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Abstract

In this article we prove that the Weinstein conjecture holds for contact manifolds (Σ,ξ) for which $\mathrm{Cont}_0(\Sigma,\xi)$ is non-orderable in the sense of Eliashberg and Polterovich [*Partially ordered groups and geometry of contact transformations*, Geom. Funct. Anal. **10** (2000), 1448–1476]. More precisely, we establish a link between orderable and hypertight contact manifolds. In addition, we prove for certain contact manifolds a conjecture by Sandon [*A Morse estimate for translated points of contactomorphisms of spheres and projective spaces*, Geom. Dedicata **165** (2013), 95–110] on the existence of translated points in the non-degenerate case.

1. Introduction

One of the driving questions in the field of contact geometry is the famous *Weinstein conjecture* [Wei79] which asserts for a closed coorientable contact manifold (Σ,ξ) that any supporting contact form admits a periodic Reeb orbit. See for instance [Hut10] for more information.

In [EP00] Eliashberg and Polterovich introduced the concept of *orderability* of contact manifolds, which is closely related to the question of contact (non-)squeezing; see [EKP06]. We denote by $\mathrm{Cont}_0(\Sigma,\xi)$ the group of contactomorphisms of (Σ,ξ) which are contact isotopic to the identity, and by $\widetilde{\mathrm{Cont}}_0(\Sigma,\xi)$ its universal cover. Eliashberg and Polterovich proved that $\mathrm{Cont}_0(\Sigma,\xi)$ is non-orderable if and only if there exists a *positive* loop φ in $\mathrm{Cont}_0(\Sigma,\xi)$, and similarly that $\widetilde{\mathrm{Cont}}_0(\Sigma,\xi)$ is non-orderable if and only if there exists a *positive contractible* loop φ; see § 2 for details. Here is our first result.

Theorem 1.1. *The Weinstein conjecture holds for any contact manifold (Σ,ξ) for which $\mathrm{Cont}_0(\Sigma,\xi)$ is non-orderable. If, in addition, $\widetilde{\mathrm{Cont}}_0(\Sigma,\xi)$ is non-orderable, then every supporting contact form admits* a contractible *closed Reeb orbit.*

Remark 1.2. Note that the original notion of orderability in [EP00] actually concerns the universal cover $\widetilde{\mathrm{Cont}}_0(\Sigma,\xi)$ which is much more restrictive than orderability of $\mathrm{Cont}_0(\Sigma,\xi)$. There are many examples of contact manifolds for which $\mathrm{Cont}_0(\Sigma,\xi)$ is non-orderable while $\widetilde{\mathrm{Cont}}_0(\Sigma,\xi)$ is orderable, e.g. $\mathbb{RP}^{2n-1}$.

Remark 1.3. Given a loop $\varphi=\{\varphi_t\}_{t\in S^1}$ of contactomorphisms we denote by $u_\varphi\in\pi_1(\Sigma)$ the homotopy class of the loop $t\mapsto\varphi_t(x)$, and by $\tilde{u}_\varphi$ the corresponding free homotopy class (i.e. the image of u_φ under the map $\pi_1\to\pi_1/\text{conjugacy}=[S^1,\Sigma]$). Theorem 1.1 can be sharpened as follows: if there exists a positive loop φ in $\mathrm{Cont}_0(\Sigma,\xi)$, then for any supporting contact form,

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either there exists a closed contractible Reeb orbit or there exists a closed Reeb orbit in the free homotopy class $\tilde{u}_\varphi$. In fact, the same assertion is true for any loop φ with spectral number $c(\varphi) \neq 0$; see Definition 4.1 and the proof of Theorem 1.1 on page 2260.

Example 1.4. For all contact manifolds (Σ, ξ) admitting a supporting contact form with periodic Reeb flow, $\mathrm{Cont}_0(\Sigma, \xi)$ is non-orderable, since in this case the Reeb flow is itself a positive loop (cf. § 2). An interesting class of examples of contact manifolds are *prequantization spaces*. Here one begins with a closed symplectic manifold (M, ω) for which the de Rham cohomology class $[\omega]$ has a primitive integral lift in $\mathrm{H}^2(M; \mathbb{Z})$. Consider a circle bundle $p : \Sigma_k \to M$ with Euler class $k[\omega]$ for some $k \in \mathbb{Z}$ with $k \neq 0$, and connection 1-form α with $p^*(k\omega) = -d\alpha$. Then (Σ_k, α) is a contact manifold whose associated Reeb flow is periodic. The closed Reeb orbits are the fibres of the bundle. The long exact homotopy sequence of the fibration is

$$\pi_2(M) \xrightarrow{q_k} \pi_1(S^1) \to \pi_1(\Sigma_k) \to \pi_1(M) \to 0. \tag{1.1}$$

The map q_k is non-trivial if and only if the homotopy class of the fibre is torsion (and note that if q_k is non-trivial, then so is q_{nk} for all $n \neq 0$). See Example 1.8 below for the relevance of this last statement.

A contact manifold (Σ, ξ) is called *hypertight* if it admits a supporting contact form without any contractible Reeb orbits; see for example [CH05] for a construction of hypertight contact manifolds. The 3-torus $\mathbb{T}^3$ (equipped with any one of the standard contact structures $\alpha_k = \cos(2\pi kr)\, ds + \sin(2\pi kr)\, dt$) is a familiar example. We point out that if a contact manifold is *not* hypertight, then all supporting contact forms possess *contractible* Reeb orbits, which is a stronger assertion than the Weinstein conjecture!

Theorem 1.5. *If (Σ, ξ) is hypertight, then for any positive loop φ in $\mathrm{Cont}_0(\Sigma, \xi)$ the class $u_\varphi \in \pi_1(\Sigma)$ is of infinite order.*

The following corollary is immediate from Theorem 1.5.

Corollary 1.6. *If (Σ, ξ) is hypertight, then all positive loops of contactomorphisms are of infinite order in $\pi_1(\mathrm{Cont}_0(\Sigma, \xi))$.*

Remark 1.7. Theorem 1.5 and Corollary 1.6 illustrate the sharp contrast with life in the symplectic world. Indeed, if (M, ω) is a compact symplectic manifold, then the evaluation map $\pi_1(\mathrm{Ham}(M, \omega)) \to \pi_1(M)$ is always trivial, cf. [MS98, Exercise 11.28].

Using Theorems 1.1 and 1.5, we can now improve Example 1.4 to obtain the following.

Example 1.8. Prequantization spaces (Σ, ξ) are hypertight if and only if the fibre is not torsion. Note that $\mathrm{Cont}_0(\Sigma, \xi)$ is always non-orderable. If in addition $\widetilde{\mathrm{Cont}}_0(\Sigma, \xi)$ is non-orderable then the homotopy class of the fibre is torsion.

Remark 1.9. Example 1.8 is sharp in the following sense: $\mathbb{RP}^{2n-1}$ is a prequantization space with torsion fibres, but $\widetilde{\mathrm{Cont}}_0(\mathbb{RP}^{2n-1}, \xi_{\mathrm{st}})$ is orderable. This can proved using Givental's non-linear Maslov index, cf. [EP00]. Otto van Koert explained to us that Example 1.8 can also be shown by using contact homology.

Finally, we prove for certain contact manifolds a conjecture by Sandon [San13, Conjecture 1.2] in the non-degenerate case. For this we recall that if $\psi \in \mathrm{Cont}_0(\Sigma, \xi)$ and α is a supporting contact form, then a point $x \in \Sigma$ is a *translated point* of ψ (with respect to α) if $\psi(x)$ belongs to the same Reeb orbit as x does, and if ψ is 'exact at x' in the sense that $\psi^*\alpha|_x = \alpha|_x$.

Theorem 1.10. *Let $(\Sigma, \xi = \ker \alpha)$ be a contact manifold such that α has no contractible closed Reeb orbits. Then every non-degenerate $\psi \in \mathrm{Cont}_0(\Sigma, \xi)$ has at least $\sum_{j=0}^{\dim \Sigma} \dim H_j(\Sigma; \mathbb{Z}_2)$ many translated points.*

Remark 1.11. The non-degeneracy hypothesis in Theorem 1.10 is a standard one, and is satisfied generically. See Definition 3.6 below. We remark also that Theorem 1.10 holds in a more general setting. Namely, for any non-degenerate Hamiltonian symplectomorphism of the symplectization of Σ, there are at least $\sum_{j=0}^{\dim \Sigma} \dim H_j(\Sigma; \mathbb{Z}_2)$ many *leaf-wise intersection points* in Σ. An even more general result is the main theorem in a forthcoming paper of Meiwes and Naef.

Remark 1.12. Sandon [San11] was the first to discover a connection between translated points and orderability and other contact rigidity phenomena. She informed us that she is working on a Floer-theoretical approach and that Zénaïdi is working on an approach based on Legendrian Contact Homology. We expect interesting interactions between this paper and the approaches followed by Sandon and Zénaïdi.

2. Preliminaries

We denote by $\mathrm{Cont}_0(\Sigma, \xi)$ the identity component of the group of contactomorphisms. Unless specified otherwise a *path* $\varphi = \{\varphi_t\}_{0 \leqslant t \leqslant 1}$ of contactomorphisms is always smoothly parametrized and begins at the identity. We denote by $\mathcal{P}\,\mathrm{Cont}_0(\Sigma, \xi)$ the set of all such paths. The universal cover $\widetilde{\mathrm{Cont}}_0(\Sigma, \xi)$ is then $\mathcal{P}\,\mathrm{Cont}_0(\Sigma, \xi)/\sim$, where $\sim$ denotes the equivalence relation of being homotopic with fixed endpoints. Suppose $\alpha \in \Omega^1(\Sigma)$ is a contact form defining ξ. To a path $\varphi = \{\varphi_t\}_{0 \leqslant t \leqslant 1} \in \mathcal{P}\,\mathrm{Cont}_0(\Sigma, \xi)$ we can uniquely associate its contact Hamiltonian h_t defined by

$$h_t \circ \varphi_t := \alpha\left(\frac{d}{dt}\varphi_t\right) : \Sigma \to \mathbb{R}, \tag{2.1}$$

see for instance [Gei08, ch. 2.3]. The following definitions are taken from [EP00]. Given $\psi \in \mathrm{Cont}_0(\Sigma, \xi)$ let us say that

$$\psi \geqslant \mathrm{id} \tag{2.2}$$

if there exists a path $\varphi = \{\varphi_t\}_{0 \leqslant t \leqslant 1}$ with $\psi = \varphi_1$ such that the contact Hamiltonian h_t of φ is everywhere non-negative. We say that $\mathrm{Cont}_0(\Sigma, \xi)$ is *orderable* if the relation $\geqslant$ induces a partial order on $\mathrm{Cont}_0(\Sigma, \xi)$. Otherwise $\mathrm{Cont}_0(\Sigma, \xi)$ is *non-orderable*. We can play the same game with $\widetilde{\mathrm{Cont}}_0(\Sigma, \xi)$: given $\Phi \in \widetilde{\mathrm{Cont}}_0(\Sigma, \xi)$ let us say that

$$\Phi \geqslant_u \mathrm{id} \tag{2.3}$$

if there exists a representative $\{\varphi_t\}_{0 \leqslant t \leqslant 1} \in \mathcal{P}\,\mathrm{Cont}_0(\Sigma, \xi)$ of Φ whose contact Hamiltonian is everywhere non-negative. Then we say that $\widetilde{\mathrm{Cont}}_0(\Sigma, \xi)$ is *orderable* if $\geqslant_u$ induces a partial order on $\widetilde{\mathrm{Cont}}_0(\Sigma, \xi)$; otherwise $\widetilde{\mathrm{Cont}}_0(\Sigma, \xi)$ is *non-orderable*.

Remark 2.1. We call a loop of contactomorphisms whose contact Hamiltonian is everywhere *strictly* positive a 'positive loop' (of contactomorphisms).

Although we used a supporting contact form α to define the notion of positivity, it is easy to see that the positivity of a path $\{\varphi_t\}_{0 \leqslant t \leqslant 1}$ is equivalent to the vector field $(d/dt)\varphi_t$ defining the given coorientation of ξ, and this of course does not depend on the choice of α. The following characterization of orderability by Eliashberg and Polterovich [EP00, § 2.1] is crucial.

Proposition 2.2 [EP00, § 2.1]. *The group $\mathrm{Cont}_0(\Sigma, \xi)$ is non-orderable if and only if there exists a positive loop of contactomorphisms. Moreover $\widetilde{\mathrm{Cont}}_0(\Sigma, \xi)$ is non-orderable if and only if there exists a positive* contractible *loop.*

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3. Rabinowitz Floer homology

Definition 3.1. Let us say that a contact form α supporting ξ is WCRO if α is *without contractible Reeb orbits*. Thus hypertight contact manifolds are exactly those which admit a WCRO contact form.

The aim of this section is to define the *Rabinowitz Floer homology* $\mathrm{RFH}_*(\Sigma, \alpha; \varphi)$ associated to a contact manifold (Σ, ξ) with a supporting WCRO contact form α and a path $\varphi = \{\varphi_t\}_{0 \leqslant t \leqslant 1}$ in $\mathcal{P}\,\mathrm{Cont}_0(\Sigma, \xi)$. The main novelty in our treatment is that we do *not* need a filling of the contact manifold. Rabinowitz Floer homology was first constructed in [CF09] by Cieliebak and Frauenfelder. The construction we present now is derived from [AM13]. We start with some notation.

Remark 3.2. By convention, in what follows a Reeb orbit of period $T < 0$ is by definition the inverse parametrization of a Reeb orbit of period $-T$.

From now on we fix a WCRO contact form α defining ξ, and denote by R its Reeb vector field and $\theta^t : \Sigma \to \Sigma$ the Reeb flow. We emphasize that we are *not* assuming that the non-contractible Reeb orbits of α are non-degenerate (cf. Remark 3.7 below). The *symplectization* $(S\Sigma, d\lambda)$ of (Σ, α) is the manifold $S\Sigma := (0, \infty) \times \Sigma$ equipped with the symplectic form $d\lambda$, where $\lambda := r\alpha$. Here r is the coordinate on $(0, \infty)$. By a common abuse of notation we identify R with the vector field $(0, R)$ on $S\Sigma$. A path $\{\varphi_t\}_{0 \leqslant t \leqslant 1} \in \mathcal{P}\,\mathrm{Cont}_0(\Sigma, \xi)$ can be lifted to a Hamiltonian isotopy on $S\Sigma$ as follows. Write $\varphi_t^* \alpha = \rho_t \varphi_t$ for $\rho_t : \Sigma \to (0, \infty)$. Then

$$(x, r) \mapsto \left(\frac{r}{\rho_t(x)}, \varphi_t(x) \right) \tag{3.1}$$

is the Hamiltonian flow of

$$H_t(x, r) := r h_t(x) : S\Sigma \to \mathbb{R}. \tag{3.2}$$

Recall from the Introduction the definition of a translated point of a contactomorphism. This notion was introduced by Sandon in [San12]. For us their relevance is that the generators of the Rabinowitz Floer homology $\mathrm{RFH}_*(\Sigma, \alpha; \varphi)$ are precisely the translated points of φ.

Definition 3.3. Fix $\psi \in \mathrm{Cont}_0(\Sigma, \xi)$, and write $\psi^* \alpha = \rho \alpha$ for a smooth positive function ρ on Σ. A *translated point* of ψ is a point $x \in \Sigma$ with the property that there exists $\eta \in \mathbb{R}$ such that

$$\psi(x) = \theta^\eta(x) \quad \text{and} \quad \rho(x) = 1. \tag{3.3}$$

We call η a *time-shift* of x. Note that if the leaf $\{\theta^t(x)\}_{t \in \mathbb{R}}$ is closed, then a time-shift η is not uniquely determined by x.

In order to define the (perturbed) Rabinowitz action functional we will work with a collection of cutoff functions depending on parameters $\kappa > \kappa' > 0$ of the following form. Let $m_{\kappa,\kappa'} : \mathbb{R} \to \mathbb{R}$ satisfy

$$m_{\kappa,\kappa'}(r) = \begin{cases} r - 1 & r \in [e^{-\kappa'}, e^{\kappa'}], \\ c_1 & r \in [e^{\kappa}, \infty), \\ c_2 & r \in (0, e^{-\kappa}], \end{cases} \qquad m'_{\kappa,\kappa'}(r) \geqslant 0, \tag{3.4}$$

for some suitable constants $c_1, c_2 \in \mathbb{R}$. Similarly let $\varepsilon_{\kappa,\kappa'} \in C^\infty((0, \infty), [0, 1])$ denote a smooth function such that

$$\varepsilon_{\kappa,\kappa'}(r) = \begin{cases} 1 & r \in [e^{-\kappa'}, e^{\kappa'}], \\ 0 & r \in (0, e^{-\kappa}] \cup [e^{\kappa}, \infty). \end{cases} \tag{3.5}$$

Next, fix a smooth function $\nu : S^1 \to \mathbb{R}$ with

$$\nu(t) = 0 \quad \text{for all } t \in \left[\frac{1}{2}, 1\right] \quad \text{and} \quad \int_0^1 \nu(t)\, dt = 1, \tag{3.6}$$

and fix a smooth monotone map $\chi : [0,1] \to [0,1]$ with $\chi(\frac{1}{2}) = 0$ and $\chi(1) = 1$. Denote by $\mathcal{L}S\Sigma$ the space of *contractible* smooth loops $u : S^1 \to S\Sigma$. The perturbed Rabinowitz action functional will be a functional defined on $\mathcal{L}S\Sigma \times \mathbb{R}$.

Definition 3.4. Fix a path $\varphi = \{\varphi_t\}_{0 \leqslant t \leqslant 1} \in \mathcal{P}\operatorname{Cont}_0(\Sigma, \xi)$. Let $h_t : \Sigma \to \mathbb{R}$ denote the contact Hamiltonian and set $H_t = rh_t : S\Sigma \to \mathbb{R}$. We define the *perturbed Rabinowitz action functional* associated to φ and a pair of real numbers $\kappa > \kappa' > 0$, written

$$\mathcal{A}_\varphi^{\kappa,\kappa'} : \mathcal{L}S\Sigma \times \mathbb{R} \to \mathbb{R}, \tag{3.7}$$

as follows: given $u \in \mathcal{L}S\Sigma$ write $u = (a, f)$, so that $a : S^1 \to (0, \infty)$ and $f \in \mathcal{L}\Sigma$. Set

$$\mathcal{A}_\varphi^{\kappa,\kappa'}(u, \eta) := \int_0^1 u^*\lambda - \eta \int_0^1 \nu(t) m_{\kappa,\kappa'}(a(t))\, dt - \int_0^1 \dot{\chi}(t) \varepsilon_{\kappa,\kappa'}(a(t)) H_{\chi(t)}(u(t))\, dt. \tag{3.8}$$

In [AM13, Proposition 2.5] we proved the following.

Lemma 3.5. *Assume $\varphi = \{\varphi_t\}_{0 \leqslant t \leqslant 1}$ is a path of contactomorphims. Write $\varphi_t^*\alpha = \rho_t\alpha$, for smooth functions $\rho_t : \Sigma \to (0, \infty)$. Set*

$$\kappa(\varphi) := \max_{t \in [0,1]} \int_0^t \max_{x \in \Sigma} \left| \frac{\dot{\rho}_\tau(x)}{\rho_\tau(x)^2} \right| d\tau. \tag{3.9}$$

If $\kappa > \kappa' > \kappa(\varphi)$ and $(u = (a, f), \eta) \in \operatorname{Crit}(\mathcal{A}_\varphi^{\kappa,\kappa'})$, then $a(S^1) \subseteq (e^{-\kappa'}, e^{\kappa'})$. Thus for $\kappa > \kappa' > \kappa(\varphi)$, a pair (u, η) is a critical point of $\mathcal{A}_\varphi^{\kappa,\kappa'}$ only if $x := f(\frac{1}{2})$ is a translated point of φ_1, with $-\eta$ a time-shift of x. Conversely every such pair (x, η) gives rise to a unique critical point of $\mathcal{A}_\varphi^{\kappa,\kappa'}$. Moreover one has

$$\mathcal{A}_\varphi^{\kappa,\kappa'}(u, \eta) = \eta. \tag{3.10}$$

In what follows we tacitly assume that $\kappa > \kappa(\varphi)$, even if this is not explicitly stated. In order to simplify the notation we define

$$\mathcal{A}_\varphi^{\kappa} := \mathcal{A}_\varphi^{\kappa,\kappa'}, \tag{3.11}$$

where κ' is any number such that $\kappa > \kappa' > \kappa(\varphi)$. The precise choice of κ' is unimportant in all of what follows. Next, we set

$$\operatorname{Spec}(\varphi) := \mathcal{A}_\varphi^{\kappa}(\operatorname{Crit}(\mathcal{A}_\varphi^{\kappa})). \tag{3.12}$$

Definition 3.6. A path φ is *non-degenerate* if $\mathcal{A}_\varphi^{\kappa} : \mathcal{L}S\Sigma \times \mathbb{R} \to \mathbb{R}$ is a Morse–Bott function for some (and hence any) $\kappa > \kappa(\varphi)$.

Remark 3.7. In [AM13] we explained why a generic path φ is non-degenerate (actually a stronger result is true: for generic φ the functional $\mathcal{A}_\varphi^{\kappa}$ is even Morse). Moreover $\operatorname{Spec}(\varphi)$ is always a nowhere dense subset of $\mathbb{R}$ (even in the degenerate case), cf. [Sch00, Lemma 3.8]. Finally given $\varphi = \{\varphi_t\} \in \mathcal{P}\operatorname{Cont}_0(\Sigma, \xi)$ the set $\operatorname{Spec}(\varphi)$ only depends on the endpoint φ_1 of the path φ. This follows from Lemma 3.5 together with the fact that the definition of a translated point only involves the map φ_1.

The non-perturbed Rabinowitz action functional $\mathcal{A}_{\mathrm{id}}^{\kappa}$ is Morse–Bott. Indeed, since α is WCRO, the critical points of $\mathcal{A}_{\mathrm{id}}^{\kappa}$ are all of the form $(u = (a, f), \eta = 0)$ with $a(t) \equiv 1$ and $f(t) \equiv x$ for some point $x \in \Sigma$. This is a Morse–Bott component, cf. [AF10, Lemma 2.12].

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Fix a family $J = \{J_t\}_{t \in S^1}$ of almost complex structures on $S\Sigma$ compatible with $d\lambda$. Here we use the (slightly unusual) sign convention that compatibility means that $d\lambda(J_t\cdot\,,\cdot)$ defines a family of Riemannian metrics on $S\Sigma$. We assume in addition that J is *SFT-like* (here and elsewhere, 'SFT' stands for Symplectic Field Theory) and independent of t outside a compact set. Here an almost complex structure on $S\Sigma$ is SFT-like if it is invariant under the translations $(r,x) \mapsto (e^c r, x)$ for $c \in \mathbb{R}$, and if it preserves ξ and satisfies $JR = r\partial_r$. We denote by $\langle\langle\cdot\,,\cdot\rangle\rangle_J$ the L^2-inner product defined by

$$\langle\langle(\hat{u},\hat{\eta}),(\hat{v},\hat{\tau})\rangle\rangle_J := \int_{S^1} d\lambda(J_t\hat{u},\hat{v})\,dt + \hat{\eta}\hat{\tau} \quad \text{for } (\hat{u},\hat{\eta}),(\hat{v},\hat{\tau}) \in T_{(u,\eta)}(\mathcal{L}S\Sigma \times \mathbb{R}). \tag{3.13}$$

We denote by $\nabla_J \mathcal{A}^{\kappa}_{\varphi}$ the gradient of $\mathcal{A}^{\kappa}_{\varphi}$ with respect to the inner product $\langle\langle\cdot\,,\cdot\rangle\rangle_J$ on $\mathcal{L}S\Sigma \times \mathbb{R}$, that is, the integro-differential operator

$$\nabla_J \mathcal{A}^{\kappa}_{\varphi}(u,\eta) = \left(J_t(u)(\partial_t u - \eta\nu X_{m_{\kappa,\kappa'}}(u) - \varepsilon_{\kappa,\kappa'}\dot{\chi}X_{H_\chi}(u)), -\int_{S^1} \nu m_{\kappa,\kappa'}(u)\,dt\right). \tag{3.14}$$

Negative gradient flow lines of $\nabla_J \mathcal{A}^{\kappa}_{\varphi}$ solve the equation

$$\partial_s(u,\eta) + \nabla_J \mathcal{A}^{\kappa}_{\varphi}(u,\eta) = 0, \tag{3.15}$$

and have energy

$$\mathbb{E}_J(u,\eta) := \int_{-\infty}^{\infty}\int_0^1 |\partial_s(u,\eta)|_J^2\,dt\,ds. \tag{3.16}$$

Definition 3.8. Denote by $\mathcal{M}(\varphi,\kappa,J)$ the set of all such flow lines with finite energy $\mathbb{E}_J(u,\eta) < e^{-\kappa}\wp(\alpha)$, where $\wp(\alpha)$ denotes the minimal period of a closed contractible Reeb orbit of α.

Of course the assumption that α is WCRO implies that $\wp(\alpha) = +\infty$, and hence in this paper $\mathcal{M}(\varphi,\kappa,J)$ simply denotes the set of *all* finite energy flow lines. Nevertheless, we state (and prove) Theorem 3.9 below without the assumption that α is WCRO, as this will be useful in a forthcoming paper. If $\mathcal{A}^{\kappa}_{\varphi}$ is Morse–Bott, then any element $(u,\eta) \in \mathcal{M}(\varphi,\kappa,J)$ converges to critical points $(u_\pm,\eta_\pm) \in \mathrm{Crit}(\mathcal{A}^{\kappa}_{\varphi})$ as $s \to \pm\infty$ and

$$\mathbb{E}_J(u,\eta) = \eta_- - \eta_+. \tag{3.17}$$

The following theorem shows how the assumption that α is WCRO implies that one can obtain compactness for flow lines of the perturbed Rabinowitz action functional $\mathcal{A}^{\kappa}_{\varphi}$. The proof uses the procedure developed by Cieliebak and Mohnke in [CM05] to establish SFT compactness (compare also [BEHWZ03]). We emphasize that in contrast to the standard SFT compactness results, in Theorem 3.9, we do *not* need to assume that the contact form α is non-degenerate.

Theorem 3.9. *Let φ be a non-degenerate path of contactomorphisms and choose $\kappa > \kappa(\varphi)$. Fix $\kappa > \kappa' > \kappa(\varphi)$ and let J be a family of almost complex structures on $S\Sigma$ compatible with $d\lambda$ which is independent of t and of SFT-type on the complement of $(e^{-\kappa'}, e^{\kappa'}) \times \Sigma$. Then there exists $\ell > 0$ such that*

$$\mathrm{im}(u) \subset [e^{-\ell}, e^{\kappa}] \times \Sigma \quad \textit{for all } (u,\eta) \in \mathcal{M}(\varphi,\kappa,J). \tag{3.18}$$

We will prove Theorem 3.9 in § 5 below. The upshot of Theorem 3.9 is that it is possible to define the *Rabinowitz Floer homology* $\mathrm{RFH}_*(\Sigma,\alpha;\varphi)$ working directly in the symplectization. This is a semi-infinite dimensional Morse theory associated to the functional $\mathcal{A}^{\kappa}_{\varphi}$ (for some $\kappa > \kappa(\varphi)$). We refer to [CF09, AF10, AM13] for more details of the construction used in this paper. Instead here we only summarize the key properties that we will need about the Rabinowitz Floer homology $\mathrm{RFH}_*(\Sigma,\alpha;\varphi)$. From now on we will assume that α is WCRO.

(i) Since α is assumed to be WCRO, the Rabinowitz Floer homology $\mathrm{RFH}_*(\Sigma,\alpha) := \mathrm{RFH}_*(\Sigma,\alpha;\mathrm{id})$ is canonically isomorphic to the singular homology $\mathrm{H}_{*+n-1}(\Sigma;\mathbb{Z}_2)$. Moreover the Rabinowitz Floer homology is independent of φ in the following strong sense: there are canonical isomorphisms

$$\zeta_\varphi : \mathrm{RFH}_*(\Sigma,\alpha) \to \mathrm{RFH}_*(\Sigma,\alpha;\varphi), \tag{3.19}$$

and given two paths φ and ψ, there is a canonical map $\zeta_{\varphi,\psi} : \mathrm{RFH}_*(\Sigma,\alpha;\varphi) \to \mathrm{RFH}_*(\Sigma,\alpha;\psi)$, the continuation homomorphism, with the property that

$$\zeta_\psi = \zeta_{\varphi,\psi} \circ \zeta_\varphi. \tag{3.20}$$

In particular, $\mathrm{RFH}_n(\Sigma,\alpha;\varphi)$ contains a non-zero class $[\Sigma_\varphi]$ which is defined by

$$\zeta_{\varphi,\psi}([\Sigma_\varphi]) = [\Sigma_\psi] \quad \text{and} \quad [\Sigma_{\mathrm{id}}] = [\Sigma] \in \mathrm{RFH}_n(\Sigma,\alpha) = \mathrm{H}_{2n-1}(\Sigma;\mathbb{Z}_2). \tag{3.21}$$

(ii) Denote by $\mathrm{RFH}^c_*(\Sigma,\alpha;\varphi)$ the Rabinowitz Floer homology generated by the subcomplex of critical points (u,η) of $\mathcal{A}^\kappa_\varphi$ with $\eta \leqslant c$. Then there is natural map

$$\iota^c_\varphi : \mathrm{RFH}^c_*(\Sigma,\alpha;\varphi) \to \mathrm{RFH}_*(\Sigma,\alpha;\varphi)$$

induced by the inclusion of critical points. Moreover, given two paths φ and ψ, there is a constant $K(\varphi,\psi)$ such that the map $\zeta_{\varphi,\psi}$ from property (i) defines a map

$$\zeta_{\varphi,\psi} : \mathrm{RFH}^c_*(\Sigma,\alpha;\varphi) \to \mathrm{RFH}^{c+K(\varphi,\psi)}_*(\Sigma,\alpha;\psi) \tag{3.22}$$

for any $c \in \mathbb{R}$. We can estimate

$$K(\varphi,\psi) \leqslant e^{\max\{\kappa(\varphi),\kappa(\psi)\}} \int_0^1 \max\Big\{\max_{x\in\Sigma}(h_t(x) - k_t(x)), 0\Big\}\, dt, \tag{3.23}$$

where h and k are the contact Hamiltonians for φ and ψ, respectively, and $\kappa(\varphi)$ is defined in (3.9).

Remark 3.10. In property (ii) we are using the fact that Theorem 3.9 also holds for s-dependent spaces $\mathcal{M}(\{\varphi_s\}_{s\in[0,1]},\kappa,\{J_s\}_{s\in[0,1]})$. In fact, in Theorem 5.3 we will prove a more general result, which we then use to deduce both Theorem 3.9 and its analogue for s-dependent solutions.

Even though it is more or less standard, the estimate (3.23) is extremely important in all that follows, and hence we prove it here. To define the continuation homomorphism $\zeta_{\varphi,\psi}$ we denote by $H_t = rh_t$ and $K_t = rk_t$ the Hamiltonian functions of φ and ψ, respectively, and choose a linear homotopy

$$L^s_t := \beta(s)H_t + (1-\beta(s))K_t \tag{3.24}$$

for a smooth function $\beta : \mathbb{R} \to [0,1]$ with $\beta(s) = 1$ for $s \leqslant -1$, $\beta(s) = 0$ for $s \geqslant 1$ and $\beta'(s) \leqslant 0$. We define the (s-dependent) action functional $\mathcal{A}_s = \mathcal{A}^{\kappa,\kappa'}_{\varphi_s}$ as in (3.8):

$$\mathcal{A}_s(u,\eta) = \int_0^1 u^*\lambda - \eta\int_0^1 \nu(t) m_{\kappa,\kappa'}(a(t))\,dt - \int_0^1 \dot\chi(t)\varepsilon_{\kappa,\kappa'}(a(t)) L^s_{\chi(t)}(u(t))\,dt, \tag{3.25}$$

where φ_s has corresponding Hamiltonian function L^s_t. Then counting solutions of

$$\partial_s(u,\eta) + \nabla_J \mathcal{A}_s(u,\eta) = 0 \tag{3.26}$$

with $(u_-,\eta_-) := (u(-\infty),\eta(-\infty)) \in \mathrm{Crit}(\mathcal{A}^\kappa_\varphi)$ and $(u_+,\eta_+) := (u(+\infty),\eta(+\infty)) \in \mathrm{Crit}(\mathcal{A}^\kappa_\psi)$ defines the continuation homomorphism. We recall that $\kappa > \max\{\kappa(\varphi),\kappa(\psi)\}$ and estimate

$$\begin{aligned}
0 &\leqslant \mathbb{E}_J(u,\eta)\\
&= \int_{-\infty}^{\infty}\int_0^1 |\partial_s(u,\eta)|_J^2\,dt\,ds\\
&= -\int_{-\infty}^{\infty}\int_0^1 \langle\langle \nabla\mathcal{A}_s(u,\eta),\partial_s(u,\eta)\rangle\rangle_J\,dt\,ds\\
&= -\int_{-\infty}^{\infty}\int_0^1 \frac{d}{ds}\mathcal{A}_s(u,\eta)\,ds + \int_{-\infty}^{\infty}\int_0^1 \frac{\partial\mathcal{A}_s}{\partial s}(u,\eta)\,dt\,ds\\
&= \mathcal{A}_\varphi(u_-,\eta_-) - \mathcal{A}_\psi(u_+,\eta_+) - \int_{-\infty}^{\infty}\int_0^1 \dot\chi(t)\varepsilon_{\kappa,\kappa'}(a(t))\frac{\partial L^s_{\chi(t)}}{\partial s}(u(t))\,dt\,ds\\
&= \mathcal{A}_\varphi(u_-,\eta_-) - \mathcal{A}_\psi(u_+,\eta_+)\\
&\quad - \int_{-\infty}^{\infty}\int_0^1 \beta'(s)\varepsilon_{\kappa,\kappa'}(a(t))\dot\chi(t)(H_{\chi(t)}(u(t)) - K_{\chi(t)}(u(t)))\,dt\,ds\\
&= \mathcal{A}_\varphi(u_-,\eta_-) - \mathcal{A}_\psi(u_+,\eta_+)\\
&\quad - \int_{-\infty}^{\infty}\int_0^1 \underbrace{\beta'(s)}_{\leqslant 0}\underbrace{\varepsilon_{\kappa,\kappa'}(a(t))a(t)}_{0\leqslant\cdot\leqslant e^\kappa}\underbrace{\dot\chi(t)}_{\geqslant 0}(h_{\chi(t)}(u(t)) - k_{\chi(t)}(u(t)))\,dt\,ds\\
&\leqslant \mathcal{A}_\varphi(u_-,\eta_-) - \mathcal{A}_\psi(u_+,\eta_+)\\
&\quad - \int_{-\infty}^{\infty}\int_0^1 \underbrace{\beta'(s)}_{\leqslant 0}\underbrace{\varepsilon_{\kappa,\kappa'}(a(t))a(t)}_{0\leqslant\cdot\leqslant e^\kappa}\underbrace{\dot\chi(t)}_{\geqslant 0}\max_{x\in\Sigma}(h_{\chi(t)}(x) - k_{\chi(t)}(x))\,dt\,ds\\
&\leqslant \mathcal{A}_\varphi(u_-,\eta_-) - \mathcal{A}_\psi(u_+,\eta_+)\\
&\quad - \int_{-\infty}^{\infty}\int_0^1 \underbrace{\beta'(s)}_{\leqslant 0}\underbrace{\varepsilon_{\kappa,\kappa'}(a(t))a(t)}_{0\leqslant\cdot\leqslant e^\kappa}\underbrace{\dot\chi(t)}_{\geqslant 0}\max\Big\{\max_{x\in\Sigma}(h_{\chi(t)}(x) - k_{\chi(t)}(x)),0\Big\}\,dt\,ds\\
&\leqslant \mathcal{A}_\varphi(u_-,\eta_-) - \mathcal{A}_\psi(u_+,\eta_+)\\
&\quad - e^\kappa\int_{-\infty}^{\infty}\int_0^1 \beta'(s)\dot\chi(t)\max\Big\{\max_{x\in\Sigma}(h_{\chi(t)}(x) - k_{\chi(t)}(x)),0\Big\}\,dt\,ds\\
&= \mathcal{A}_\varphi(u_-,\eta_-) - \mathcal{A}_\psi(u_+,\eta_+)\\
&\quad - e^\kappa\underbrace{\int_{-\infty}^{\infty}\beta'(s)\,ds}_{=-1}\int_0^1 \dot\chi(t)\max\Big\{\max_{x\in\Sigma}(h_{\chi(t)}(x) - k_{\chi(t)}(x)),0\Big\}\,dt\\
&= \mathcal{A}_\varphi(u_-,\eta_-) - \mathcal{A}_\psi(u_+,\eta_+) + e^\kappa\int_0^1 \max\Big\{\max_{x\in\Sigma}(h_t(x) - k_t(x)),0\Big\}\,dt.
\end{aligned} \tag{3.27}$$

This proves estimate (3.23). We conclude this section by proving Theorem 1.10 from the Introduction, which we restate here for the convenience of the reader.

Theorem 3.11. *Let $(\Sigma,\xi=\ker\alpha)$ be a contact manifold such that α has no contractible closed Reeb orbits. Then every non-degenerate $\psi\in\mathrm{Cont}_0(\Sigma,\xi)$ has at least $\sum_{j=0}^{\dim\Sigma}\dim H_j(\Sigma;\mathbb{Z}_2)$ many translated points.*

Proof. Since $\mathrm{RFH}_*(\Sigma,\alpha;\varphi)\cong \mathrm{H}_{*+n-1}(\Sigma;\mathbb{Z}_2)$, and the translated points of φ_1 are the generators of $\mathrm{RFH}_*(\Sigma,\alpha;\varphi)$, the theorem is almost immediate. There is a slight subtlety, however, coming from the fact that φ_1 could have translated points lying on a closed Reeb orbit. Indeed, suppose

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x is a translated point of φ_1 lying on a closed Reeb orbit of period T (note by assumption this orbit is necessarily non-contractible). Let $0 \leqslant \eta < T$ denote a time-shift of x. Let $y_k : \mathbb{R}/2\mathbb{Z} \to \Sigma$ denote the continuous and piecewise smooth map defined by

$$y_k(t) := \begin{cases} \theta^{(kT+\eta)t}(\theta^{-\eta}(x)) & t \in [0,1], \\ \varphi_{t-1}(x) & t \in [1,2]. \end{cases} \tag{3.28}$$

Next we observe that there is at most one value of k for which y_k is contractible, and for this value of k we obtain a generator $(u_k, \eta + kT)$ of $\mathrm{RFH}_*(\Sigma, \alpha; \varphi)$. This proves the theorem. □

4. Spectral invariants

We now explain how to use the Rabinowitz Floer homology groups $\mathrm{RFH}_*(\Sigma, \alpha; \varphi)$ to define *spectral numbers* $c(\varphi) \in \mathbb{R}$ associated to any path $\varphi = \{\varphi_t\}_{0 \leqslant t \leqslant 1} \in \mathcal{P}\,\mathrm{Cont}_0(\Sigma, \xi)$. In particular, we still assume that α is WCRO.

Definition 4.1. Let φ denote a non-degenerate path. We define its *spectral number* by

$$c(\varphi) := \inf\{c \in \mathbb{R} \mid [\Sigma_\varphi] \in \iota^c_\varphi(\mathrm{RFH}^c_*(\Sigma, \alpha; \varphi))\}. \tag{4.1}$$

The fact that $c(\varphi) \in \mathbb{R}$, that is, the infimum in (4.1) is not $-\infty$, follows immediately from Proposition 4.2 and Lemma 4.3 below.

Proposition 4.2. *Let φ and ψ be two non-degenerate paths. Then we have the estimate*

$$\begin{aligned} c(\psi) &\leqslant c(\varphi) + K(\varphi, \psi) \\ &\leqslant c(\varphi) + e^{\max\{\kappa(\varphi), \kappa(\psi)\}} \int_0^1 \max\Big\{\max_{x \in \Sigma}(h_t(x) - k_t(x)), 0\Big\}\, dt \end{aligned} \tag{4.2}$$

where h and k are the contact Hamiltonians of φ and ψ, respectively. In particular, we have

$$h_t(x) \leqslant k_t(x) \quad \text{for all } x \in \Sigma, t \in [0,1] \Longrightarrow c(\varphi) \geqslant c(\psi). \tag{4.3}$$

Proof. This follows immediately from the definition of the spectral number together with (3.22) and the estimate (3.23). □

Lemma 4.3. *For any non-degenerate path $\varphi \in \mathcal{P}\,\mathrm{Cont}_0(\Sigma, \xi)$ its spectral number is a critical value of $\mathcal{A}_\varphi$, i.e. $c(\varphi) \in \mathrm{Spec}(\varphi)$. In particular $c(t \mapsto \theta^{tT}) = -T$ where as always θ^t is the Reeb flow.*

Moreover c admits a unique extension to all of $\mathcal{P}\,\mathrm{Cont}_0(\Sigma, \xi)$: given a degenerate path φ, set

$$c(\varphi) := \lim_k c(\varphi_k), \tag{4.4}$$

where $\varphi_k \to \varphi$ is any sequence of non-degenerate paths converging to φ in C^2. The extension still satisfies $c(\varphi) \in \mathrm{Spec}(\varphi)$ and the estimates (4.2) and (4.3). In particular, $c : \mathcal{P}\,\mathrm{Cont}_0(\Sigma, \xi) \to \mathbb{R}$ is a continuous function when we equip $\mathcal{P}\,\mathrm{Cont}_0(\Sigma, \xi)$ with the C^2-topology.

Proof. The assertion $c(\varphi) \in \mathrm{Spec}(\varphi)$ follows immediately from the fact that $\mathrm{RFH}^c_*(\Sigma, \alpha; \varphi)$ only changes if c crosses a critical value of $\mathcal{A}_\varphi$. Moreover $\mathrm{Spec}(t \mapsto \theta^{tT}) = \{-T\}$.

To prove the existence of the extension we are required to prove that the limit exists and is independent of the choice of approximating sequence φ_k. We denote by h_k the corresponding contact Hamiltonians. Since we assume that φ_k converges to φ in C^2 it follows that $\kappa(\varphi_k) \to \kappa(\varphi)$

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and $h_k \to h$, the contact Hamiltonian of φ. From Proposition 4.2 we conclude that $(c(\varphi_k))$ converges and in the same way independence of the approximating sequence (φ_k) is proved. That $c(\varphi) \in \operatorname{Spec}(\varphi)$ and that the estimates (4.2) and (4.3) hold follows from the definition of c as a limit. □

Corollary 4.4. *Suppose that φ has contact Hamiltonian h_t with $h_t \leqslant -\delta < 0$ for all $t \in [0,1]$. Then $c(\varphi) > 0$. Similarly if $h_t \geqslant \delta > 0$, then $c(\varphi) < 0$.*

Proof. Note that the constant function δ generates the path $\{t \mapsto \theta^{t\delta}\}$. Thus Proposition 4.2 and Lemma 4.3 together with $h_t \leqslant -\delta < 0$ imply

$$c(\varphi) \geqslant c(t \mapsto \theta^{-t\delta}) = \delta > 0. \tag{4.5}$$

Similarly, $h_t \geqslant \delta > 0$ implies

$$c(\varphi) \leqslant c(t \mapsto \theta^{t\delta}) = -\delta < 0. \tag{4.6}$$

□

Lemma 4.5. *The map $c : \mathcal{P}\operatorname{Cont}_0(\Sigma,\xi) \to \mathbb{R}$ descends to give a well-defined map $c : \widetilde{\operatorname{Cont}}_0(\Sigma,\xi) \to \mathbb{R}$.*

Proof. We recall from Remark 3.7 that $\operatorname{Spec}(\varphi) \subset \mathbb{R}$ is nowhere dense and actually only depends on the endpoint φ_1 of the path φ. Moreover, Lemma 4.3 implies that c is a continuous map. If we vary the path φ while keeping the endpoints fixed, the continuous map c takes values in the fixed, nowhere dense set $\operatorname{Spec}(\varphi_1)$, and thus is constant. This proves the Lemma. □

The following result pertains specifically to loops $\varphi = \{\varphi_t\}_{t\in S^1}$ of contactomorphisms. Recall from Remark 1.3 that to such a loop we have associated classes $u_\varphi \in \pi_1(\Sigma)$, as well as a class $\tilde{u}_\varphi \in [S^1, \Sigma]$. We remind the reader of the convention adopted in Remark 3.2.

Proposition 4.6. *Suppose φ is a loop of contactomorphisms. Then $c(\varphi) = 0$ if and only if $\tilde{u}_\varphi$ is the class of contractible loops. Moreover if $c(\varphi) \neq 0$, then there exists a Reeb orbit of period $-c(\varphi)$ belonging to the free homotopy class $\tilde{u}_\varphi$.*

Proof. We first remind the reader that, since φ is assumed to be a loop, if $(u = (a, f), \eta)$ is a critical point of $\mathcal{A}^\kappa_\varphi$, then the loop $f : S^1 \to \Sigma$ is obtained by concatenating a closed Reeb orbit $t \mapsto \theta^{t\eta}(x)$ of period η with the loop $x \mapsto \varphi_t(x)$ (modulo reparametrization).

Suppose that $c(\varphi) = 0$. Then there is a critical point $(u = (a, f), \eta = 0)$ of $\mathcal{A}^\kappa_\varphi$ which (modulo reparametrization) is of the form $f(t) = \varphi_t(x)$ for some point $x \in \Sigma$. Since $\mathcal{A}^\kappa_\varphi$ is defined on the space of contractible loops, u is contractible, and thus the class $\tilde{u}_\varphi$ is necessarily trivial. If $c(\varphi) \neq 0$, then there exists a critical point of $\mathcal{A}^\kappa_\varphi$ of the form $(u = (a, f), \eta)$ where f is the concatenation of a (non-constant) closed Reeb orbit and the loop $x \mapsto \varphi_t(x)$ for some point $x \in \Sigma$. Since u is a contractible loop by assumption, this Reeb orbit must belong to the free homotopy class $-\tilde{u}_\varphi$. This proves the remaining two statements. □

We can now prove the main results of this paper.

Proof of Theorem 1.1 and of the sharpened statement from Remark 1.3. If (Σ, ξ) is not hypertight, then there is nothing to prove. Otherwise there exists α which is WCRO. We first show that there are no non-trivial contractible positive loops. Indeed, if φ is a contractible loop, then $c(\varphi) = c(\mathrm{id}) = 0$ by Lemma 4.5. But from Corollary 4.4, a positive loop φ satisfies $c(\varphi) < 0$.

Now assume that $\mathrm{Cont}_0(\Sigma,\xi)$ is non-orderable. Thus there exists a positive loop φ (which is necessarily non-contractible). Then again by Corollary 4.4 we have $c(\varphi)<0$, and Proposition 4.6 implies that $\tilde{u}_\varphi$ contains a Reeb orbit of period $-c(\varphi)$. The final statement from Remark 1.3 follows again from Proposition 4.6. ☐

Proof of Theorem 1.5. Choose a supporting contact form α for (Σ,ξ) which is WCRO. We argue by contradiction: assume that there is a positive loop φ and $x\in\Sigma$ such that the class $u_\varphi\in\pi_1(\Sigma,x)$ is torsion. Thus there is $k\in\mathbb{N}$ with

$$1=(u_\varphi)^k=u_{\varphi^k}\in\pi_1(\Sigma). \tag{4.7}$$

This implies that $\tilde{u}_{\varphi^k}\in[S^1,\Sigma]$ is the class of contractible loops. Hence by Proposition 4.6 one has $c(\varphi^k)=0$. But φ^k is still a positive loop, and hence $c(\varphi^k)<0$ by Corollary 4.4; contradiction. ☐

5. SFT compactness

The aim of this section is to prove Theorem 3.9. In fact, we will prove in Theorem 5.3 below a result on pseudoholomorphic curves which will imply Theorem 3.9, and the generalization mentioned in Remark 3.10. Before stating Theorem 5.3, we need to introduce various definitions.

In this section, it will be more convenient to view the symplectization of Σ as the manifold $\mathbb{R}\times\Sigma$ endowed with the symplectic form $d(e^s\alpha)$, where s is the coordinate on $\mathbb{R}$. This identification is justified by the canonical map

$$i:\mathbb{R}\times\Sigma\longrightarrow(0,\infty)\times\Sigma,\quad (s,x)\mapsto(r,x):=(e^s,x) \tag{5.1}$$

which satisfies $i^*(r\alpha)=e^s\alpha$ and is therefore an exact symplectomorphism. Under this identification an almost complex structure on $(0,\infty)\times\Sigma$ of SFT-type is identified with an almost complex structure i^*J on $\mathbb{R}\times\Sigma$ which is invariant under $\mathbb{R}$-translations, preserves the contact distribution and satisfies $JR=\partial_s$, where $R=(0,R)$ still denotes the Reeb vector field for α. The set of such almost complex structures will be denoted by $\mathcal{J}_{\mathrm{SFT}}(\mathbb{R}\times\Sigma,d(e^s\alpha))$. We next recall the definition of the *Hofer energy* of a J-holomorphic map $u:Z\to\mathbb{R}\times\Sigma$.

Definition 5.1. Let (Z,j) denote a compact Riemann surface (possibly disconnected and with boundary). Orient Z by declaring (jv,v) to be a positively oriented basis of T_zZ whenever $0\neq v\in T_zZ$. Fix $J\in\mathcal{J}_{\mathrm{SFT}}(\mathbb{R}\times\Sigma,d(e^s\alpha))$. Suppose $u:Z\to\mathbb{R}\times\Sigma$ is a (j,J)-holomorphic map. Write $u=(a,f)$ so that $f:Z\to\Sigma$ and $a:Z\to\mathbb{R}$. Define the *Hofer energy* $E(u)$ of u as

$$E(u)=\sup_{\nu\in\mathcal{S}}\int_Z u^*\,d(\nu\alpha)=\sup_{\nu\in\mathcal{S}}\left(\int_Z u^*(\nu\,d\alpha)+\int_Z u^*(\nu'(s)\,ds\wedge\alpha)\right)\in[0,\infty], \tag{5.2}$$

where $\mathcal{S}:=\{\nu\in C^\infty(\mathbb{R},[0,1])\mid\nu'\geqslant 0\}$.

Given $J\in\mathcal{J}_{\mathrm{SFT}}(\mathbb{R}\times\Sigma,d(e^s\alpha))$, we can define a compatible $\mathbb{R}$-invariant Riemannian metric g_J on $\mathbb{R}\times\Sigma$ given by $g_J(\cdot,\cdot)=d\alpha(J\cdot,\cdot)+\alpha^2(\cdot,\cdot)+(dr)^2(\cdot,\cdot)$.

Warning: Since the compatibility condition imposed on J follows the (slightly unusual) sign convention adopted throughout this paper, the first factor of the metric g_J takes a different form to the corresponding metric in [CM05].

Example 5.2. Assume γ is a T-periodic orbit of α and let $u:\mathbb{R}\times S^1\to\mathbb{R}\times\Sigma$ have the form $u(s,t)=(c\pm Ts,\gamma(\pm tT))$ for some $c\in\mathbb{R}$. If the complex structure j on $\mathbb{R}\times S^1$ satisfies $j\partial_t=\partial_s$ for coordinates $(s,t)\in\mathbb{R}\times S^1$, then u is a J-holomorphic map for any $J\in\mathcal{J}_{\mathrm{SFT}}(\mathbb{R}\times\Sigma,d(e^s\alpha))$, and $E(u)=T$. Such a map u is called a *trivial cylinder over the periodic orbit* γ.

Next we state the main result of this section. Results to a similar effect were previously obtained in [Hof93], [Lau95, § 4] and [EHS95, § 3.9] and it is probably well known to those who studied work related to SFT compactness as in [BEHWZ03, CM05]. Nevertheless, to the best of our knowledge, this result has not appeared explicitly in the literature so far. We emphasize that we make *no* non-degeneracy assumptions on the Reeb orbits of α. We give below a proof following the method of [CM05], but it is also possible to prove this result using Fish's *target local compactness* [Fis11], as we explain in Remark 5.5 below.

Theorem 5.3. *Let (Σ, α) be a cooriented contact manifold and let $J \in \mathcal{J}_{\mathrm{SFT}}(\mathbb{R} \times \Sigma, d(e^s \alpha))$. Suppose (Z_k, j_k) is a family of compact (possibly disconnected) Riemann surfaces with boundary and uniformly bounded genus. Assume that*

$$u_k = (a_k, f_k) : Z_k \to \mathbb{R} \times \Sigma \tag{5.3}$$

is a sequence of (j_k, J)-holomorphic maps which have uniformly bounded Hofer energy $E(u_k) \leqslant E$, are non-constant on each connected component of Z_k, and satisfy $a_k(\partial Z_k) \subset [0, \infty)$.

Assume that $\inf_k \inf_{Z_k} a_k = -\infty$. Then there exists a subsequence k_n and cylinders $C_n \subset Z_{k_n}$, biholomorphically equivalent to standard cylinders $[-L_n, L_n] \times S^1$, such that $L_n \to \infty$ and such that $u_{k_n}|_{C_n}$ converges (up to an $\mathbb{R}$-shift) in $C^\infty_{\mathrm{loc}}(\mathbb{R} \times S^1, \mathbb{R} \times \Sigma)$ to a trivial cylinder over a Reeb orbit of period at most E.

We now show how Theorem 3.9 follows from Theorem 5.3.

Proof of Theorem 3.9. Observe that in the situation of Theorem 3.9, J is independent of t on the complement of $(e^{-\kappa'}, e^{\kappa'}) \times \Sigma$ and the restriction $u|_{u^{-1}(((0,\infty)\setminus(e^{-\kappa}, e^{\kappa}))\times\Sigma)}$ of any flow line $(u, \eta) \in \mathcal{M}(\varphi, \kappa, J)$ is a J-holomorphic curve by equation (3.14). Since the asymptotic ends $u_\pm$ of u are contained in $(e^{-\kappa'}, e^{\kappa'}) \times \Sigma$, we can apply the maximum principle to the J-holomorphic curve $u|_{u^{-1}(((0,\infty)\setminus(e^{-\kappa}, e^{\kappa}))\times\Sigma)}$ to see that $\operatorname{im} u \subset (0, e^\kappa] \times \Sigma$.

Recall the map i from (5.1) and observe that there exists $\overline{J} \in \mathcal{J}_{\mathrm{SFT}}(\mathbb{R} \times \Sigma, d(e^s\alpha))$ such that if $(u, \eta) \in \mathcal{M}(\varphi, \kappa, J)$, then the restriction $u|_{u^{-1}((-\infty, -\kappa]\times\Sigma)}$ gives rise to a $\overline{J}$-holomorphic map $\overline{u}$ into $\mathbb{R} \times \Sigma$.

Claim. The Hofer energy $E(\overline{u})$ for $(u, \eta) \in \mathcal{M}(\varphi, \kappa, J)$ is uniformly bounded by $e^\kappa(\eta_- - \eta_+)$.

Indeed, if $\lim_{s\to\pm\infty}(u, \eta) = (u_\pm, \eta_\pm)$, we have from (3.17) that

$$\eta_- - \eta_+ = \int_{-\infty}^{\infty} |\nabla \mathcal{A}^\kappa_\varphi(u, \eta)|^2_J \geqslant \int_{\overline{u}^{-1}((-\infty,-\kappa]\times\Sigma)} \overline{u}^* \, d(e^s\alpha). \tag{5.4}$$

On the other hand, by Stokes' theorem we have for any $\nu \in \mathcal{S}$ that

$$\int_{\overline{u}^{-1}((-\infty,-\kappa]\times\Sigma)} \overline{u}^* \, d(\nu\alpha) \leqslant \int_{\overline{u}^{-1}((-\infty,-\kappa]\times\Sigma)} \overline{u}^* \, d\alpha = e^\kappa \int_{\overline{u}^{-1}((-\infty,-\kappa]\times\Sigma)} \overline{u}^* \, d(e^s\alpha) \tag{5.5}$$

and therefore, by the definition of $E(\overline{u})$ (cf. Definition 5.1), the claimed bound follows.

Assume now by contradiction that there is no ℓ as in Theorem 3.9, and hence there exists a sequence of gradient flow lines $(u_k, \eta_k) \in \mathcal{M}(\varphi, \kappa, J)$ such that the corresponding maps $\overline{u}_k = (\overline{a}_k, \overline{f}_k)$ satisfy $\lim_k \inf \overline{a}_k = -\infty$. Then, for a $K < -\kappa$ which is a regular value of all the $\overline{a}_k$, we consider the $\overline{J}$-holomorphic curves $v_k := \overline{u}_k|_{\overline{u}_k^{-1}((-\infty,K]\times\Sigma)}$ and let $Z_k := (\overline{u}_k)^{-1}((-\infty, K] \times \Sigma)$.

Since the gradient flow lines are asymptotic to critical points contained in $(e^{-\kappa'}, e^{\kappa'}) \times \Sigma$, each Z_k is a compact Riemann surface of genus 0. Moreover the $\overline{J}$-holomorphic curves v_k have no constant components and their Hofer energy is uniformly bounded by $e^\kappa(\eta_- - \eta_+)$.

By applying Theorem 5.3 to the pseudoholomorphic curves v_k (shifted by K in the $\mathbb{R}$-direction) it follows that there exists a map u_{k_0} (in fact a whole subsequence of the u_k of such maps) with the following property: there is an embedded circle S in the domain $\mathbb{R}\times S^1$ of u_{k_0}, such that the restriction of f_{k_0} to S parametrizes a circle in Σ homotopic to a Reeb orbit γ of period less than $e^{\kappa}(\eta_- - \eta_+)$. Since the domain of u_{k_0} is $\mathbb{R}\times S^1$, this circle S bounds a disk D in $\mathbb{R}\times S^1$, or it is isotopic to a circle $\{s\}\times S^1\subset\mathbb{R}\times S^1$. In either case γ is contractible, since $u_{k_0}(S)$ is contractible. In the latter case this follows since $f_{k_0}(S)$ is homotopic to the asymptotic end of f_{k_0}, and this is contractible since the asymptotic ends $(u_{k_0})_\pm$ of $u_{k_0}=(a_{k_0},f_{k_0})$ lie in $\mathcal{L}S\Sigma$. This contradiction shows that images of all the maps u (for $(u,\eta)\in\mathcal{M}(\varphi,\kappa,J)$) must be contained in a compact subset $[e^{-\ell},e^{\kappa}]\times\Sigma$ as claimed.

Adaptations for Remark 3.10. We note that in the s-dependent case (i.e. for moduli spaces $\mathcal{M}(\{\varphi_s\}_{s\in[0,1]},\kappa,\{J_s\}_{s\in[0,1]})$) the proof goes through verbatim, since the almost complex structures involved are by assumption both s and t independent on the complement of a compact subset of the symplectization. □

The proof of Theorem 5.3 depends on the following proposition, which we will prove in the next section as Proposition 6.10 by following the methods of [CM05].

Proposition 5.4. *Under the assumptions of Theorem 5.3 we can find, by passing to a subsequence, compact subcylinders $C_k\subset Z_k$ such that an $\mathbb{R}$-shift $v_k=(b_k,g_k)$ of the restriction of u_k to C_k has the following properties:*

(1) *C_k is biholomorphic to $[-L_k,L_k]\times S^1$ and $L_k\to\infty$ as $k\to\infty$;*
(2) *$\int_{C_k} v_k^*\,d\alpha\to 0$;*
(3) *there is a sequence $\sigma_k\to\infty$ such that $\pm b_k(\pm L_k,t)\geqslant\sigma_k$ for each $t\in S^1$.*

Proof of Theorem 5.3. Theorem 5.3 is an immediate consequence of Proposition 5.4 and the following claim.

Claim. *If $v_k=(b_k,g_k):[-L_k,L_k]\times S^1\to\mathbb{R}\times\Sigma$ is a sequence of pseudoholomorphic cylinders as in Proposition 5.4 above, then there exists a subsequence which converges, after possibly an $\mathbb{R}$-shift, in the C^∞_{loc}-topology to a trivial cylinder over a periodic Reeb orbit.*

Proof of Claim. First observe that away from the boundary the derivatives are uniformly bounded. That is, there is $D>0$ such that $\|dv_k(s,t)\|\leqslant D$ for all $(s,t)\in[-L_k+1,L_k-1]\times S^1$. Otherwise, by applying a bubbling off procedure on disks with radius $1/3$ around points where the gradient blows up, one would obtain a non-constant pseudoholomorphic map $v:\mathbb{C}\to\mathbb{R}\times\Sigma$ with $\int v^*\,d\alpha=0$; this, however, contradicts [Hof93, Lemma 28]. Since we have uniform gradient bounds, and thereby also uniform bounds on the higher derivatives, we find by the theorem of Arzelà and Ascoli that, for a sequence $\tau_k\in\mathbb{R}$, a subsequence of $\tilde v_k:=(b_k-\tau_k,g_k)$ converges in C^∞_{loc} to a pseudoholomorphic cylinder $\tilde v:\mathbb{R}\times S^1\to\mathbb{R}\times\Sigma$. Since $\int\tilde v^*\,d\alpha=0$, it follows from [Hof93, p. 538] that there are only two options: either $\tilde v$ is constant or $\tilde v$ is a trivial cylinder over a periodic Reeb orbit. Therefore it remains to show that $\tilde v$ is non-constant. To achieve this, we use arguments from [HWZ02].

If $\tilde v$ is constant, then in particular, $\tilde v_k(0,0)\to p_0\in\mathbb{R}\times\Sigma$ and $\int_{S^1}\tilde v_k(0,\cdot)^*\alpha\to 0$. As $\int\tilde v_k^*\,d\alpha\to 0$, Stokes' theorem implies that $\int_{S^1}\tilde v_k(s,\cdot)^*\alpha\to 0$ uniformly in $s\in[-L_k,L_k]$. We will show now that in this situation, there is $h>0$ such that

$$\tilde v_k(s,t)\in B_1(p_0)\quad\text{for all }(s,t)\in[-L_k+h,L_k-h].\tag{5.6}$$

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Assume not. Then (up to passing to a subsequence of $\tilde v_k$), there is a sequence $(s_k, t_k) \in [-L_k+k, L_k-k]\times S^1$ such that $\tilde v_k(s_k,t_k) \notin B_1(p_0)$; we may assume, however, without loss of generality, that $\tilde v_k(s_k,t_k) \in B_2(p_0)$ for sufficiently large k (since $[-L_k+k, L_k-k]\times S^1$ is connected and $\tilde v_k(0,0) \to p_0$). Consider now the sequence $\tilde v'_k : [-k,k]\times S^1 \to \mathbb{R}\times\Sigma$ defined by $\tilde v'_k(s,t) := \tilde v_k(s_k+s, t_k+t)$. Since the derivatives are bounded away from the boundary and $\tilde v'_k(0,0) \in B_2(p_0)$, a subsequence of $\tilde v'_k$ (without any $\mathbb{R}$-shift) converges to a pseudoholomorphic map $\tilde v' : \mathbb{R}\times S^1 \to \mathbb{R}\times\Sigma$ with $\int \tilde v'^* \, d\alpha = 0$. Lemma 28 in [Hof93] combined with $\int_{S^1} \tilde v_k(s_k,\cdot)^*\alpha \to 0$ now show that $\tilde v'$ is the constant map $\tilde v'(s,t) = \tilde v'(0,0) =: p_1 \neq p_0$. Let us assume without loss of generality $s_k > 0$ and consider the pseudoholomorphic cylinders $w_k : [0,s_k]\times S^1 \to \mathbb{R}\times\Sigma$ defined by restriction: $w_k(s,t) := \tilde v_k(s,t)$. They have the following properties.

$$\int w_k^* \, d\alpha \to 0, \quad \sup_{s\in[0,s_k]} \int w_k(s,\cdot)^*\alpha \to 0, w_k(s_k,t) \to p_1, w_k(0,t) \to p_0 \quad \text{for all } t \in S^1. \tag{5.7}$$

There are now $(\sigma_k, \tau_k) \in [0,s_k]\times S^1$ such that if $q_k := w_k(\sigma_k,\tau_k)$, then $\operatorname{dist}(q_k,p_0) \geqslant 1/3$ and $\operatorname{dist}(q_k,p_1) \leqslant 1/3$. This allows us to apply the monotonicity Lemma 6.1 centered at q_k which implies that $\operatorname{area}(w_k) \geqslant C/9$. On the other hand, the area of w_k is bounded by its Hofer energy $E(w_k)$ which goes to 0 by the first two conditions above, and we have a contradiction. In conclusion, there is an $h > 0$ as claimed in equation (5.6) above.

Thus the map $\tilde v_k = (\tilde b_k = b_k - \tau_k, g_k)$ has the following properties: $\tilde v_k([-L_k+h, L_k-h]\times S^1) \subset B_1(p_0)$, but $\tilde b_k(L_k,t) \geqslant \sigma_k - \tau_k$ and $\tilde b_k(-L_k,t) \leqslant -\sigma_k - \tau_k$. Since (at least) one of the sequences $\pm\sigma_k - \tau_k$ is unbounded, at least one of the pseudoholomorphic cylinders $\tilde v_k^+ : [L_k-h, L_k]\times S^1 \to \mathbb{R}\times\Sigma$ and $\tilde v_k^- : [-L_k, -L_k+h]\times S^1 \to \mathbb{R}\times\Sigma$ will have arbitrary large conformal modulus (see Remark 6.8 below for the definition of conformal modulus) by Lemma 6.9. However, the conformal modulus of each of them is constant equal to h, a contradiction which proves the claim. □

Remark 5.5 (communicated by Fish). Theorem 5.3 is also a direct consequence of Fish's work on target local compactness [Fis11, Theorem A]. An outline of his argument is as follows.

First we can 'trim' the curves so that their new domains are given by $Z_k := a_k^{-1}((-\infty,0])$ and $a_k(Z_k) = [A_k, 0]$ with $\lim_{k\to\infty} A_k = -\infty$. Since the Hofer energy is bounded, so is the $d\alpha$ energy, and the latter is additive. Consequently by the pigeon-hole principle, we may find constants $c_k < 0$ and $\ell_k > 0$ such that $\mathcal{I}_k := [c_k - \ell_k, c_k + \ell_k] \subset [A_k, 0]$, and such that

$$\int_{a_k^{-1}(\mathcal{I}_k)} f_k^* \, d\alpha \to 0 \quad \text{and} \quad \ell_k \to \infty. \tag{5.8}$$

Now, after $\mathbb{R}$-shifting the curves by c_k (or $-c_k$), one can apply target local compactness inductively on the region $[-1,1]\times\Sigma$, then on the region $[-2,2]\times\Sigma$ and so forth. By passing to a diagonal subsequence we construct a sequence which converges on regions mapped into $[-c,c]\times\Sigma$ for any c in some open dense subset of $\mathbb{R}$. Since the $d\alpha$ energy tends to zero, the image of the limit curves must lie in an orbit cylinder. Since the genus is bounded and the limit is a branched cover of a finite collection of orbit cylinders, we can conclude that the number of branch points is *a priori* bounded. Hence we can find, after shifting and trimming again (and possibly passing to another subsequence), a sequence $c'_k < 0$ such that, for the subsequence of curves shifted by c'_k (or $-c'_k$), we have convergence in the region $[-c,c]\times\Sigma$ to an unbranched multiple cover of a collection of orbit cylinders for arbitrary $c > 0$. To complete the proof, one needs to estimate the conformal modulus of the cylinders; this can be done by using results in [Fis07] or by using Lemma 6.9 below.

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It is perhaps worth noting that if one argues using target local compactness, the topological bounds established in Proposition 6.7 are a consequence of the compactness result; in our proof we first establish the topological bounds (by following [CM05]) and then use elementary compactness arguments to find a trivial cylinder.

6. Proof of Proposition 5.4

In this section we will give a proof of Proposition 5.4 by following the procedure used by Cieliebak and Mohnke in [CM05] to establish SFT compactness. We will present below a slight adaptation of the relevant part of their procedure.

First we quote the following *monotonicity lemma* from [CM05, Lemma 5.1]. A detailed proof in the case of a compact target manifold can be found in [Hum97, ch. 2]. Since both J and g_J are $\mathbb{R}$-invariant the proof in our situation is easily reduced to that case.

Lemma 6.1. *Fix $J \in \mathcal{J}_{\mathrm{SFT}}(\mathbb{R}\times\Sigma, d(e^s\alpha))$. Then there exist constants $C, 1 > 10\varepsilon > 0$ such that for any J-holomorphic map $u : Z \to \mathbb{R}\times\Sigma$ defined on a (possibly disconnected) compact Riemann surface Z and $0 < \delta < \varepsilon$*

$$\mathrm{area}_{g_J}(u|_{B_{g_J}(y;\delta)}) \geqslant C\delta^2 \tag{6.1}$$

whenever there is a $y \in u(Z)$ with $B_{g_J}(y;\delta) \cap u(\partial Z) = \emptyset$ such that u is non-constant on a component $Z_0 \subset Z$ whose image contains y.

The next several lemmata allow us to achieve with Proposition 6.7 some control over the topology of pseudoholomorphic curves in a symplectization.

Let us now fix constants $0 < \delta < \varepsilon$ and C as in Lemma 6.1 and let Z be a (not necessarily connected) compact Riemann surface Z and $u = (a, f) : Z \to \mathbb{R}\times\Sigma$ a J-holomorphic curve which is non-constant on all its components and satisfies $u(\partial Z) \subset (K + 4\delta, \infty) \times \Sigma$ for some $K \in \mathbb{R}$. Then similarly to Cieliebak and Mohnke in [CM05, § 5.3], we introduce surfaces $Z^S_R(u)$ and $Z^R_S(u)$ where $R < S < K$ are such that $R, S, R \pm \delta, S \pm \delta$ are regular values of a and $S - R \geqslant 2\delta$.

Namely, first define $\mathcal{C}_R$ to be the collection of connected components of $a^{-1}([R, R + \delta])$ and $a^{-1}([R - \delta, R])$. Then define subsets $\mathcal{C}^{\pm}_R \subset \mathcal{C}_R$ as follows. First, we include in $\mathcal{C}^+_R$ all components that meet $a^{-1}(R + \delta)$, as well as those in $a^{-1}([R, R + \delta])$ that do not meet $a^{-1}(R)$. Similarly, we include in $\mathcal{C}^-_R$ all components that meet $a^{-1}(R - \delta)$ as well as those in $a^{-1}([R - \delta, R])$ that do not meet $a^{-1}(R)$. In the next step, we include in $\mathcal{C}^+_R$ all components in $\mathcal{C}_R$ which can be connected to $\mathcal{C}^+_R$ without passing through $\mathcal{C}^-_R$, and then we include in $\mathcal{C}^-_R$ all components in $\mathcal{C}_R$ which can be connected to $\mathcal{C}^-_R$ without passing through (the previously extended) $\mathcal{C}^+_R$. This last step is repeated as long as the size of the collections $C^{\pm}_R$ increases; this is a finite process since R is a regular value. Since there are no closed components, we see that $\mathcal{C}_R = \mathcal{C}^+_R \cup \mathcal{C}^-_R$. By an abuse of notation we will denote by $\mathcal{C}^{\pm}_R$ also the subsets of Z which are the union of the components in $\mathcal{C}^{\pm}_R$.

Now fix $b \in (K + 2\delta, K + 3\delta)$ such that $b, b \pm \delta$ are regular values of a (a choice depending on $u = (a, f)$). Depending on b we set for $R < S < K$ as above

$$Z^S_R(u) := a^{-1}([R + \delta, S - \delta]) \cup \mathcal{C}^+_R \cup \mathcal{C}^-_S \quad \text{and} \quad Z^R_S(u) := (a^{-1}((-\infty, b - \delta]) \cup \mathcal{C}^-_b)\backslash Z^S_R(u). \tag{6.2}$$

Observe that since there are no closed components each connected component of $Z^S_R(u)$ and $Z^R_S(u)$ has a boundary component. These boundary components can *a priori* lie either in $a^{-1}(R), a^{-1}(S)$ or in $a^{-1}(b)$ (this last option $a^{-1}(b)$ does not occur in the setting of [CM05]).

In the following lemma we bound the number of components in $Z^S_R(u)$ and $Z^R_S(u)$.

Lemma 6.2 [CM05, Lemma 5.5]. *For a pseudoholomorphic curve $u = (a, f)$ and regular values R, S as above, the number of connected components of the surfaces $Z_R^S(u)$ and $Z_S^R(u)$ is at most $8E(u)/C\delta^2$.*

Proof. Let Z_0 be a connected component of $Z_R^S(u)$ (respectively $Z_S^R(u)$ which has a boundary component away from $a^{-1}(b)$). Then Z_0 has a boundary component in $a^{-1}(R)$ or $a^{-1}(S)$ and by construction there exists therefore a point $z_0 \in Z_0$ with $a_0 := a(z_0) \in \{R + \delta/2, S - \delta/2\}$ (respectively $a_0 := a(z_0) \in \{R - \delta/2, S + \delta/2\}$). On the other hand, if (some or) all boundary components of Z_0 meet $a^{-1}(b)$ (and therefore $Z_0 \subset Z_S^R(u)$), then it meets both $a^{-1}(b)$ and $a^{-1}(b - \delta)$. Therefore there exists a point $z_0 \in Z_0$ with $a_0 := a(z_0) = b - \delta/2$. Fix a $\nu \in \mathcal{S}$ satisfying $\nu'(s) \geqslant 1$ whenever s is in a $\delta/2$-ball around $R \pm \delta/2$, $S \pm \delta/2$ or $b - \delta/2$. (The existence of ν is clear since $10\varepsilon < 1$.) We now see that the ball of radius $\delta/2$ around $y_0 := u(z_0)$ does not intersect $u(\partial Z_0)$ and so by the monotonicity Lemma 6.1

$$\frac{C\delta^2}{4} \leqslant \mathrm{area}_{g_J}(u|_{a^{-1}(a_0 - \delta/2, a_0 + \delta/2) \cap Z_0}) \leqslant \int_{Z_0} u^*(d\alpha) + \int_{Z_0} u^*(\nu'(s)\, ds \wedge \alpha) \leqslant 2E(u). \tag{6.3}$$

Here we used that

$$\begin{aligned} \left|\frac{du(z)}{dx}\right|^2_{g_J} &= d\alpha\left(J\frac{du(z)}{dx}, \frac{du(z)}{dx}\right) \\ &\quad + \nu'\, ds \wedge \alpha\left(J\frac{du(z)}{dx}, \frac{du(z)}{dx}\right) \quad \text{for } z \in a^{-1}\left(a_0 - \frac{\delta}{2}, a_0 + \frac{\delta}{2}\right) \end{aligned} \tag{6.4}$$

(or in other words that we can choose the taming constant $C_T = 1$ in the notation of [CM05, Lemma 2.5]). By summing these inequalities over the different components and using the definition of the Hofer energy we see that

$$\frac{C\delta^2}{4} \cdot \#\{\text{connected components in } Z_R^S(u), Z_S^R(u)\} \leqslant 2E(u). \tag{6.5}$$

□

Following [CM05] we call a subset $P_0 \subset Z$ a *δ-essential local minimum on level* R_0 of $u : Z \to \mathbb{R} \times \Sigma$ if P_0 is a connected component of $a^{-1}((-\infty, R_0 + \delta])$ and $R_0 = \min_{P_0} a$. Similarly, a *δ-essential local maximum* P_0 *on level* R_0 is a connected component P_0 of $a^{-1}([R_0 - \delta, \infty))$ with $R_0 = \max_{P_0} a$. Observe that any two different δ-essential local minima are disjoint.

Lemma 6.3 [CM05, Lemma 5.6]. *For any J-holomorphic curve $u : Z \to \mathbb{R} \times \Sigma$ with $u(\partial Z) \subset (K + 4\delta, \infty) \times \Sigma$ the number of δ-essential local minima of u in $(-\infty, K) \times \Sigma$ is bounded above by $2E(u)/C\delta^2$. Furthermore there are no δ-essential local maxima in $(-\infty, K) \times \Sigma$.*

Proof. Let $P_i, i = 1, \ldots, p$ denote the δ-essential local minima of u with critical values R_i. By definition, the P_i are pairwise disjoint and therefore

$$\int_Z u^*\, d\alpha \geqslant \sum_{i=1}^p \int_{P_i} u^*\, d\alpha. \tag{6.6}$$

On the other hand if we choose a function $\nu \in \mathcal{S}$ with $\nu'(s) = 1$ for $s \in [R_i, R_i + \delta]$ and $\nu(R_i + \delta) = 1$ we see that

$$2\int_{P_i} u^*\, d\alpha \geqslant \int_{P_i} u^*\, d\alpha + \int_{P_i} u^*(\nu'\, ds \wedge \alpha) \geqslant \mathrm{area}_{g_J}(u|_{a^{-1}((R_i - \delta, R_i + \delta)) \cap P_i}) \geqslant C\delta^2 \tag{6.7}$$

where we use first Stokes' theorem, and then the compatibility of J with g_J and the monotonicity Lemma 6.1 as above. Summing over the different local minima the claimed bound follows immediately by the definition of $E(u)$. It is well known that a non-constant pseudoholomorphic curve in the symplectization $\mathbb{R}\times\Sigma$ does not have any (interior) local maxima by the maximum principle, therefore in particular no δ-essential local maxima in $(-\infty,K)\times\Sigma$. This can also be seen by choosing $\nu\in\mathcal{S}$ with $\nu'(s)=1$ in $[R_0-\delta,R_0]$ and $\nu(R_0-\delta)=0$ for a local maximal value R_0. The computation above implies that the curve has no area near the local maximum and therefore the map is constant on this connected component, a contradiction to our assumption that u has no constant components. □

Lemma 6.4 [CM05, Lemma 5.7]. *Let $u=(a,f)$ be a pseudoholomorphic curve of total genus at most g and denote by $A:=a^{-1}((-\infty,b-\delta])\cup\mathcal{C}_b^-$. If we make the assumptions as above, then we have*

$$\chi(A)\geqslant 2-3g-12E(u)/C\delta^2 \tag{6.8}$$

and the following estimates on the Euler characteristic for $R<S<K$:

$$\chi(A)-8E(u)/C\delta^2\leqslant\chi(Z_R^S(u))\leqslant 8E(u)/C\delta^2, \tag{6.9}$$
$$\chi(A)-8E(u)/C\delta^2\leqslant\chi(Z_S^R(u))\leqslant 8E(u)/C\delta^2. \tag{6.10}$$

Proof. Using the same procedure as in Lemma 6.2, we first observe that the number p of components of A is at most $8E(u)/C\delta^2$. In the next step we want to get an upper bound on the number of q of boundary components of A (on level b), and therefore a lower bound on the Euler characteristic of A. In order to do that we look at how the boundary components of A on level b behave, when we extend our domain of consideration to the extended surface $\tilde{A}:=a^{-1}((-\infty,b])\cup\mathcal{C}_{b+\delta}^-$. Since there are no local maxima, there are only two options. Some of these boundary components will connect with each other in $a^{-1}(b,b+\delta)$, and some of them will stay apart. Since the genus is at most g, and any connection of two boundary components which lie in the same component of A will increase the genus or reduce the number of connected components of (the extended) A, there are at most $p+g$ new connections which join the different boundary components of A. Let us call two boundary components of A *equivalent* if they lie in the same connected component of $\tilde{A}$. By the above discussion, there are at most $q-(p+g)$ equivalence classes of boundary components of A, each of them representing a component of $\mathcal{C}_{b+\delta}^-$ which intersects both the levels b and $b+\delta$. By the monotonicity Lemma 6.1 we see that there can be at most $4E(u)/C\delta^2$ such components, i.e. $q\leqslant p+g+4E(u)/C\delta^2$. Finally since $\chi(A)\geqslant 2-2g-q$ we obtain the first inequality. By Lemma 6.2, both $Z_R^S(u)$ and $Z_S^R(u)$ have at most $8E(u)/C\delta^2$ components. Furthermore, all of these components have Euler characteristic at most 1, therefore the upper bounds follow. Next, since the Euler characteristic is additive under gluing along common boundary, we see that $\chi(A)=\chi(Z_S^R(u))+\chi(Z_R^S(u))$. From the lower bounds on $\chi(A)$ and the upper bounds on each of the terms on the right-hand side, we find the claimed lower bounds for each of the terms on the right-hand side. □

Lemma 6.5 [CM05, Lemma 5.8]. *With the notation from above,*

$$\chi(Z_R^S(u))\leqslant\#\{\delta\text{-essential local minima } Z_0\subset Z_R^S(u)\}. \tag{6.11}$$

If $Z_R^S(u)$ contains no δ-essential local minima and $\chi(Z_R^S(u))=0$, then $Z_R^S(u)$ is the disjoint union of cylinders connecting levels R and S.

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Proof. If a component in $Z_R^S(u)$ has positive Euler characteristic, then it is necessarily a disk with boundary on level S since no local maxima can occur. This disk will (by definition of $Z_R^S(u)$) contribute a δ-essential local minimum. If there are no δ-essential minima, then by the argument above, all components of $Z_R^S(u)$ have non-positive Euler characteristic. If $\chi(Z_R^S(u)) = 0$, then this implies that the Euler characteristic of all components is equal to 0, and that they are therefore cylinders connecting the R and S levels, since there are no δ-essential extrema. □

We now introduce the function $\chi_u : (-\infty, K]_{\mathrm{reg}} \to \mathbb{Z}$ on the set of regular values of a by

$$\chi_u(r) := \chi(Z_r^b(u)). \tag{6.12}$$

A value $r \in (-\infty, K]$ is called an *upward jump*, if

$$\limsup_{S \searrow r} \chi_u(S) - \liminf_{R \nearrow r} \chi_u(R) > 0, \tag{6.13}$$

where the limits are taken over regular values S and R. Similarly, one can define a *downward jump*.

Lemma 6.6 [CM05, Lemma 5.9]. *The number of downward jumps of χ_u is at most $E(u)/C\delta^2$. The number of all upward jumps of χ_u is at most $3g + 5E(u)/C\delta^2$.*

Proof. Consider regular values $R < S$. The difference $\chi_u(R) - \chi_u(S)$ is the Euler characteristic of $Z_R^b(u) \backslash Z_S^b(u)$. Any of the components of this surface has its boundary components on level R or S. It contributes some positive Euler characteristic exactly if it is a disk C with boundary component on level S. By construction of $Z_R^b(u)$ and $Z_S^b(u)$, we know that $R - \delta < \min_C a < S - \delta$ (otherwise it would either not belong to $Z_R^b(u)$ or it would belong to $Z_R^b(u)$). This implies that C contains an δ-essential local minimum in the interval $[R - \delta, S - \delta]$. Choosing $R \nearrow r$ and $S \searrow r$, we find that at any downward jump $r \leqslant K$ there must be a δ-essential local minimum on level $r - \delta$. Now the first claim follows from Lemma 6.3. Furthermore, from the estimate in Lemma 6.4, we know that the Euler characteristic of $Z_{\min u-1}^S(u)$ lies always in a interval of length at most $3g + 28E(u)/C\delta^2$, and therefore, since $Z_{\min u-1}^S(u) \cup Z_S^b(u) = Z_{\min u-1}^b(u)$, so does $Z_S^b(u)$. Using our previous bound on the number of upward jumps, the result follows. □

In conclusion the bounded function χ_u extends to a locally constant function (also denoted by) $\chi_u : (-\infty, K] \to \mathbb{Z}$ with finitely many jumps. We summarize the results of Lemmas 6.2–6.6 in the following Proposition.

Proposition 6.7. *Let $E > 0, \delta > 0$ and $g \in \mathbb{N}$ and $K \in \mathbb{R}$. Then there exists an $N_0 \in \mathbb{N}$ such that for any pseudoholomorphic curve $u = (a, f) : Z \to \mathbb{R} \times \Sigma$ which:*

(1) *is defined on a compact manifold Z of genus at most g;*
(2) *has Hofer energy $E(u) \leqslant E$;*
(3) *is non-constant on all components of Z and satisfies $u(\partial Z) \subset (K + 4\delta, \infty) \times \Sigma$,*

there are at most N_0 jumps of $\chi_u : (-\infty, K] \to \mathbb{Z}$ and the number of δ-essential minima of u below level K is bounded by N_0. Furthermore, if $R < S < K, R \pm \delta, S \pm \delta$ are all regular values of a, $\chi(Z_R^S(u)) = 0$, and there are no δ-essential minima in $Z_R^S(u)$, then $Z_R^S(u)$ is a union of at most N_0 cylinders connecting the level R with the level S.

Proof. The bound on the number of jumps comes from Lemma 6.6, the bound on the number of δ-essential local minima below level K follows from Lemma 6.5. In the last statement all components are cylinders connecting level R with level S again by Lemma 6.5, and the bound on the number of such cylinders follows immediately from Lemma 6.2. □

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Remark 6.8. Recall that any compact Riemann surface C which is diffeomorphic to a (finite) cylinder is biholomorphically equivalent to a standard cylinder $([0, L] \times \mathbb{R}/\mathbb{Z}, i) = [0, L] \times S^1$ for a uniquely determined L. The number L is called *the conformal modulus* of C and $\sqrt{L^{-1}}$ is called *the conformal length* of C (see for instance [Ahl73] for more information).

In order to prove Proposition 5.4, we need to bound the conformal modulus of a pseudoholomorphic cylinder from below. This is achieved by the next lemma.

LEMMA 6.9 [CM05, Lemmas 4.20–4.22]. *Let $u = (a, f) : [0, L] \times S^1 \to \mathbb{R} \times \Sigma$ be a pseudoholomorphic cylinder with $a(0, t) \leqslant R$ and $a(L, t) \geqslant S$ for all $t \in S^1$ and some $R < S$. Then the conformal modulus L of the cylinder is bounded below by $(S - R)/2E(u)$.*

Proof. The proof we outline is based on a method to bound the *conformal length* of a family Γ of curves from above, which can be found in [Gro83, pp. 55–56]. For each $s \in [R, S]$ which is a regular value of a, denote by $\gamma_s := a^{-1}(s)$ its preimage, consisting of circles which separate $[0, L] \times S^1$. Now let $\Gamma = \{\gamma_s \mid R < s < S \text{ is a regular value of } a\}$ be the collection of those curves and let g_0 be the (possibly singular) metric on $[0, L] \times S^1$ induced by u. The conformal length of Γ is by definition

$$\text{conf length}(\Gamma) := \sup_g \inf_s |\gamma_s|_g \tag{6.14}$$

where g runs over all conformal metrics $g = \varphi^2 g_0$ on $[0, L] \times S^1$ with area $\int_{[0,L]\times S^1} \varphi^2 \, d\,\text{vol}_{g_0} \leqslant 1$. Now fix such a g and compute

$$\begin{aligned} \inf_s |\gamma_s|_g \leqslant \frac{1}{S - R} \int_R^S |\gamma_s|_g \, ds &= \frac{1}{S - R} \int_R^S \int_{\gamma_s} \varphi(v) \, d\gamma_s(v) \, ds \\ &= \frac{1}{S - R} \int_{a^{-1}([R,S])} \varphi |\nabla a|_{g_0} \, d\,\text{vol}_{g_0}. \end{aligned} \tag{6.15}$$

Here the last step follows from the coarea formula and the fact that $\gamma_s = a^{-1}(s) \subset [0, L] \times S^1$ is endowed with the metric induced by g_0. Since $|\nabla a|_{g_0} \leqslant 1$, $\text{area}_g([0, L] \times S^1) \leqslant 1$, and $\text{area}_{g_0}(a^{-1}([s, s + 1])) \leqslant 2E(u)$, the Cauchy–Schwarz inequality and a computation as in (6.3) implies that

$$\inf_s |\gamma_s|_g \leqslant \frac{1}{S - R} \sqrt{\text{area}_{g_0}(a^{-1}([R, S]))} \leqslant \frac{1}{S - R} \sqrt{(S - R) 2E(u)}. \tag{6.16}$$

On the other hand, if the metric is non-singular, then a special choice of metric is given by $g_L = L^{-1} g_{\text{Euclidean}}$, and in this case

$$\inf_s |\gamma_s|_{g_L} \geqslant L^{-1/2}. \tag{6.17}$$

Combining these inequalities gives the claim in the case where g_0 is non-singular. By concentrating on the full measure, open subset of $s \in [R, S]$ of regular values of a, a suitable subdivision of the interval $[R, S]$ makes the above proof also applicable in the singular case. Indeed, the cylinder contains disjoint open subcylinders of the form $a^{-1}((r_i, s_i))$ with $r_{i+1} = s_i$ and $[R, S] = \bigcup_i [r_i, s_i]$ with non-singular induced metrics. Therefore we have, as above, bounds on the conformal moduli L_i of these subcylinders. Summing up these lower bounds gives the claimed lower bound for the conformal modulus L of the cylinder. □

We finally come to the proof of Proposition 5.4.

Proposition 6.10. *Suppose (Z_k, j_k) is a family of compact (possibly disconnected) Riemann surfaces with boundary and uniformly bounded genus. Assume that*

$$u_k = (a_k, f_k) : Z_k \to \mathbb{R} \times \Sigma \tag{6.18}$$

is a sequence of (j_k, J)-holomorphic maps which have uniformly bounded Hofer energy $E(u_k) \leqslant E$, are non-constant on each connected component of Z_k, and satisfy $a_k(\partial Z_k) \subset [0, \infty)$.

Assume that $\inf_k \inf_{Z_k} a_k = -\infty$. Then there exists a subsequence k_n and cylinders $C_n \subset Z_{k_n}$ such that an $\mathbb{R}$-shift $v_n = (b_n, g_n)$ of the restriction of u_{k_n} to C_n has the following properties:

(1) *C_n is biholomorphic to $[-L_n, L_n] \times S^1$ and $L_n \to \infty$ as $n \to \infty$;*
(2) *$\int_{C_n} v_n^* \, d\alpha \to 0$;*
(3) *after a suitable identification of C_n with $[-L_n, L_n] \times S^1$ there is a sequence $\sigma_n \to \infty$ such that $\pm b_n(\pm L_n, t) \geqslant \sigma_n$ for each $t \in S^1$.*

Proof. We now follow [CM05, § 5.4] and call a level $r \in (-\infty, K]$ *essential for u_k* if it satisfies one of the following conditions:

- $r = \min u_k$ or $r = K$;
- u_k has a δ-essential minimum on level $r - \delta$;
- χ_k has a jump at level r.

Observe that the u_k satisfy the assumptions of Proposition 6.7 for $K = -5\delta$. Therefore the number of essential levels is bounded independently of k by Proposition 6.7. Since $\inf \min_{Z_k} u_k = -\infty$, this implies in particular that for a subsequence k_n, there are intervals $[\rho_n, \sigma_n] \subset [\min u_{k_n}, K]$ with $\sigma_n - \rho_n \geqslant n^2$ which do not contain any critical level. Next observe that if we cut the interval $[\rho_n, \sigma_n]$ into n pieces I_n^j of equal length, then, for at least one $j = j(n) \in \{1, \ldots, n\}$,

$$\int_{a_{k_n}^{-1}(I_n^{j(n)})} u_{k_n}^* \, d\alpha \leqslant \frac{1}{n} \int_{Z_{k_n}} u_{k_n}^* \, d\alpha \leqslant E/n \quad \text{and} \quad \text{length}(I_n^{j(n)}) \geqslant n. \tag{6.19}$$

Subdivide $I_n^{j(n)}$ into three subintervals of equal length and denote the middle one by $I_n := [\rho_n', \sigma_n']$. By possibly shrinking I_n slightly, but keeping the notation the same, we may assume that ρ_n', σ_n', $\rho_n' \pm \delta, \sigma_n' \pm \delta$ are all regular values of a_{k_n} and (in the notation of (6.2)) $Z_{\rho_n'}^{\sigma_n'}(u_{k_n}) \subset a_{k_n}^{-1}([\rho_n' - \delta, \sigma_n' + \delta]) \subset a_{k_n}^{-1}(I_n^{j(n)})$ if $n \geqslant 3$, and hence

$$\int_{a_{k_n}^{-1}(I_n)} u_{k_n}^* \, d\alpha \leqslant E/n \quad \text{and} \quad \text{length}(I_n) \geqslant n/4. \tag{6.20}$$

We infer from the last statement in Proposition 6.7 that $a_{k_n}^{-1}(I_n)$ is parametrized by a disjoint union of cylinders (which is necessarily non-empty, since by the maximum principle each component of u_k meets the level $a_k^{-1}(K)$).

For each n, choose a component $C_n \subset a_{k_n}^{-1}(I_n)$. Then $a_{k_n}(\partial C_n) = \{\rho_n', \sigma_n'\}$ and so we derive from Lemma 6.9 that the conformal modulus of C_n tends to infinity. It is now immediate that the restriction of u_{k_n} to C_n satisfies properties (1) and (2). Finally if we set $b_{k_n} := a_{k_n} + \rho_n' + (\sigma_n' - \rho_n')/2$, then the $\mathbb{R}$-shift $v_n := (b_n, f_{k_n})$ of u_{k_n} restricted to C_n in addition satisfies property (3), after possibly switching the boundary components of C_n by a biholomorphic map. □

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