lines are disjoint, this map is weakly monotone.

COROLLARY 5.6. *For all $y \in \mathcal{U}$, $\eta^+(y) \neq \eta^-(y)$ in $S^1(L_y)$.*

Proof. If $\eta^+(y) = \eta^-(y)$, then the flow line γ_y bounds a disk $\mathcal{D}$ on $L_y \cup S^1(L_y)$ such that the closure of $\mathcal{D}$ in $L \cup S^1(L)$ intersects $S^1(L)$ only in $\eta^+(y) = \eta^-(y)$. For any z in the interior of $\mathcal{D}$, the flow line γ_z is contained in $\mathcal{D}$, and hence $\eta^+(z) = \eta^-(z) = \eta^+(y) = \eta^-(y)$. This contradicts Proposition 5.4, because if $z, y \in \mathcal{U}$, and $\gamma_z \neq \gamma_y$, then $\eta^+(z) \neq \eta^+(y)$. □

We now extend the map η^+ to a map from $\widetilde{\mathcal{M}}$ to $S^1(\widetilde{\mathcal{M}})$. For each x in $\widetilde{\mathcal{M}}$, $\eta^+(x)$ is in $S^1(L_x) \subset S^1(\widetilde{\mathcal{M}})$.

PROPOSITION 5.7. *η^+ and η^- are continuous on $\widetilde{\mathcal{M}}$.*

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Proof. In this proof, we again use a Candel metric in M.

Suppose $x_i \to x_0$ in $\widetilde{\mathcal{M}}$. We will show that $\eta^+(x_i) \to \eta^+(x_0)$ in $S^1(\widetilde{\mathcal{M}})$. There are two different cases depending on whether $x_0 \in \widetilde{\mathcal{R}}$ or $x_0 \notin \widetilde{\mathcal{R}}$.

We first prove the result for $x_0 \notin \widetilde{\mathcal{R}}$. As $x_0 \notin \widetilde{\mathcal{R}}$, the forward ray starting at x_0 is asymptotic to a forward flow ray in $\widetilde{\mathcal{A}}$. Therefore, it is enough to assume that $\{x_i\}$ and x_0 belong to a neighborhood $\mathcal{U}$ as constructed above, since this is true for every ray asymptoptic to $\widetilde{\mathcal{A}}$.

For z in $\widetilde{\mathcal{M}}$, let L_z be the leaf of $\widetilde{\mathcal{F}}$ containing z.

For $i \in \mathbb{N} \cup \{0\}$, let γ_i^+ denote the forward flow ray starting from x_i and let ζ_i denote the geodesic ray on L_{x_i} starting at x_i and with ideal point $\eta^+(x_i)$ in $S^1(L_{x_i})$. Each ζ_i defines the ideal point $\eta^+(x_i)$ on $S^1(L_{x_i})$, therefore it is enough to show that any convergent subsequence of (ζ_i) converges to ζ_0 in the compact open topology. Since all x_i are contained in a compact subset of $\widetilde{\mathcal{M}}$, existence of convergent subsequences of $\{\zeta_i\}$ is assured.

Suppose that a subsequence $(\zeta_{i(k)})$ converges to ζ'. We have to prove that $\zeta' = \zeta_0$. We assume that the neighborhood $\mathcal{U}$ constructed above has a point $x \in \widetilde{\mathcal{A}}$, as in the construction of $\mathcal{U}$. Then all flow rays in $L_x \cap \mathcal{U}$ are (K, s)-quasigeodesics in L_x for some fixed K, s. Since all flow rays in $\mathcal{U}$ are forward asymptotic to flow rays in L_x, there are K', s' so that all flow rays in $\mathcal{U}$ are (K', s')-quasigeodesics in their respective $\widetilde{\mathcal{F}}$ leaves. It follows that there exists a constant $d' > 0$ such that

$$\gamma_{i(k)}^+ \subset \mathcal{N}_{d'}(\zeta_{i(k)}) \quad \text{and} \quad \gamma_0^+ \subset \mathcal{N}_{d'}(\zeta_0),$$

where $\mathcal{N}_{d'}$ denotes the neighborhood of radius d in the respective leaf of $\widetilde{\mathcal{F}}$. For any $d_1 > 0$, the segment of length d_1 on $\gamma_{i(k)}^+$ starting at $x_{i(k)}$ is within d'-distance from $\zeta_{i(k)}$. Therefore in the limit, the segment of γ_0^+ of length d_1 starting from x_0 is contained in $\mathcal{N}_{d'}(\zeta')$ in the respective leaf. This is true for all d_1, so ζ' is at Hausdorff distance d' from γ_0^+ on L_{x_0}. However, γ_0^+ is also at a bounded distance from ζ_0 on L_{x_0}; therefore, ζ' and ζ_0 are at a finite Hausdorff distance from each other on L_{x_0}. Hence $\zeta' = \zeta_0$, because they have the same starting point. As this is true for all convergent subsequences of (ζ_i), we get our result for x_0 not in $\widetilde{\mathcal{R}}$.

Before dealing with the remaining case, let us note the following.

Observation 5.8. By the construction of $\mathcal{U}$ starting with x in $\widetilde{\mathcal{A}}$ and continuity of η^+ near $\widetilde{\mathcal{A}}$, we observe that the set $\mathcal{U} \cup \{\eta^+(z) | z \in \mathcal{U}\}$ is homeomorphic to $[0, 1] \times [0, 1] \times [0, 1]$ inside $\mathcal{W} = \bigcup_{y \in \lambda}(L_y \cup S^1(L_y))$, which is homeomorphic to a compact solid cylinder $[0, 1] \times \{\text{theclosedunitdisc}\mathbb{D}\}$.

The set $\mathcal{U} \cup \{\eta^+(z) | z \in \mathcal{U}\}$ above is saturated by forward flow lines and all the ideal points contained in this neighborhood are defined by forward flow rays. Hence, we conclude the following.

(1) If $L \in \widetilde{\mathcal{F}}|_{\widetilde{\mathcal{A}}}$ and $p \in L$, then there exists a neighborhood N_p of $\eta^+(p)$ in $\bigcup_{L \in \widetilde{\mathcal{F}}} S^1(L)$ such that $N_p \subset \eta^+(\widetilde{\mathcal{M}})$ and $N_p \cap \eta^-(\widetilde{\mathcal{M}}) = \emptyset$.

(2) Similarly, for $q \in L_q \in \widetilde{\mathcal{R}}$, there exists a neighborhood N_q of $\eta^-(q)$ in $\bigcup_{L \in \widetilde{\mathcal{F}}} S^1(L)$ such that $N_q \subset \eta^-(\widetilde{\mathcal{M}})$ and $N_p \cap \eta^+(\widetilde{\mathcal{M}}) = \emptyset$. Moreover, the backward ray γ_q^- starting from q is contained in an infinite cubical neighborhood in $\widetilde{\mathcal{M}}$ saturated by backward flow rays.

To continue the proof of Proposition 5.7, we next assume that $x_0 \in \widetilde{\mathcal{R}}$. Suppose that a subsequence $(\eta^+(x_{i(k)}))$ converges to q where q is not $\eta^+(x_0)$. As x_0 is in $\widetilde{\mathcal{R}}$, then L_{x_0} is a leaf of the weak stable foliation $\widetilde{\mathcal{F}}^{ws}$. Hence by Property 2.4 on L_{x_0}, all the forward flow rays converge to a single ideal point in $S^1(L_{x_0})$ and all the other ideal points in $S^1(L_{x_0})$ are ideal points of backward flow rays. As $q \neq \eta^+(x_0)$, q is defined by a backward ray, that is, $q = \eta^-(z)$ for some z in L_{x_0}. By Observation 5.8(2) starting with z in $\widetilde{\mathcal{R}}$ (notice that z is in the repeller, not the attractor), there exits a neighborhood $\mathcal{V}$ saturated by backward flow rays around z in $\bigcup\{L_y \cup S^1(L_y) | y \in \lambda'\}$ for some transversal λ'. By Observation 5.8(2), all limit points are backward ideal points in $\mathcal{V}$ and no limit point is a forward ideal point. This contradicts the fact that the forward rays $\gamma^+_{i(k)}$ have ideal points in these intervals of ideal points for k big enough by construction. This contradiction shows that a subsequence $(\eta^+(x_{i(k)}))$ converging to $q \neq \eta^+(x_0)$ is not possible, and hence $q = \eta^+(x_0)$.

Hence η^+ is continuous on $\widetilde{\mathcal{M}}$. If we consider the flow $\Psi_t = \Phi_{-t}$, then backward ideal points of Φ_t are forward ideal points of Ψ_{-t} and the continuity of η^- follows. This completes the proof of Proposition 5.7. □

6. *Flow lines are leafwise quasigeodesic*

In this section, we prove a general result of quasigeodesic behavior of some subfoliations. This result will imply that in the examples we constructed associated with non-transitive Anosov flows, the flow lines are uniform quasigeodesics in their respective two-dimensional leaves. We first consider some general continuity properties.

Definition 6.1. (Continuity properties) Let $\mathcal{G}$ be a one-dimensional oriented subfoliation of a two-dimensional foliation $\mathcal{F}$ with Gromov hyperbolic leaves on a 3-manifold $\mathcal{M}$. Suppose that leaves of $\mathcal{G}$ are C^1 curves in leaves of $\mathcal{F}$. Suppose that the following three properties are satisfied.

(1) For each x in $\widetilde{M}$, let ℓ be the leaf of $\widetilde{\mathcal{G}}$ containing it, and L the leaf of $\widetilde{\mathcal{F}}$ containing x. Then in the forward direction (given by the orientation of $\widetilde{\mathcal{G}}$), the leaf ℓ has a unique limiting point in $S^1(L)$ and this is denoted by $\eta^+(x)$. Similarly, in the negative direction, there is a unique limiting point in $S^1(L)$ denoted by $\eta^-(x)$.
(2) For each x in $\widetilde{M}$, the points $\eta^+(x), \eta^-(x)$ are distinct points in $S^1(L)$ (L the leaf of $\widetilde{\mathcal{F}}$ containing x).
(3) The functions $\eta^+, \eta^- : \widetilde{\mathcal{M}} \to \bigcup_{L \in \widetilde{\mathcal{F}}} S^1(L)$ are continuous.

Then we say that $(\mathcal{F}, \mathcal{G})$ has the continuity properties.

From the foliation $\mathcal{G}$, we can produce a flow with flow lines which are the leaves of $\mathcal{G}$: for example, just flow forward along leaves of $\mathcal{G}$ with speed 1 in the positive direction. Any reparameterization of the flow produces a time parameter which is quasi-isometric with this one, so the result on the quasigeodesic behavior of flow lines depends only on $\mathcal{G}$ and not the particular parameterization, or description of $\mathcal{G}$ as the flow foliation of a flow.

Notice that the two-dimensional foliation $\mathcal{F}$ constructed in §3 with the one-dimensional subfoliation $\mathcal{G}$ by the flow lines of Φ_t satisfies the *continuity properties* as follows. In the previous section, we proved the pair $(\mathcal{F}, \mathcal{G})$ satisfies the properties (1), (2), and (3)

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of Definition 6.1: property (1) was proved in Lemma 5.3, property (2) was proved in Corollary 5.6, and property (3) was proved in Proposition 5.7.

The next result is a general result that will imply that the foliations we constructed in §3 are leafwise quasigeodesic foliations.

THEOREM 6.2. *Suppose that $\widetilde{\mathcal{G}}$ is a one-dimensional subfoliation of a two-dimensional foliation $\widetilde{\mathcal{F}}$ satisfying the continuity properties of Definition 6.1. Then $\widetilde{\mathcal{G}}$ is a leafwise quasigeodesic foliation.*

The proof will be attained by the following three results. In the next lemma, we combine all the results of the previous section to obtain a key property that will be used to show that all the flow lines are quasigeodesic on their respective leaves of $\widetilde{\mathcal{F}}$.

We stress that the quasigeodesic behavior is proved using only the continuity properties, irrespective of how these continuity properties are obtained. Therefore, Theorem 6.2 is applicable not only to the examples constructed in §3, but theoretically to many other situations as well.

To prove Theorem 6.2, again we use a Candel metric. Given x in $\widetilde{M}$, let γ_x be the leaf of $\widetilde{\mathcal{G}}$ through it. In addition, let L_x be the leaf of $\widetilde{\mathcal{F}}$ containing x. Using property (i) of Definition 6.1, we let $\eta^+(x)$, $\eta^-(x)$ be the unique limiting points of the two rays of γ_x in $S^1(L_x)$. Notice that they are distinct points in $S^1(L_x)$ by property (ii) of Definition 6.1. Since L_x has a hyperbolic metric, there is a unique geodesic in L_x, denoted by g_x, whose ideal points in $S^1(L_x)$ are $\eta^+(x)$, $\eta^-(x)$.

LEMMA 6.3. *There exists $\delta > 0$ such that for all $x \in \widetilde{M}$, we have that*

$$\gamma_x \subset \mathcal{N}_\delta(g_x),$$

where g_x is the geodesic on L_x connecting $\eta^+(x)$ and $\eta^-(x)$ and $\mathcal{N}_\delta(g_x)$ is the δ-neighborhood of g_x on L_x.

Proof. Suppose that there does not exist any such δ. Then there exists a sequence (x_i) in $\widetilde{M}$ with x_i in leaves L_{x_i} of $\widetilde{\mathcal{F}}$ such that $d_{L_{x_i}}(x_i, g_{x_i}) > i$. Up to deck transformations, there exists a convergent subsequence of (x_i) which we assume is the original sequence, and we assume $x_i \to x$. By property (iii) of Definition 6.1, we know that

$$\eta^+(x_i) \to \eta^+(x) \quad \text{and} \quad \eta^-(x_i) \to \eta^-(x).$$

Since x_i converges to x, we assume that all x_i are in leaves of $\widetilde{\mathcal{F}}$ which intersect a fixed transversal λ to $\widetilde{\mathcal{F}}$.

Since $\eta^+(x_i)$ converges to $\eta^+(x)$, $\eta^-(x_i)$ converges to $\eta^-(x)$, and $\eta^+(x) \neq \eta^-(x)$, it follows that $\{g_{x_i}\}$ converges to g_x. This uses that the topology defined on $\bigcup_{y\in\lambda}(S^1(L_y))$ is given by the trivialization of the unit tangent bundle to $\widetilde{\mathcal{F}}$ along λ. By convergence we mean convergence in the compact open topology. However, this contradicts that $d_{L_{x_i}}(x_i, g_{x_i})$ converges to infinity, since $d_{L_x}(x, g_x)$ is finite and the sequence $d_{L_{x_i}}(x_i, g_{x_i})$ converges to it. This finishes the proof. □

We now prove a weak quasigeodesic property of the leaves of $\widetilde{\mathcal{G}}$ in the leaves of $\widetilde{\mathcal{F}}$ containing them.

PROPOSITION 6.4. *For any $b > 0$, there exists a $c_b > 0$ such that if γ is the segment in a leaf of $\widetilde{\mathcal{G}}$ connecting x and y with $length(\gamma) > c_b$, then $d_{L_x}(x, y) > b$, where L_x is the leaf of $\widetilde{\mathcal{F}}$ which contains x.*

Proof. Fix $b > 0$. We do the proof by contradiction. Suppose the statement is not true for some $b > 0$. Then for all $i \in \mathbb{N}$, there exists two points x_i and y_i in leaves L_i of $\widetilde{\mathcal{F}}$, with x_i, y_i in the same flow line defining a flow line segment γ_i satisfying $length(\gamma_i) > i$ but $d_{L_i}(x_i, y_i) < b$. Up to deck transformations and a subsequence, we assume that (x_i) is convergent and $x_i \to x_0$. Since $d_{L_i}(x_i, y_i) < b$, we can similarly assume that (y_i) is convergent and let $y_i \to y_0$.

CLAIM 2. *x_0 and y_0 are on the same leaf L_0 of $\widetilde{\mathcal{F}}$.*

Proof. If we consider a compact ball B_{x_0} on L_0 containing x_0 and a product neighborhood $N(B_{x_0})$ with respect to $\widetilde{\mathcal{F}}$, then for all large i, L_i intersects $N(B_{x_0})$ and $x_i \in L_i \cap N(B_{x_0})$. If we consider B_{x_0} sufficiently large, the assumption $d_{L_i}(x_i, y_i) < b$ for all i forces that y_i has to be contained in $N(B_{x_0})$. Hence by the product structure on $N(B_{x_0})$, y_0 also has to lie on L_0 as $y_i \to y_0$. □

CLAIM 3. *x_0 and y_0 cannot be on the same flow line in L_0.*

Proof. If not, then there exists a flow line segment γ connecting x_0 and y_0 and consider a compact neighborhood $\mathcal{N}$ around γ which has a product structure with respect to the flow lines. As $x_i \to x_0$ and $y_i \to y_0$, the flow segments γ_i are contained in $\mathcal{N}$ for all large i. By continuity of length of flow lines, $length(\gamma_i) \to length(\gamma)$. However, that is not possible as $length(\gamma_i) \to \infty$ and γ is compact, a contradiction. □

CLAIM 4. *x_0 and y_0 cannot be connected by a curve on L_0 everywhere transversal to the flow lines in L_0.*

Proof. Suppose that there exists a line segment σ on L_0 everywhere transversal to the flow lines on L_0 and connecting x_0 and y_0. By the local product structure of $\widetilde{\mathcal{F}}$ near $\sigma \in L_0$, there should be a segment σ_i in L_i connecting x_i and y_i, and everywhere transversal to flow lines on L_i. Up to taking a sub-segment of γ_i if necessary and then a sub-segment of σ_i, we may assume that γ_i does not intersect the interior of σ_i. It follows that the union of σ_i and γ_i bounds a disk $\mathcal{D}_i$ on L_i as their end points are the same. All the flow lines which enter $\mathcal{D}_i$ transversally intersecting σ_i have to exit $\mathcal{D}_i$ transversally intersecting σ_i. The Poincaré–Hopf theorem says that there exists at least one flow line tangent to σ_i, a contradiction. □

By Lemma 5.5, the leaf space of the flow foliation in L_0 is homeomorphic to the reals. Hence any two distinct flow lines in L_0 are connected by a transversal.

This contradiction proves Proposition 6.4. □

Now we are ready to prove our final claim.

PROPOSITION 6.5. *The leaves of $\widetilde{\mathcal{G}}$ are uniformly quasigeodesics in their respective leaves of $\widetilde{\mathcal{F}}$.*

Proof. We prove the theorem by contradiction. Recall that we are using a Candel metric in leaves of $\mathcal{F}$. We assume that the leaves of $\widetilde{\mathcal{G}}$ are not uniform quasigeodesic on their leaves. From this assumption, we will construct a sequence of pairs $\{(x_i, y_i)\}$ such that x_i and y_i are connected by a flow segment γ_i, where $\text{length}(\gamma_i) \to \infty$ but $d_{L_i}(x_i, y_i)$ is bounded. Here L_i is the $\widetilde{\mathcal{F}}$ leaf containing both x_i, y_i. However, this will contradict Proposition 6.4. A very similar result was proved in [**FM01**], we reconstruct the same arguments in our specific case.

By our assumption, the leaves of $\widetilde{\mathcal{G}}$ are not uniform quasigeodesics. We get that for any $K > 0$, there exists a segment of a leaf of $\widetilde{\mathcal{G}}$ with endpoints x, y denoted by $\gamma_{[x,y]}$, contained in a leaf of $\widetilde{\mathcal{F}}$ denoted by L_x, and such that

$$\text{length}(\gamma_{[x,y]})/d_{L_x}(x, y) > 2K \text{ and } \text{length}(\gamma_{[x,y]}) > K.$$

For each K, one can find such x, y, L_x, which obviously depend on K, but we omit the explicit dependence on K for notational simplicity. Consider the geodesic $g_x = g_y$ on L_x with ideal points

$$\eta^+(x) = \eta^+(y) \quad \text{and} \quad \eta^-(x) = \eta^-(y) \quad \text{or } S^1(L_x).$$

By Lemma 6.3, there exists $\delta > 0$ such that $\gamma \subset \mathcal{N}_\delta(g_x)$, where the neighborhood is in L_x. This δ is global, it works for any segment in a leaf of $\widetilde{\mathcal{G}}$ in its respective leaf of $\widetilde{\mathcal{F}}$. Let $\rho : L_x \to g_x$ be the 'closest point map projection', which means $\rho(p)$ is the orthogonal projection in L_x to g_x, a bi-infinite length-minimizing geodesic on L_x. The map is well defined as the leaves L_x are of constant curvature -1 and so isometric to the hyperbolic plane: the 'closest point map' on to a length-minimizing geodesic is well defined in the hyperbolic plane. It follows that:

$$d_{L_x}(\rho(x), \rho(y)) \le d_{L_x}(x, y) \le d_{L_x}(\rho(x), \rho(y)) + 2\delta. \tag{6.1}$$

Let us assume that $d_{L_x}(x, y) > 1 + 2\delta$. Hence, $d_{L_x}(\rho(x), \rho(y)) > 1$ by equation (6.1) and

$$\frac{\text{length}(\gamma_{[x,y]})}{d_{L_x}(\rho(x), \rho(y))} \ge \frac{\text{length}(\gamma_{[x,y]})}{d_{L_x}(x, y)} \ge 2K > K + \frac{K}{d_{L_x}(\rho(x), \rho(y))}.$$

Therefore,

$$\frac{\text{length}(\gamma_{[x,y]})}{K} > d_{L_x}(\rho(x), \rho(y)) + 1 > \lceil d_{L_x}(\rho(x), \rho(y)) \rceil,$$

where $\lceil a \rceil$ denotes the integer n such that $n - 1 < a \le n$.

Suppose $n_0 = \lceil d_{L_x}(\rho(x), \rho(y)) \rceil$, then $\text{length}(\gamma_{[x,y]}) > n_0 K$. Also,

$$n_0 - 1 < \lceil d_{L_x}(\rho(x), \rho(y)) \rceil \le n_0,$$

and hence we can construct a sequence $\{\rho(x) = z_0, z_1, \ldots, z_n = \rho(y)\}$ of points in g_x, such that $d_{L_x}(z_{i-1}, z_i) = 1$ for all $i < n_0$ and $d_{L_x}(z_{n_0-1}, z_{n_0}) \le 1$. Next we consider the sequence $x = x_0, x_1, \ldots, x_{n_0}$, where x_i is the last point on $\gamma_{[x,y]}$ such that $\rho(x_i) = z_i$.

If γ_i denotes the flow segment joining x_{i-1} and x_i, we have $\gamma_{[x,y]} = \gamma_1 * \gamma_2 * \cdots * \gamma_{n_0}$. Hence,

$$\sum_{n=1}^{n_0} \text{length}(\gamma_i) = \text{length}(\gamma_{[x,y]}) > n_0 K.$$

By the pigeonhole principle, there exists x_{i-1} and x_i such that $\text{length}(\gamma_{[x_{i-1},x_i]}) > K$. However, from $(*)$, we get that for all i,

$$d_{L_x}(x_{i-1}, x_i) \leq d_{L_x}(\rho(x_{i-1}, x_i)) + 2\delta = d_{L_x}(z_{i-1}, z_i)) + 2\delta < 1 + 2\delta.$$

As the choice of $K > 0$ was arbitrary, this proves that the 'weak quasigeodesic property' in Lemma 6.4 is not true for $b = 1 + 2\delta$, a contradiction. We conclude that leaves of $\widetilde{\mathcal{G}}$ are uniformly quasigeodesic on their respective leaves of $\widetilde{\mathcal{F}}$.

This finishes the proof of Proposition 6.5. □

7. *Conclusion*

We now apply the results of this section to the two-dimensional foliation $\mathcal{F}$ with a subfoliation $\mathcal{G}$ as constructed in §3. Section 4 shows that every leaf in $\mathcal{F}$ is Gromov hyperbolic when lifted to the universal cover. Proposition 6.5 proves that the flow foliation (that is, the foliation $\mathcal{G}$) is a leafwise quasigeodesic subfoliation of $\mathcal{F}$. Moreover, Proposition 5.4 proves that all leaves of $\mathcal{F}$ which are not contained in $\mathcal{A}$ or $\mathcal{R}$ are *non-funnel*. This is because of the following: if γ_a, γ_b are distinct flow lines in some leaf L of $\widetilde{\mathcal{F}}$ which are not in the lift of the attractor or the repeller, then Proposition 5.4 shows that $\eta^+(a) \neq \eta^+(b)$ in $S^1(L)$. Applying the same result to negative flow rays, one obtains that $\eta^-(a) \neq \eta^-(b)$ in $S^1(L)$. Hence L cannot be a funnel leaf. However, all leaves in $\mathcal{A}$ or $\mathcal{R}$ are *funnel* by Corollary 2.6. This completes the proof of the Theorem 1.1.

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