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A Short Proof of Paouris' Inequality

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Abstract. We give a short proof of a result of G. Paouris on the tail behaviour of the Euclidean norm $|X|$ of an isotropic log-concave random vector $X \in \mathbb{R}^n$, stating that for every $t \geq 1$,

$$\mathbb{P}\big(|X| \geq ct\sqrt{n}\big) \leq \exp(-t\sqrt{n}).$$

More precisely we show that for any log-concave random vector X and any $p \geq 1$,

$$(\mathbb{E}|X|^p)^{1/p} \sim \mathbb{E}|X| + \sup_{z \in S^{n-1}} \big(\mathbb{E}|\langle z, X\rangle|^p\big)^{1/p}.$$

1 Introduction

Let X be a random vector in the Euclidean space $\mathbb{R}^n$ equipped with its Euclidean norm $|\cdot|$ and its scalar product $\langle \cdot, \cdot \rangle$. Assume that X has a log-concave distribution (a typical example of such a distribution is a random vector uniformly distributed on a convex body). Assume further that it is centered and that its covariance matrix is the identity; such a random vector will be called *isotropic*. A famous and important result, [14, Theorem 1.1], states the following.

Theorem 1.1 *There exists an absolute constant $c > 0$ such that if X is an isotropic log-concave random vector in $\mathbb{R}^n$, then for every $t \geq 1$,*

$$\mathbb{P}\big(|X| \geq ct\sqrt{n}\big) \leq \exp(-t\sqrt{n}).$$

This result had a huge impact on the study of log-concave measures and has a lot of applications in that subject.

A Borel probability measure on $\mathbb{R}^n$ is called *log-concave* if for all $0 < \theta < 1$ and all compact sets $A, B \subset \mathbb{R}^n$ one has

$$\mu\big((1-\theta)A + \theta B\big) \geq \mu(A)^{1-\theta}\mu(B)^{\theta}.$$

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We refer to [5, 6] for a general study of this class of measures. Clearly, the affine image of a log-concave probability is also log-concave. The Euclidean norm of an n-dimensional log-concave random vector has moments of all orders (see [5]). A log-concave probability is supported on some convex subset of an affine subspace where it has a density. In particular when the support of the probability generates the whole space $\mathbb{R}^n$ (in which case we talk, in short, about full-dimensional probability) a characterization of Borell (see [5,6]) states that the probability is absolutely continuous with respect to the Lebesgue measure and has a density that is log-concave. We say that a random vector is log-concave if its distribution is a log-concave measure.

Let $X \in \mathbb{R}^n$ be a random vector. Denote the weak p-th moment of X by

$$\sigma_p(X) = \sup_{z \in S^{n-1}} \left(\mathbb{E}|\langle z, X\rangle|^p \right)^{1/p}.$$

The purpose of this article is to give a short proof of the following theorem.

Theorem 1.2 *For any log-concave random vector $X \in \mathbb{R}^n$ and any $p \geq 1$,*

$$(\mathbb{E}|X|^p)^{1/p} \leq C\left(\mathbb{E}|X| + \sigma_p(X)\right),$$

where C is an absolute positive constant.

This result may be deduced directly from Paouris' work [14]. Indeed, it is a consequence of Theorem 8.2 combined with Lemma 3.9 in that paper. As formulated here, Theorem 1.2 first appeared as [3, Theorem 2]. Note that because trivially a converse inequality is valid (with constant $1/2$), Theorem 1.2 states in fact an equivalence for $(\mathbb{E}|X|^p)^{1/p}$.

It is noteworthy that the following strengthening of Theorem 1.2 is still open: $(\mathbb{E}|X|^p)^{1/p} \leq \mathbb{E}|X| + C\sigma_p(X)$, where C is an absolute positive constant.

If X is a log-concave random vector, then so is $\langle z, X\rangle$ for every $z \in S^{n-1}$. It follows that there exists an absolute constant $C' > 0$ such that for any $p \geq 1$, $\sigma_p(X) \leq C' p\, \sigma_2(X)$ ([5]). (In fact one can deduce this inequality with $C' = 1$ from [4] or from [11, Remark 5]; see also Remark 1 following [2, Theorem 3.1].) If, moreover, X is isotropic, then $\mathbb{E}|X| \leq (\mathbb{E}|X|^2)^{1/2} = \sqrt{n}$ and $\sigma_2(X) = 1$; thus

$$(\mathbb{E}|X|^p)^{1/p} \leq C\left(\sqrt{n} + C' p\right).$$

From Markov's inequality for $p = t\sqrt{n}$, Theorem 1.2 implies Theorem 1.1 with $c = (C' + 1)eC$.

Let us recall the idea underlying the proof by Paouris. Let $X \in \mathbb{R}^n$ be an isotropic log-concave random vector. Let $p \sim \sqrt{n}$ be an integer (for example, $p = [\sqrt{n}]$). Let $Y = PX$, where P is an orthogonal projection of rank p and let $G \in \operatorname{Im} P$ be a standard Gaussian vector. By rotation invariance, $\mathbb{E}|Y|^p \sim \mathbb{E}|\langle G/\sqrt{p}, Y\rangle|^p$. If the linear forms $\langle z, X\rangle$ with $|z| = 1$ had a sub-Gaussian tail behaviour, the proof would be straightforward. But in general they only obey a sub-exponential tail behaviour. The first step of the proof consists of showing that there exists some z for which $(\mathbb{E}|\langle z, Y\rangle|^p)^{1/p}$ is in fact small compared to $\mathbb{E}|Y|$. The second step uses a concentration principle to show that $(\mathbb{E}_X|\langle z, PX\rangle|^p)^{1/p}$ is essentially constant on the sphere for

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a random orthogonal projection of rank $p \sim \sqrt{n}$, and thus comparable to the minimum. Thus for these *good* projections, one has a good estimate of $(\mathbb{E}|Y|^p)^{1/p}$, and the result follows by averaging over P. Our proof follows the same scheme, at least for the first step, but whereas the proof of the first step in [14] is the most technical part, our argument is very simple. Then the estimate for $\min_{|z|=1} \mathbb{E}|\langle z, Y\rangle|^p$ brings us to a minimax problem precisely in the form answered by Gordon's inequality ([9]).

Finally we would like to note that our proof can be generalized to the case of convex measures in the sense of [5, 6]. Of course the proof is longer and more technical. We provide the details in [1].

2 Proof of Theorem 1.2

First let us notice that it is enough to prove Theorem 1.2 for symmetric log-concave random vectors. Indeed, let X be a log-concave random vector and let X' be an independent copy. By Jensen's inequality we have for all $p \geq 1$,

$$(\mathbb{E}|X|^p)^{1/p} \leq \left(\mathbb{E}|X - \mathbb{E}X|^p\right)^{1/p} + |\mathbb{E}X| \leq \left(\mathbb{E}|X - X'|^p\right)^{1/p} + \mathbb{E}|X|.$$

On the other hand $\mathbb{E}|X - X'| \leq 2\mathbb{E}|X|$ and for $p \geq 1$ one has $\sigma_p(X - X') \leq 2\sigma_p(X)$. Since $X - X'$ is log-concave (see [8]) and symmetric, we obtain that the symmetric case proved with a constant C' implies the non-symmetric case with the constant $C = 2C' + 1$.

Lemma 2.1 *Let $Y \in \mathbb{R}^q$ be a random vector. Let $\|\cdot\|$ be a norm on $\mathbb{R}^q$. Then for all $p > 0$,*

$$\min_{|z|=1}\left(\mathbb{E}|\langle z, Y\rangle|^p\right)^{1/p} \leq \frac{(\mathbb{E}\|Y\|^p)^{1/p}}{\mathbb{E}\|Y\|}\,\mathbb{E}|Y|.$$

Proof Let r be the largest number such that $r\|t\| \leq |t|$ for all $t \in \mathbb{R}^q$. Using duality, pick $z \in \mathbb{R}^q$ such that $|z| = 1$ and $\|z\|_* \leq r$ (the dual norm of $\|\cdot\|$). Then $|\langle z, t\rangle| \leq r\|t\| \leq |t|$ for all $t \in \mathbb{R}^q$. Therefore, $(\mathbb{E}|\langle z, Y\rangle|^p)^{1/p} \leq r(\mathbb{E}\|Y\|^p)^{1/p}$ for any $p > 0$, and the proof follows from $r\mathbb{E}\|Y\| \leq \mathbb{E}|Y|$. ■

Lemma 2.2 *Let Y be a full-dimensional symmetric log-concave $\mathbb{R}^q$-valued random vector. Then there exists a norm $\|\cdot\|$ on $\mathbb{R}^q$ such that*

$$\left(\mathbb{E}\|Y\|^q\right)^{1/q} \leq 500\,\mathbb{E}\|Y\|.$$

Remark In fact, the constant 500 can be significantly improved. To keep the presentation short and transparent we omit the details.

Proof From Borell's characterization, Y has an even log-concave density g_Y. Thus $g_Y(0)$ is the maximum of g_Y. Define a symmetric convex set by

$$K = \left\{t \in \mathbb{R}^q : g_Y(t) \geq 25^{-q} g_Y(0)\right\}.$$

Since K clearly has a non-empty interior, it is the unit ball of a norm that we denote by $\|\cdot\|$. On one hand, $1 \geq \mathbb{P}(Y \in K) = \int_K g_Y \geq 25^{-q} g_Y(0)\mathrm{vol}(K)$, thus

$$\mathbb{P}\big(\|Y\| \leq 1/50\big) = \int_{K/50} g_Y \leq g_Y(0)50^{-q}\mathrm{vol}(K) \leq 2^{-q} \leq 1/2.$$

Therefore, $\mathbb{E}\|Y\| \geq \mathbb{P}(\|Y\| > 1/50)/50 \geq 1/100$. On the other hand, by the log-concavity of g_Y,

$$\forall t \in \mathbb{R}^q \setminus K \qquad g_{2Y}(t) = 2^{-q} g_Y(t/2) \geq 2^{-q} g_Y(t)^{1/2} g_Y(0)^{1/2} \geq (5/2)^q g_Y(t).$$

Therefore,

$$\mathbb{E}\|Y\|^q \leq 1 + \mathbb{E}(\|Y\|^q 1_{Y \in \mathbb{R}^q \setminus K}) \leq 1 + (2/5)^q \mathbb{E}\|2Y\|^q = 1 + (4/5)^q \mathbb{E}\|Y\|^q.$$

We conclude that $(\mathbb{E}\|Y\|^q)^{1/q} \leq 5$ and $(\mathbb{E}\|Y\|^q)^{1/q}/\mathbb{E}\|Y\| \leq 500$. ∎

Lemma 2.3 *Let $n, q \geq 1$ be integers and $p \geq 1$. Let X be an n-dimensional random vector, let G be a standard Gaussian vector in $\mathbb{R}^n$, and let Γ be an $n \times q$ standard Gaussian matrix. Then*

$$(\mathbb{E}|X|^p)^{1/p} \leq \alpha_p^{-1}\Big(\mathbb{E}\min_{|t|=1} |||\Gamma t||| + (\alpha_p + \sqrt{q})\,\sigma_p(X)\Big),$$

where $|||z||| = (\mathbb{E}|\langle z, X\rangle|^p)^{1/p}$ and α_p^p is the p-th moment of an $N(0,1)$ Gaussian random variable (so that $\lim_{p\to\infty}(\alpha_p/\sqrt{p}) = 1/\sqrt{e}$).

Proof By rotation invariance, $\mathbb{E}|\langle G, X\rangle|^p = \alpha_p^p\,\mathbb{E}|X|^p$. Notice that

$$\sigma^2 := \sup_{|||t|||_* \leq 1} \mathbb{E}|\langle G, t\rangle|^2 = \sup_{|||t|||_* \leq 1} |t|^2 = \sigma_p^2(X),$$

where $|||\cdot|||_*$ denotes the norm on $\mathbb{R}^n$ dual to the norm $|||\cdot|||$. Denote the median of $|||G|||$ by M_G. The classical deviation inequality for a norm of a Gaussian vector ([7, 15], see also [12, Theorem 12.2]) states

$$\forall s \geq 0 \qquad \mathbb{P}\Big(\big|\,|||G||| - M_G\big| \geq s\Big) \leq 2\int_{s/\sigma}^{\infty} \exp\big(-t^2/2\big)\,\frac{dt}{\sqrt{2\pi}}$$

and since $M_G \leq \mathbb{E}\,|||G|||$ ([10], see also [12, Lemma 12.2]), this implies

$$(\mathbb{E}|X|^p)^{1/p} = \alpha_p^{-1}(\mathbb{E}\,|||G|||^p)^{1/p} \leq \alpha_p^{-1}\big(\mathbb{E}\,|||G||| + \alpha_p\sigma_p(X)\big)$$

(*cf.* [13, Statement 3.1]).

The Gordon minimax lower bound (see [9, Theorem 2.5]) states that for any norm $|||\cdot|||$,

$$\mathbb{E}\,|||G||| \leq \mathbb{E}\min_{|t|=1} |||\Gamma t||| + \big(\mathbb{E}|H|\big)\max_{|z|=1} |||z||| \leq \mathbb{E}\min_{|t|=1} |||\Gamma t||| + \sqrt{q}\,\sigma_p(X),$$

where H is a standard Gaussian vector in $\mathbb{R}^q$. This concludes the proof. ∎

Proof of Theorem 1.2 Assume that X is log-concave symmetric. We use the notation of Lemma 2.3 with q the integer such that $p \le q < p+1$. We first condition on Γ. Let $Y = \Gamma^* X$. Note that Y is log-concave symmetric and that

$$|||\Gamma t||| = \left(\mathbb{E}_X |\langle \Gamma t, X\rangle|^p\right)^{1/p} = \left(\mathbb{E}_X |\langle t, \Gamma^* X\rangle|^p\right)^{1/p}.$$

If $\Gamma^* X$ is supported by a hyperplane then $\min_{|t|=1} (\mathbb{E}_X |\langle t, \Gamma^* X\rangle|^p)^{1/p} = 0$. Otherwise Lemma 2.2 applies and combined with Lemma 2.1 gives that

$$\min_{|t|=1} |||\Gamma t||| \le \min_{|t|=1} (\mathbb{E}_X |\langle t, \Gamma^* X\rangle|^p)^{1/p} \le 500\, \mathbb{E}_X |\Gamma^* X|.$$

By taking expectation over Γ we get

$$\mathbb{E} \min_{|t|=1} |||\Gamma t||| \le 500\, \mathbb{E}|\Gamma^* X| = 500\, \mathbb{E}|H|\, \mathbb{E}|X| \le 500 \sqrt{q}\, \mathbb{E}|X|,$$

where $H \in \mathbb{R}^q$ is a standard Gaussian vector. Applying Lemma 2.3 we obtain

$$(\mathbb{E}|X|^p)^{1/p} \le 500\, \alpha_p^{-1} \sqrt{q}\, \mathbb{E}|X| + (1 + \alpha_p^{-1} \sqrt{q}) \sigma_p(X).$$

This implies the desired result, since $q \le p+1$ and hence $\alpha_p^{-1}\sqrt{q} \le c$ for some numerical constant c (recall that $\lim_{p\to\infty} \left(\alpha_p/\sqrt{p}\right) = 1/\sqrt{e}$). ∎

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