ON MULTIPLE INTEGRAL GEOMETRIC INTEGRALS AND THEIR APPLICATIONS TO PROBABILITY THEORY

FRANZ STREIT

Dedicated to Professor H. Hadwiger on the occasion of his sixtieth birthday

It has been pointed out repeatedly in the literature that the methods of integral geometry (a mathematical theory founded by Wilhelm Blaschke and considerably extended by several mathematicians) provide highly suitable means for the solution of problems concerning "geometrical probabilities" [**2**; **6**; **12**; **15**]. The possibilities for the application of these integral geometric results to the evaluation of probabilities, satisfying certain conditions of invariance with respect to a group of transformations which acts on the probability space, are obviously not yet exhausted. In this article, such applications are presented. First, some concepts and notation are introduced (§ 1). In the next section we derive some integral geometric relations (§ 2). These results are generalizations of known systems of formulae and they are valid in the k-dimensional Euclidean space. In § 3, we determine mean-value formulae for the fundamental characteristics of point-sets, generated by randomly placed convex bodies. In particular, the rotational mean values of the fundamental characteristics of a zonotope (and of a parallelotope) with given lengths of its edges are investigated (§ 4). Finally, in § 5 we deal with the probability laws for certain agglomerations of planes, distributed at random in the k-dimensional Euclidean space. The results in this section are the k-dimensional extensions of some formulae in the special case $k = 3$, which have been developed by Hadwiger and myself in an earlier paper [**11**].

In the present work, some elementary knowledge of integral geometry is assumed to be known; in particular, we do not explain the concept of integral geometric densities. Details of invariant measures of sets of geometrical figures may be found in the introductory texts [**1**; **6**; **8**; **15**].

1. Basic concepts and notation. In order to be able to give precise statements about probabilities related to geometrical objects, we have to make some assumptions about the class of geometrical entities taken into consideration. Usually it is necessary to impose certain conditions on the form of these point-sets. Such a class of sets, which is especially suitable for integral

---

Received October 28, 1968. This research was supported by a grant of the "Conseil des Arts du Canada", Ottawa, Canada.

https://doi.org/10.4153/CJM-1970-018-4 Published online by Cambridge University Press

geometric calculations, is the class of convex bodies of the k-dimensional Euclidean space $\mathscr{R}^k$. In the following, we introduce some concepts and symbols of the theory of convex bodies.

First we define by the relations (1)–(3) some sets of auxiliary numbers, which we shall use quite frequently:

(1) $\omega_k := \dfrac{\pi^{k/2}}{\Gamma(k/2+1)}$, volume of the k-dimensional unit ball $[k = 0, 1, 2, \ldots]$,

(2) $c_k := \dfrac{k!}{2}\, \omega_1 \ldots \omega_k$, complete integral over the group of rotations in $\mathscr{R}^k$ $[k = 1, 2, \ldots]$,

(3) $c_{ik} := \dbinom{k}{i} \dfrac{\omega_{k-1} \ldots \omega_{k-i}}{\omega_1 \ldots \omega_i}$ $\quad [i = 1, \ldots, k], c_{0k} = 1 \quad [k = 1, 2, \ldots]$.

A convex body is by definition a compact, convex point-set in $\mathscr{R}^k$. We represent by $\mathscr{A}^k$ the class of all convex bodies of $\mathscr{R}^k$, by $\chi(A)$ the functional on $\mathscr{A}^k$, which assigns the value 0 to the empty set and the value 1 to every other element of $\mathscr{A}^k$ (characteristic of Euler-Poincaré, see [**7**]), by E_i an i-dimensional plane (linear subspace) of $\mathscr{R}^k$ $[i = 0, 1, \ldots, k]$, by $K_r^{(k)}$ a closed k-dimensional ball of radius r (for $k = 0$ and any $r \geqq 0$, $K_r^{(k)}$ degenerates, by definition, to a point), and by Z an arbitrarily chosen point of $\mathscr{R}^k$, which serves as an origin of $\mathscr{R}^k$.

It is well known that convex bodies may be generated from other convex bodies by certain geometric operations. For instance, it is easy to verify that

(4) $$A, B \in \mathscr{A}^k \rightarrow A \cap B \in \mathscr{A}^k,$$

(5) $$A, B \in \mathscr{A}^k \rightarrow A \times B \in \mathscr{A}^k.$$

Here the signs $\cap$ and $\times$ stand for the set-theoretic formation of the intersection and the Minkowski addition, respectively. The Minkowski sum $A \times B$ of the sets A and B is by definition the union of all points, whose radius vector p with respect to Z can be represented by $p = a + b$, where a is a radius vector of a point of A, and b is a radius vector of a point of B. Of course, $A \times B$ depends on the relative position of A and B with respect to Z. However, it is easy to prove that the convex sets, obtained for the different relative positions of A and B, can be mapped by translations into one another [**8**, pp. 13 ff., pp. 142 ff.]. The set $A_r := A \times K_r^{(k)}(Z)$ is conventionally called the "exterior parallel set of A in the distance r". A_r consists of points of the union $\{\bigcup K_r^{(k)}(p) \colon p \in A\}$, where p indicates the centre of the ball. Analogously, we understand the "interior parallel set of A in the distance r" to be the set $A_{-r} := \{\bigcup p \colon K_r^{(k)}(p) \subset A\}$. It is well known that if $A \in \mathscr{A}^k$, then $A_{-r} \in \mathscr{A}^k$.

The fundamental characteristics (*Minkowskische Quermassintegrale*) are often used to describe properties of convex bodies. They comprise a scale of

https://doi.org/10.4153/CJM-1970-018-4 Published online by Cambridge University Press

$k + 1$ non-negative numbers, assigned uniquely to every convex set. These numbers may be defined in the following way using the abbrevations (1) and (3):

$$(6) \qquad W_i(A) := c_{ik}^{-1} \int \chi(A \cap E_i)\, dE_i \qquad [i = 0, 1, \ldots, k-1],$$
$$W_k(A) := \omega_k \chi(A).$$

In the first relation, the integration of E_i is extended over all possible positions of E_i in $\mathscr{R}^k$. Note, however, that $\chi(A \cap E_i) = 0$ if $A \cap E_i = \emptyset$. The differential dE_i, the so-called "kinematic density of E_i", indicates how, in fact, the process of integration has to be carried out. (We have $dE_i = d\bar{E}_i\, d\tilde{E}_i$, where $d\bar{E}_i$ is the point density (volume element) in a plane E_{k-i} totally orthogonal to E_i, and $d\tilde{E}_i$ denotes the density with respect to the group of rotations (for details see [**8**, pp. 227, 240]). The fundamental characteristics are of special importance since, for every $A \in \mathscr{A}^k$, the following relations hold:

$$(7) \qquad W_0(A) = V(A), \qquad kW_1(A) = F(A), \qquad kW_{k-1}(A) = N(A).$$

Here $V(A)$ denotes the volume, $F(A)$ the surface area, and $N(A)$ the norm of A. For the dimensions $k = 1, 2$, and 3, the functions W_i are related to well-known geometrical quantities which we quote from [**8**, p. 210]:

(8)
(a) $k = 1$ (A: linear segment): $W_0(A) = V(A) = s(A)$, length of segment, $W_1(A) = \omega_1\chi(A)$,
(b) $k = 2$ (A: planar convex body): $W_0(A) = f(A)$, planar area, $2W_1(A) = l(A)$ length of circumference, $W_2(A) = \omega_2\chi(A)$,
(c) $k = 3$ (A: 3-dimensional convex body): $W_0(A) = V(A)$, volume, $3W_1(A) = F(A)$ surface area, $3W_2(A) = M(A)$ integral of mean curvature, $W_3(A) = \omega_3\chi(A)$.

It may be useful to remark that most of the results derived in what follows remain valid for many sets which are not convex. With suitable modifications, our investigations could also be carried out within the class of so-called "normal bodies", which contains the elements of $\mathscr{A}^k$; see [**10**].

Finally, we explain two other notations: By $\binom{n}{\lambda_0, \ldots, \lambda_n}$ we mean the multinomial coefficient

$$\binom{n}{\lambda_0, \ldots, \lambda_n} := \frac{n!}{\lambda_0! \ldots \lambda_n!} \qquad [n, \lambda_\nu \geqq 0, \text{integers}; \lambda_0 + \ldots + \lambda_n = n].$$

Furthermore, we call the function

$$(9) \qquad R_k(x_1, \ldots, x_n) := \sum_{\nu_1 < \nu_2 < \ldots < \nu_n} x_{\nu_1} \ldots x_{\nu_k}$$
$$[x_i \in \mathscr{R} \quad [i = 1, \ldots, n], \nu_1, \ldots, \nu_k \in \{1, \ldots, n\}]$$

the "symmetric polynomial of degree k in the variables x_i". In particular, definition (9) implies that

$$R_0(x_1, \ldots, x_n) := 1, \qquad R_1(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i,$$

and

$$R_n(x_1, \ldots, x_n) = \prod_{i=1}^{n} x_i.$$

2. Multiple kinematic and rotational integrals. For our purpose we need a generalization of the following two systems of formulae which are of great importance for the theory of integral geometry:

$$(10) \quad \mathscr{W}_\mu[A_0, A_1] = \int W_\mu(A_0 \cap A_1)\, dA_1 = \frac{c_k}{\omega_k} \sum_{\lambda=0}^{\mu} D_{\lambda,\mu} W_{\mu-\lambda}(A_0) W_\lambda(A_1)$$

$$[\mu = 0, 1, \ldots, k; A_0, A_1 \in \mathscr{A}^k], \quad \text{with } D_{\lambda,\mu} = \binom{\mu}{\lambda} \frac{\omega_{k-\lambda}\,\omega_\mu\,\omega_{k+\lambda-\mu}}{\omega_\lambda\,\omega_{k-\mu}\,\omega_{\mu-\lambda}},$$

$$(11) \quad \mathscr{W}_\mu[A_0 \times A_1] = \int W_\mu(A_0 \times A_1)\, d\tilde{A}_1$$

$$= \frac{c_k}{\omega_k} \sum_{\lambda=0}^{k-\mu} \binom{k-\mu}{\lambda} W_{k-\lambda}(A_1) W_{\mu+\lambda}(A_0)$$

$$[\mu = 0, 1, \ldots, k; A_0, A_1 \in \mathscr{A}^k].$$

Relation (10) is called the "principal theorem of integral geometry" and is due to Blaschke and Santaló [**8**, p. 244]. It reveals an interesting relationship between the integrals of the fundamental characteristics of the intersection of A_0 and A_1 and the fundamental characteristics of these bodies. Note that these integrals are extended over all possible positions of A_1 in $\mathscr{R}^k$, while A_0 remains fixed, and therefore the result does not depend on the position of A_0 in $\mathscr{R}^k$. dA_1 denotes the kinematic density of A_1. In a similar way, (11) expresses a connection between the rotational integrals of the fundamental characteristics of a convex body generated by Minkowski addition, and the fundamental characteristics of the involved bodies [**8**, p. 231]. In this case, the integration extends over all rotations of A_1 around Z with fixed A_0. $d\tilde{A}_1$ designates the rotational density of A_1.

In order to obtain suitable generalizations of (10) and (11), we now allow, not only one, but several movable convex bodies to participate in the formation of the integrals. We therefore study the multiple integrals:

$$\mathscr{W}_\mu[A_0, A_1, \ldots, A_n] = \int \ldots \int W_\mu(A_0 \cap A_1 \cap \ldots \cap A_n)\, dA_1 \ldots dA_n$$

$$[\mu = 0, 1, \ldots, k; n = 1, 2, \ldots; A_0, A_1, \ldots, A_n \in \mathscr{A}^k],$$

$$\mathscr{W}_\mu[A_0 \times A_1 \times \ldots \times A_n] = \int \ldots \int W_\mu(A_0 \times A_1 \times \ldots \times A_n)\, d\tilde{A}_1 \ldots d\tilde{A}_n$$

$$[\mu = 0, 1, \ldots, k; n = 1, 2, \ldots; A_0, A_1, \ldots, A_n \in \mathscr{A}^k].$$

Using (10) and (11), we obtain the recursive formulae

(12) $\mathscr{W}_\mu[A_0, A_1, \ldots, A_n]$

$$= \frac{c_k}{\omega_k} \sum_{\lambda_1=0}^{\mu} D_{\lambda_1,\mu} W_{\lambda_1}(A_1) \mathscr{W}_{\mu-\lambda_1}[A_0, A_2, \ldots, A_n],$$

(13) $\mathscr{W}_\mu[A_0 \times A_1 \times \ldots \times A_n]$

$$= \frac{c_k}{\omega_k} \sum_{\lambda_1=0}^{k-\mu} \binom{k-\mu}{\lambda_1} W_{k-\lambda_1}(A_1) \mathscr{W}_{\mu+\lambda_1}[A_0 \times A_2 \times \ldots \times A_n].$$

Repeated application of (12) and (13) leads to the explicit representations:

$$\mathscr{W}_\mu[A_0, A_1, \ldots, A_n] = \left(\frac{c_k}{\omega_k}\right)^n \sum_{\lambda_0,\ldots,\lambda_n}^{\mu}{}^{*} \left(\prod_{i=1}^{n} D_{\lambda_i,\mu-\sigma_{i-1}}\right)\left(\prod_{i=0}^{n} W_{\lambda_i}(A_i)\right)$$

$$\text{with } \sigma_i = \sum_{p=1}^{i} \lambda_p,\ \sigma_0 = 0,$$

and, after some simple transformations, we have:

(14) $$\mathscr{W}_\mu[A_0, A_1, \ldots, A_n] = \left(\frac{c_k}{\omega_k}\right)^n \sum_{\lambda_0,\ldots,\lambda_n}^{\mu}{}^{*} K(\lambda_0, \ldots, \lambda_n; \mu) \prod_{i=0}^{n} W_{\lambda_i}(A_i)$$

with

(15) $$K(\lambda_0, \ldots, \lambda_n; \mu) = \binom{\mu}{\lambda_0, \ldots, \lambda_n} \frac{\omega_\mu}{\omega_{k-\mu}} \prod_{i=0}^{n} \frac{\omega_{k-\lambda_i}}{\omega_{\lambda_i}}$$

and

(16) $$\mathscr{W}_\mu[A_0 \times A_1 \times \ldots \times A_n] = \left(\frac{c_k}{\omega_k}\right)^n \sum_{\lambda_0,\ldots,\lambda_n}^{k-\mu}{}^{*} \binom{k-\mu}{\lambda_0, \ldots, \lambda_n} \prod_{i=0}^{n} W_{k-\lambda_i}(A_i).$$

The $\overset{s}{\sum}{}^*$ in (14) and (16) is used to indicate that addition is extended over all ordered $(n + 1)$-tuples $(\lambda_0, \ldots, \lambda_n)$ of non-negative integers satisfying the additional condition $\sum_{i=1}^{n} \lambda_i = s$. As we see, it is also possible to represent the multiple kinematic and rotational integrals by the fundamental characteristics of the involved convex bodies. (14) and (16) are generalizations of (10) and (11) since we obtain the latter relations from the former by putting $n = 1$. In the special case $k = 3$, a derivation of (14) has been given by Santaló [**14**, pp. 32 ff.]. The corresponding formula for $k = 2$ may be found in [**1**, § 23; **13**, pp. 231 ff.]. Finally, we would mention that (14) and (16) assume special simple forms in the following cases:

$$\mathscr{W}_0[A_0, A_1, \ldots, A_n] = c_k^n \prod_{i=0}^{n} V(A_i),$$

$$\mathscr{W}_k[A_0 \times A_1 \times \ldots \times A_n] = \omega_k c_k^n \qquad [A_i \neq \emptyset, i = 0, 1, \ldots, n],$$

$$\mathscr{W}_{k-1}[A_0 \times A_1 \times \ldots \times A_n] = c_k^n \sum_{i=0}^{n} W_{k-1}(A_i).$$

The last result may be deduced directly from the well-known linearity of the norm with respect to Minkowski addition.

3. Mean-value formulae for randomly generated sets. Given the convex bodies A_i $[i = 0, \ldots, n]$, the fundamental characteristics of the sets $A_0 \cap A_1 \cap \ldots \cap A_n$ and $A_0 \times A_1 \times \ldots \times A_n$ depend only on the position of the generating convex bodies A_i. Using (14) and (16), we readily obtain statements about the mean values, $\bar{W}_\mu(A_0 \cap A_1 \cap \ldots \cap A_n)$ and $\bar{W}_\mu(A_0 \times A_1 \times \ldots \times A_n)$, of the fundamental characteristics

$$W_\mu(A_0 \cap A_1 \cap \ldots \cap A_n) \text{ and } W_\mu(A_0 \times A_1 \times \ldots \times A_n).$$

In a typical probabilistic application, we assume the convex bodies A_i $[i = 1, \ldots, n]$ to be in random position in $\mathscr{R}^k$. The fundamental characteristic of $\bigcap_{i=0}^n A_i$ may then be interpreted as random variables. In the following, we calculate the expectations of these random variables, having restricted the formation of the mean to all positions of the A_i which satisfy some condition with simple intuitive interpretation. As is usually done when problems about geometrical probabilities occur, we assume the positions taken into consideration to be equivalent in the probabilistic sense. (For an axiomatic foundation of the group-invariant probability theory, see [**9**].)

Under the assumption that $A_0 \cap A_i \neq \emptyset$ $[i = 1, \ldots, n]$ and that the position of A_0 is kept fixed, we obtain the following formulae for the means of the fundamental characteristics of $\bigcap_{i=0}^n A_i$:

$$(17) \quad \bar{W}_\mu(A_0 \cap A_1 \cap \ldots \cap A_n) = \frac{\int \ldots \int W_\mu(A_0 \cap A_1 \cap \ldots \cap A_n)\, dA_1 \ldots dA_n}{\prod_{i=1}^n \int \chi[A_0 \cap A_i]\, dA_i} = \omega_k^n \mathscr{W}_\mu[A_0, A_1, \ldots, A_n]\left(\prod_{i=1}^n \mathscr{W}_k[A_0, A_i]\right)^{-1}.$$

Provided that the fundamental characteristics of the generating sets A_i are known, the right-hand side of (17) can easily be evaluated.

Example. (For similar problems and their practical applications, see [**12**, pp. 109 ff.].) In the case $k = 1$, $\mu = 0$, the convex bodies are linear segments. We denote their lengths by s_i and assume that these values are positive.

$$A_i = S_i, \quad W_0(S_i) = s_i \qquad (s_i > 0) \qquad [i = 0, 1, \ldots, n].$$

From (17) it follows that the mean length of the intersection $\bigcap_{i=1}^n S_i$ (determined under equivalent consideration of all cases $S_i \cap S_0 \neq \emptyset$ $[i = 1, \ldots, n]$) is

$$(18) \qquad \bar{W}_0\left(\bigcap_{i=0}^n A_i\right) = \bar{s}\left(\bigcap_{i=0}^n S_i\right) = s_0\left[\prod_{i=1}^n \left(1 + \frac{s_0}{s_i}\right)\right]^{-1}.$$

This shows that $\bar{s}$ increases monotonically with increasing length of the segments S_i $[i = 1, \ldots, n]$. This is also true for s_0 if $n = 1$. In the case

$n = 2$, however, $\bar{s}$ assumes its maximum for $s_0 = \sqrt{(s_1 s_2)}$, provided that s_1 and s_2 are kept fixed. More generally, we find that, for $n \geqq 2$, $\bar{s}$ reaches its absolute maximum with respect to s_0 when s_0 satisfies the equation

$$s_0^{-1} = \sum_{i=1}^{n} [s_0 + s_i]^{-1}.$$

Formula (17) may be used to solve many similar problems. We limit ourselves to pointing out that the expression $\omega_k^{-1} W_k(\bigcap_{i=1}^{n} A_i)$ may be interpreted as the probability that the common intersection of the n convex bodies A_i hitting A_0 is not empty. This follows immediately from (17).

$$(19)\quad \Pr\left(A_0 \cap A_i \neq \emptyset \quad [i = 1, \ldots, n] \to \bigcap_{i=0}^{n} A_i \neq \emptyset\right) = \omega_k^{-1} \bar{W}_k\left(\bigcap_{i=0}^{n} A_i\right).$$

Applying (19) to our previously mentioned example, we see that

$$(20)\quad \Pr\left(S_0 \cap S_i \neq \emptyset \quad [i = 1, \ldots, n] \to \bigcap_{i=0}^{n} S_i \neq \emptyset\right)$$
$$= \prod_{i=1}^{n} (s_0 + s_i)^{-1} \sum_{p=0}^{n} \left(s_p^{-1} \prod_{i=0}^{n} s_i\right).$$

In another approach to the problem of finding meaningful mean values, we may take into consideration only those positions with $\bigcap_{i=0}^{n} A_i \neq \emptyset$. Instead of (17), we then obtain the modified values

$$(17')\quad \bar{W}_\mu^*(A_0 \cap A_1 \cap \ldots \cap A_n) = \frac{\int \ldots \int W_\mu\left(\bigcap_{i=0}^{n} A_i\right) dA_1 \ldots dA_n}{\int \ldots \int \chi\left(\bigcap_{i=0}^{n} A_i\right) dA_1 \ldots dA_n}$$
$$= \omega_k \mathscr{W}_\mu[A_0, \ldots, A_n](\mathscr{W}_k[A_0, \ldots, A_n])^{-1}.$$

This system of formulae is only of interest for $\mu \neq k$; since $\mu = k$ yields the trivial result $\bar{W}_k^* = \omega_k$.

Relation (16) is also useful for calculations of expectations. Especially interesting are the results obtained by extending the integration over all relative positions of $A_1, \ldots, A_n$ (similar types of expectations are discussed in [**4**]). In order to avoid degenerate cases, we assume in the following that none of the considered sets is empty. Under this restriction, we obtain:

$$(21)\quad \bar{W}_\mu(A_0 \times A_1 \times \ldots \times A_n)$$
$$= \frac{\int \ldots \int W_\mu(A_0 \times A_1 \times \ldots \times A_n)\, d\tilde{A}_1 \ldots d\tilde{A}_n}{\int \ldots \int \chi(A_0 \times A_1 \times \ldots \times A_n)\, d\tilde{A}_1 \ldots d\tilde{A}_n}$$
$$= \frac{\omega_k \mathscr{W}_\mu(A_0 \times \ldots \times A_n)}{\mathscr{W}_k(A_0 \times \ldots \times A_n)} = \frac{1}{c_k^{\,n}} \mathscr{W}_\mu[A_0 \times A_1 \times \ldots \times A_n].$$

Because of (16), this is equivalent to

$$(21') \quad \bar{W}_\mu(A_0 \times A_1 \times \ldots \times A_n) = \frac{1}{\omega_k^{\,n}} \sum_{\lambda_0,\ldots,\lambda_n}^{k-\mu}{}^{*} \binom{k-\mu}{\lambda_0, \ldots, \lambda_n} \prod_{\nu=0}^{n} W_{k-\lambda_\nu}(A_\nu).$$

In particular, we find for $\mu = k$ and $\mu = k - 1$ the simple representations:

$$\bar{W}_k(A_0 \times A_1 \times \ldots \times A_n) = \omega_k,$$

$$\bar{W}_{k-1}(A_0 \times A_1 \times \ldots \times A_n) = \sum_{i=0}^{n} W_{k-1}(A_i).$$

The results discussed in this section may serve to yield statements of a purely geometrical nature. A simple consequence of (17′) and (21) is, for instance, that it is possible to arrange $n + 1$ convex bodies with prescribed fundamental characteristics $W_\mu(A_i)$ in such a way in $\mathscr{R}^k$ that

$$W_\mu\Big(\bigcap_{i=0}^{n} A_i\Big) \geqq \bar{W}_\mu^*\Big(\bigcap_{i=0}^{n} A_i\Big)$$

and

$$W_\mu(A_0 \times A_1 \times \ldots \times A_n) \geqq \bar{W}_\mu(A_0 \times A_1 \times \ldots \times A_n),$$

respectively.

Inequalities of similar form have often been applied with success (see for instance [**2**]).

It may be helpful sometimes to know estimates in simple form for the mean values. One way to obtain suitable inequalities is to take advantage of the inequalities of Minkowski which indicate limits for the fundamental characteristics. For $A_\nu \neq \emptyset$ and $\lambda_\nu = 0, 1, \ldots k$, the following relations hold (see [**8**, p. 278]):

$$\omega_k^{(k-\lambda_\nu)/k} V^{\lambda_\nu/k}(A_\nu) \leqq W_{k-\lambda_\nu}(A_\nu) \leqq \omega_k^{1-\lambda_\nu} W_{k-1}^{\lambda_\nu}(A_\nu).$$

If we substitute this, for instance, in (21′) we obtain the double inequality:

$$\omega_k^{\mu/k} \sum_{\lambda_0,\ldots,\lambda_n}^{k-\mu}{}^{*} \binom{k-\mu}{\lambda_0, \ldots, \lambda_n} \prod_{\nu=0}^{n} V^{\lambda_\nu/k}(A_\nu) \leqq \bar{W}_\mu(A_0 \times \ldots \times A_n),$$

$$\bar{W}_\mu(A_0 \times \ldots \times A_n) \leqq \omega_k^{1-(k-\mu)} \sum_{\lambda_0,\ldots,\lambda_n}^{k-\mu}{}^{*} \binom{k-\mu}{\lambda_0, \ldots, \lambda_n} W_{k-1}^{\lambda_\nu}(A_\nu).$$

Finally, we would like to mention that, by a well-known procedure [**5**], it is possible to obtain from (14) mean-value formulae related to a lattice of convex point-sets.

4. Rotational means of the fundamental characteristics of a zonotope. A zonotope in $\mathscr{R}^k$ may be represented by n linear segments in the following way;

$$T = S_1 \times \ldots \times S_n \qquad \text{(see [3, p. 323]).}$$

For fixed lengths s_i of the segments S_i, the quantities $W_\mu(T)$ depend only on the directions of the generating edges. The expectation $\bar{W}_\mu(T)$ with respect to all combinations of directions is, according to (21) and (21′), given by:

$$\bar{W}_\mu(T) = \bar{W}_\mu(S_1 \times \ldots \times S_n) = \frac{1}{c_k^{n-1}} \mathscr{W}_\mu[S_1 \times S_2 \times \ldots \times S_n]$$
$$= \frac{1}{\omega_k^{n-1}} \sum_{\lambda_1,\ldots,\lambda_n}^{k-\mu}{}^{*} \binom{k-\mu}{\lambda_1, \ldots, \lambda_n} \prod_{i=1}^{n} W_{k-\lambda_i}(S_\nu).$$

A short calculation shows that

$$\text{(22)} \qquad W_k(S_i) = \omega_k, \quad W_{k-1}(S_i) = \frac{\omega_{k-1}}{k} s_i, \quad W_\nu(S_i) = 0 \qquad [\nu = 0, \ldots, k-2].$$

Therefore, we obtain:

$$\bar{W}_\mu(T) = 0 \qquad [0 \leqq \mu < k - n],$$

(23)

$$\bar{W}_\mu(T) = \frac{(k-\mu)!\,\omega_{k-1}{}^{k-\mu}}{k^{k-\mu}\,\omega_k{}^{k-\mu-1}} R_{k-\mu}(s_1, \ldots, s_n) \qquad [\mu \geqq k - n \text{ and } 0 \leqq \mu \leqq k].$$

The polynomials $R_{k-\mu}(s_1, \ldots, s_n)$, appearing in this formula, are the functions defined by (9). In the special case $n = k$, the zonotope is sometimes called parallelotope [**8**, p. 16]. (23) shows, for instance, that the rotational mean $\bar{V}(T)$ of a k-dimensional parallelotope with lengths of edges s_i $[i = 1, \ldots, k]$ differs only by a multiplicative constant $g(k)$ from the volume of the corresponding orthogonal parallelotope having edges of the same length s_i. Note that $g(k)$ depends exclusively on k. We see that

$$\bar{W}_0(T) = \bar{V}(T) = g(k), \qquad R_k(s_1, \ldots, s_k) = g(k) \prod_{i=1}^{k} s_i.$$

The factor $g(k)$ is given by

$$\text{(24)} \qquad g(k) = \frac{k!\,\omega_{k-1}{}^{k}}{k^k\,\omega_k{}^{k-1}} = \frac{k![\Gamma(\frac{1}{2}k + 1)]^{k-1}}{k^k[\Gamma(\frac{1}{2}(k-1) + 1)]^k}.$$

For $k = 1, 2, 3$, this quantity assumes the values

$$g(1) = 1, \qquad g(2) = 2/\pi, \qquad g(3) = \pi/8.$$

5. Probabilities for agglomerations in the k-dimensional Euclidean space. In this section we derive probabilistic statements concerning agglomerations of planes (linear subspaces) which all intersect a given convex set. Such questions have been studied in [**11**] in the special case $\mathscr{R}^3$. In the following, some of these problems are investigated for a Euclidean space with arbitrary dimension k.

We assume that $B \in \mathscr{A}^k$ and $B \neq \emptyset$. We let $E_l^{(1)}, \ldots, E_l^{(n)}$ denote n arbitrary l-dimensional planes in $\mathscr{R}^k$, subjected to the restriction that they have a point in common with B, so that $E_l^{(i)} \cap B \neq \emptyset$ $[i = 1, \ldots, n]$.

Provided that the expression

$$\rho_l: = \inf(t; \exists\ K_t^{(k)}: K_t^{(k)} \subset B, E_l^{(i)} \cap K_t^{(k)} \neq \emptyset\ [i = 1, \ldots, n])$$
$$[k = 1, 2, \ldots; l = 1, \ldots, k]$$

exists, we denote the so-defined number by ρ_l and call it "agglomeration radius of the planes $E_l^{(i)}$ with respect to B". It is important to note that, by characterizing ρ_l, we require only that a ball $K_t^{(k)}$, with the specified properties, exists; in particular, we do not prescribe the position of $K_t^{(k)}$ within B. The quantity ρ_l may be considered as a measure for the local density of the planes within B, and hence comparatively small values of ρ_l indicate special bundling effects.

We now assume the positions of the planes $E_l^{(i)}$ in $\mathscr{R}^k$ (except for the conditional restriction $E_l^{(i)} \cap B \neq \emptyset$) to be random in the sense of the theory of geometrical probabilities. This implies that the construction of the underlying measure for the possibilities of position of the $E_l^{(i)}$ has to be based on the kinematic densities $dE_l^{(i)}$. When ρ_l exists and does not exceed r, where r is a fixed number in the interval $[0, r_B]$ (r_B is the radius of the greatest ball contained in B), we say that the planes $E_l^{(i)}$ are bundled by a ball of radius r with respect to B. We devote the end of this article to the proof that the probability for the occurrence of this event is given by:

$$\begin{aligned} \Pr(\rho_l \leqq r) &= \Pr(E_l^{(i)} \cap B \neq \emptyset \quad [i = 1, \ldots, n] \\ &\qquad \to \exists\ K_r^{(k)} \subset B: E_l^{(i)} \cap K_r^{(k)} \neq \emptyset \quad [i = 1, \ldots, n]), \\ &= r^{(n-1)k-nl}\left[\binom{k}{l}^n \omega_k W_l(B)^n\right]^{-1} \cdot Q(r; k, n, l, B) \end{aligned} \tag{25}$$

with

$$Q(r; k, n, l, B) = \sum_{\lambda_0, \ldots, \lambda_n}^{k}{}^{*[k-l]}\left(K(\lambda_0, \ldots, \lambda_n; k)\left[\prod_{i=1}^{n}\binom{k-\lambda_i}{l}\right]W_{\lambda_0}(B_{-r})r^{\lambda_0}\right)$$
$$[k = 1, 2, \ldots; l = 0, \ldots, k; n = 1, 2, \ldots; B \in \mathscr{A}^k, B \neq \emptyset; 0 \leqq r \leqq r_B].$$

The $\sum^{s}{}^{*[k-l]}$ in (25) indicates that addition has to be extended over all $(n + 1)$-tuples of non-negative integers $(\lambda_0, \ldots, \lambda_n)$ with the properties

$$\sum_{i=0}^{n} \lambda_i = k \qquad [\lambda_1, \ldots, \lambda_k \leqq k - l]$$

(no further restriction is made about λ_0). The constants ω_k and $K(\lambda_0, \ldots, \lambda_n; k)$ are defined by (1) and (15). We notice that the agglomeration probabilities $\Pr(\rho_l \leqq r)$ are simple functions of the chosen level r, the numbers of dimension k and l, and the number n of planes taken into consideration. It is especially worthwhile to observe that the dependence on the set B appears only indirectly in the formula by means of the fundamental characteristics of B and B_{-r}.

Some simple special cases of (25) are:

$l = 0$: $\Pr(\rho_0 \leqq r)$

$$= r^{(n-1)k}[\omega_k W_0(B)^n]^{-1} \sum_{\lambda_0,\ldots,\lambda_n}^{k}{}^{*} (K(\lambda_0, \ldots, \lambda_n; k)\, W_{\lambda_0}(B_{-r}) r^{\lambda_0})$$

(agglomeration of points)

and

$$l = k: \Pr(\rho_k \leqq r) = \frac{W_k(B_{-r})}{\omega_k} = 1 \qquad [0 \leqq r \leqq r_B]$$

(agglomeration of the total space).

The distribution (25) may be used in practical experiments in order to help decide whether the hypothesis that the planes are distributed uniformly at random in $\mathscr{R}^k$ is reasonable or not.

Proof of (25). Since the ideas of the reasoning essentially follow the pattern of the deduction in the previously investigated case $k = 3$, we give only the main steps of the proof and refer to [**11**] for further details. In order to be able to apply the results (10) and (14) for the evaluation of the required agglomeration probabilities, we have to represent the relevant random events as intersections of suitably chosen convex bodies. This can be done in the following way (the symbol $\simeq$ is used to denote congruence):

$$\begin{aligned} &A_0 = B_{-r}, && A_0^* = B, \\ &A_i \simeq A = E_l \times K_r^{(k)} && A_i^* \simeq E_l = \lim_{R\to\infty} A^*(R) \\ (26)\quad &= \lim_{R\to\infty} A(R) \quad [i = 1, \ldots, n] && [i = 1, \ldots, n] \\ &\text{with } A(R) = K_R^{(l)} \times K_r^{(k)}, && \text{with } A^*(R) = K_R^{(l)}. \end{aligned}$$

A simple argument shows that

$$\Pr(\rho_l \leqq r) = \Pr\left(A_i^* \cap A_0^* \neq \emptyset \to \bigcap_{i=0}^{n} A_i \neq \emptyset\right). \tag{27}$$

Therefore

$$\Pr(\rho_l \leqq r) = \lim_{R\to\infty} \frac{\int \ldots \int \chi(A_0 \cap A_1(R) \cap \ldots \cap A_n(R))\, dA_1(R) \ldots dA_n(R)}{[\int \chi(A_0^* \cap A^*(R))\, dA^*(R)]^n}$$

(28) or

$$\Pr(\rho_l \leqq r) = \lim_{R\to\infty} \frac{\omega_k^{-1} \mathscr{W}_k[B_{-r}, K_R^{(l)} \times K_r^{(k)}, \ldots, K_R^{(l)} \times K_r^{(k)}]}{\omega_k^{-n} (\mathscr{W}_k[B, K_R^{(l)}])^n}.$$

Thus, using (6) and (26), we have found an expression for $\Pr(\rho_l \leqq r)$ in terms of the multiple kinematic integrals discussed in § 2. In order to actually be able to take the limit $R \to \infty$, we need explicit expressions for the

https://doi.org/10.4153/CJM-1970-018-4 Published online by Cambridge University Press

k-dimensional fundamental characteristics of $K_R^{(l)}$ and $K_R^{(l)} \times K_r^{(k)}$. A standard calculation leads to the auxiliary formulae

$$W_\mu(K_R^{(l)}) = \begin{cases} \binom{k}{l}^{-1} \binom{\mu}{k-l} \frac{\omega_\mu \omega_l}{\omega_{\mu-(k-l)}} R^{k-\mu} & [\mu \geqq k-l], \\ 0 & [\mu < k-l], \end{cases} \tag{29}$$

$$W_\mu(K_R^{(l)} \times K_r^{(k)}) = \sum_{m=\max(0,k-l-\mu)}^{k-\mu} \binom{k}{l}^{-1} \binom{k-\mu}{m} \binom{\mu+m}{k-l} \times \frac{\omega_{\mu+m}\omega_l}{\omega_{\mu+m-(k-l)}} R^{k-\mu-m} r^m. \tag{30}$$

We now substitute these data in (10) and (14), and hence reach the following representation for the numerator and denominator of (28):

$$\omega_k^{-1} \mathscr{W}_k[B_{-r}, K_R^{(l)} \times K_r^{(k)}, \ldots, K_R^{(l)} \times K_r^{(k)}]$$
$$= \omega_k^{-1} \left(\binom{k}{l}^{-1} \frac{c_k}{\omega_k} \omega_l \omega_{k-l} \right)^n r^{(n-1)k-nl} \cdot \left(\sum_{\lambda_0, \ldots, \lambda_n}^{k} {}^{*[k-l]} K(\lambda_0, \ldots, \lambda_n; k) \right.$$
$$\left. \times \left[\prod_{i=1}^{n} \binom{k-\lambda_i}{l} \right] W_{\lambda_0}(B_{-r}) r^{\lambda_0} \right) R^{nl} + o(R^{nl}) \qquad [R \to \infty],$$

$$\omega_k^{-n} (\mathscr{W}_k[B, K_R^{(l)}])^n = (\omega_k^{-1} c_k \omega_l \omega_{k-l} W_l(B))^n R^{nl} + o(R^{nl}) \qquad [R \to \infty].$$

From this, we obtain (25) by forming the quotient and taking the limit as prescribed in (28).

For practical applications, the agglomeration probabilities related to the one-dimensional, two-dimensional, and three-dimensional case may be especially useful. For $k = 3$, one obtains from (25) the results derived in [**11**]. We conclude this article by giving a survey of the results in the cases $k = 1$ and $k = 2$.

The case $k = 1$. $l = 0$: agglomeration of points in $\mathscr{R}$:

$$B = S, \quad W_0(B) = s, \quad W_1(B) = \frac{C(B)}{2\pi}, \quad C(B) = 4\pi\chi(B),$$

$$\Pr(\rho_0 \leqq r) = \frac{2^{n-1} r^{n-1}}{s^n} \left[nV(B_{-r}) + \frac{C(B_{-r})}{2\pi} r \right]. \tag{31}$$

Therefore the probability that n points chosen at random in a segment of length s lie in a segment of length t is

$$(1-n)\left(\frac{t}{s}\right)^n + n\left(\frac{t}{s}\right)^{n-1} \qquad [t \leqq s]. \tag{32}$$

Similar formulae and their connection to the Poisson distribution are compiled in [**16**].

The case $k = 2$. $l = 0$: agglomeration of points in $\mathscr{R}^2$:

$$W_0(B) = f(B),\quad W_1(B) = \tfrac{1}{2}l(B),\quad W_2(B) = \tfrac{1}{4}C(B),\quad C(B) = 4\pi\chi(B),$$

$$\Pr(\rho_0 \leqq r) = \frac{\pi^{n-1}r^{2n-2}}{[f(B)]^n}\,[n^2 f(B_{-r}) + nl(B_{-r})r + \tfrac{1}{4}C(B_{-r})r^2]. \tag{33}$$

$l = 1$: agglomeration of straight lines in $\mathscr{R}^2$:

$$\Pr(\rho_1 \leqq r) = \frac{(2\pi)^{n-1}r^{n-2}}{[l(B)]^n}\left[\binom{n}{2}f(B_{-r}) + \binom{n}{1}l(B_{-r})r + \tfrac{1}{2}C(B_{-r})r^2\right]. \tag{34}$$

References

1. W. Blaschke, *Vorlesungen über Integralgeometrie.* I und II. (Teubner, Leipzig, 1936–37; Chelsea, New York, 1949; VEB Deutscher Verlag der Wissenschaften, Berlin, 1955).
2. L. Fejes Tóth and H. Hadwiger, *Mittlere Treffzahlen und geometrische Wahrscheinlichkeiten,* Experientia *3* (1947), 366–369.
3. B. Grünbaum, *Convex polytopes* (Interscience, London, 1967).
4. H. Hadwiger, *Ueber die mittlere mittlere Breite zufallsartig gestalteter Polygone,* Comment. Math. Helv. *11* (1938), 321–329.
5. ——— *Ueber Mittelwerte im Figurengitter,* Comment. Math. Helv. *11* (1939), 221–233.
6. ——— *Altes und Neues über konvexe Körper* (Birkhäuser, Basel, 1955).
7. ——— *Eulers Charakteristik und kombinatorische Geometrie.* J. Reine Angew. Math. *194* (1955), 101–110.
8. ——— *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie* (Springer, Berlin, 1957).
9. ——— *Zur Axiomatik der innermathematischen Wahrscheinlichkeitstheorie,* Mitt. Verein. Schweiz. Versich.-Math. *58* (1958), 159–165.
10. ——— *Normale Körper im euklidischen Raum und ihre topologischen und metrischen Eigenschaften,* Math. Z. *71* (1959), 124–140.
11. H. Hadwiger and F. Streit, *Ueber Wahrscheinlichkeiten räumlicher Bündelungserscheinungen,* Monatsh. Math. *74* (1970), 30–40.
12. M. G. Kendall and P. A. P. Moran, *Geometrical probability* (Griffin, London, 1963).
13. L. A. Santaló, *Geometria Integral.* 4: *Sobre la medida cinemática en el plano,* Abh. Math. Sem. Univ. Hamburg *11* (1936), 222–236.
14. ——— *Integralgeometrie.* 5: *Ueber das kinematische Mass im Raum* (Hermann, Paris, 1936).
15. ——— *Introduction to integral geometry* (Hermann, Paris, 1953).
16. F. Streit, *Anordnungswahrscheinlichkeiten bei poissonverteilten Ereignissen,* Ph.D. Thesis, Universität Bern, Bern, Switzerland, 1966.

University of Toronto,
Toronto, Ontario