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On combinatory complete sets of proper combinators

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Abstract

A combinatory system (or equivalently the set of its basic combinators) is called combinatorially complete for a functional system, if any member of the latter can be defined by an entity of the former system. In this paper the decision problem of combinatory completeness for finite sets of proper combinators is studied for three subsystems of the pure lambda calculus. Precise characterizations of proper combinator bases for the linear and the affine λ-calculus are given, and the respective decision problems are shown to be decidable. Furthermore, it is determined which extensions with proper combinators of bases for the linear λ-calculus are combinatorially complete for the λI-calculus.

Capsule Review

Let $\mathscr{X}$ be a set of combinators (closed lambda terms) closed under application. Such a set is called a *subsystem* of λ, the untyped lambda calculus. A subset $\mathscr{B} \subseteq \mathscr{X}$ is called *complete* for $\mathscr{X}$ if every element of $\mathscr{X}$ can be written as an applicative combination of elements of $\mathscr{B}$.

The following subsystems are treated: λA (the affine lambda calculus), λI (the λI calculus) and λL (the linear lambda calculus). In these subsystems the number of (free) occurrences of x in M in context $\lambda x.M$ is always ≤ 1, ≥ 1 or $=1$, respectively. We have $\lambda\text{A}\cup\lambda\text{I} = \lambda$ and $\lambda\text{A}\cap\lambda\text{I} = \lambda\text{L}$.

A *proper combinator* is of the form $\lambda\vec{x}.M$ with M built up from the $\vec{x}$ using application only. The following results are proved.

1. In λA it is decidable whether a finite set $\mathscr{B}$ of proper combinators is complete.
2. In λL it is decidable whether a finite set $\mathscr{B}$ of proper combinators is complete.
3. Given a complete set $\mathscr{B}$ for λL it can be characterized which finite extensions with proper combinators make $\mathscr{B}$ complete for the subsystem λI.

Note that the third result is different from the first two. Statman (1986) has proved that completeness is undecidable for subsets $\mathscr{B} \subseteq \lambda$I consisting of normal terms, hence for sets of combinators in general. The decidability problem for sets of proper combinators remains open. The proof in this paper uses an ingenious translation of problems concerning completeness to linear Diophantine equations over the natural numbers. The latter problems are known to be decidable, as proved by Gauss. As an application it is shown that the set $\{\lambda x.x, \lambda xyzu.y(zu)x\}$ is complete for λL but $\{\lambda xyz.x, \lambda xyzuvw.vw(ux)y\}$ is not complete for λA.

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1 Introduction

The first combinator bases for the lambda calculus were introduced by Schönfinkel (1924) and Curry (1929), who proved combinatory completeness by exhibition of abstraction algorithms which transform lambda terms into equivalent combinatory terms of the respective systems. In general, a combinatory system (or equivalently the set of its basic combinators) is called combinatorially complete for a functional system, such as the lambda calculus, if any member of the latter can be defined by an entity of the former system. A finite set of combinators which is combinatorially complete is called a basis for the corresponding functional system. Up to now, many bases are known for the pure lambda calculus (see Barendregt (1981), Abdali (1976) and Turner (1979)), as well as for many of its subsystems (see Barendregt (1981), Hindley (1989) and Trigg *et al.* (1994)). One interesting problem is the decision problem of combinatory completeness for finite sets of combinators: given a finite set of combinators, decide if it forms a basis for the lambda calculus or one of its subsystems. One knows already that this problem is undecidable for the pure lambda calculus (see Statman, 1986); moreover, his proof can easily be adapted to show the same result for the λI-calculus. On the other hand, many necessary conditions for finite sets of proper combinators forming a basis have been given by Craig in Curry and Feys (1958), as well as by Statman (1986).

In this paper we study the decision problem for finite sets of proper combinators for three subsystems of the pure lambda calculus: the linear λ-calculus, the affine λ-calculus and Church's original λI-calculus. All these systems have their own interest, in particular because of their relation to propositional calculus via type-assignment. In fact, their type schemes form respectively the positive implicational linear, affine and relevance logic (see Hindley (1989), Fitch (1936), Meredith and Prior (1963) and Anderson and Belnap (1975)). The relationship between the four calculi is illustrated in Fig.1, where each calculus is written down together with one of its bases. Here, an arrow between two calculi that is adorned with a combinator X means that the source calculus is a proper subsystem of the target calculus, and the former can be extended to the latter by extension of any of its bases with the combinator X.

In section 2 we define some basic concepts that will be used in the rest of the paper. In sections 3 and 4 we give precise characterizations of proper combinator bases respectively for the linear and the affine λ-calculus and thereby show that the corresponding decision problems are decidable. Furthermore, the proofs of these results, which for reason of length and readability are postponed to the appendix, are constructive in the following sense: given a (linear or affine) lambda term M and a basis of proper combinators Σ for the corresponding calculus, one can, following the proof of completeness for Σ, define M in terms of a combination of elements of Σ. In section 5 we determine which extensions with proper combinators of bases for the linear λ-calculus are combinatorially complete for the λI-calculus. Throughout we shall assume the reader is familiar with the basic concepts of lambda calculus, as well as combinatory logic, which can all be found in Barendregt (1981).

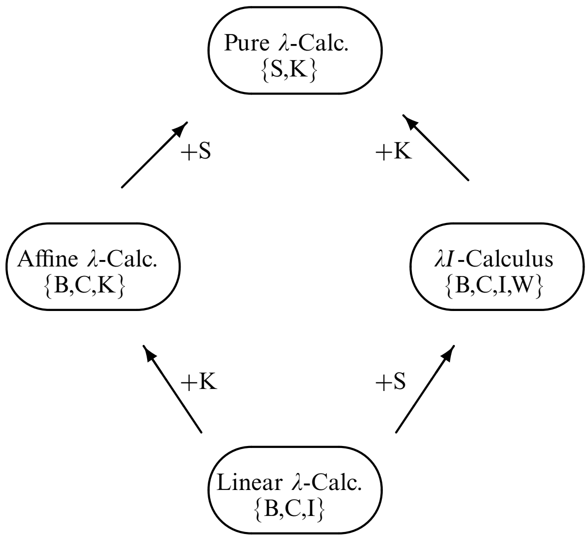

Fig. 1.

2 Basic concepts

A lambda term M is called linear if, for each subterm $\lambda x.N$, the variable x occurs exactly once in N, while all free variables of M occur free only once. If for each subterm $\lambda x.N$ the variable x occurs at most once in N, then M is affine. By combinator we mean a closed term, which is called proper if the term Z in the corresponding reduction rule $Mx_1 \dots x_n \geq_1 Z$ is an applicative term and contains no constants and no other variables than $x_1, \dots, x_n$. Then M is a linear (resp. affine) combinator if every variable $x_1, \dots, x_n$ occurs exactly (resp. at most) once in Z. Let $\mathscr{S} = \{0, 1\}^*$ be the set of finite, possibly empty, sequences of 0's and 1's. From now on we denote by α, β, γ, etc. the elements of $\mathscr{S}$, while a and b stand for either 0 or 1. When a is 0, then $\overline{a}$ is 1, and *vice versa.*

Definition 2.1
Consider two occurrences of applicative terms X and Y. If ϵ and $\cdot$ are respectively the empty sequence and the concatenation operation, then the position of Y in X is denoted by $p(Y, X) \in \mathscr{S} \cup \{\otimes\}$ and is defined as follows:

- $p(Y, Y) = \epsilon$;
- $p(Y, UV) = 0 \cdot p(Y, U)$ if the occurrence of Y is in U;
- $p(Y, UV) = 1 \cdot p(Y, V)$ if the occurrence of Y is in V;
- $p(Y, X) = \otimes$ if the occurrence of Y is not in X.

To obtain an alternative and more helpful representation for applicative terms and proper combinators, we introduce the notion of A-domain. We call the readers' attention on the fact that A-domains are very similar to tree-domains which can be used to represent 'totally' labelled trees as described in Guessarian (1981).

Definition 2.2
An applicative domain (A-domain) is a finite subset $A \subseteq \mathscr{S}$ such that

(a) $A \neq \emptyset$;
(b) $\alpha \in A \;\wedge\; \alpha\beta \in A \;\Rightarrow\; \beta = \epsilon$;
(c) $\alpha a \beta \in A \;\Rightarrow\; \exists \gamma \in \mathscr{S} : \; \alpha \bar{a} \gamma \in A$.

If A is an A-domain, let $A(\alpha)$ denote

$$A(\alpha) = \{\beta : \alpha\beta \in A\}.$$

Note that $A(\alpha)$ is an A-domain whenever $A(\alpha) \neq \emptyset$. Let $\mathscr{V}$ and $\mathscr{C}$ be respectively the sets of variables and constants of our calculus. Then, any applicative term X can be represented by a unique function $t_X : A_X \longrightarrow \mathscr{V} \cup \mathscr{C}$, where A_X is an A-domain and for all $\alpha \in A_X$ there is $p(t_X(\alpha), X) = \alpha$. Conversely, any such function represents one and only one term X. This becomes obvious when we represent, for instance, the term $X = x(By)z$ as in Fig. 2.

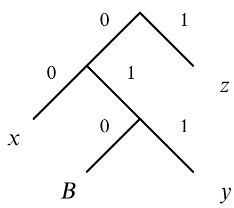

Fig. 2.

Then, $A_X = \{00, 010, 011, 1\}$ and t_X is given by $t_X(00) = x$, $t_X(010) = B$, $t_X(011) = y$ and $t_X(1) = z$.

Definition 2.3
Let A_X and A_Y be A-domains and consider the functions $t_X : A_X \longrightarrow \mathscr{V} \cup \mathscr{C}$ and $t_Y : A_Y \longrightarrow \mathscr{V} \cup \mathscr{C}$. Then t_Y is a subterm of t_X at position α if and only if

$$A_X(\alpha) = A_Y \quad \wedge \quad \forall \beta \in A_Y .\; t_X(\alpha\beta) = t_Y(\beta).$$

Definition 2.4
If t_Y is a subterm of t_X at position α, then the substitution of t_Y in t_X by another term t_Z is given by $t : A \longrightarrow \mathscr{V} \cup \mathscr{C}$, where

$$A = (A_X \setminus A_X(\alpha)) \cup \{\alpha\beta : \beta \in A_Z\}$$

and

$$t(\gamma) = \begin{cases} t_X(\gamma) & \text{for } \gamma \in A_X \setminus A_X(\alpha) \\ t_Z(\gamma') & \text{for } \gamma = \alpha\gamma' \in \{\alpha\beta : \beta \in A_Z\}. \end{cases}$$

Now consider a linear (or affine) proper combinator M with reduction rule

$$M x_0 \dots x_n \geq_1 Z, \qquad (1)$$

and let

$$\begin{array}{rcl} \alpha_0 & = & p(x_n, Z) \\ \alpha_1 & = & p(x_{n-1}, Z) \\ & \vdots & \\ \alpha_n & = & p(x_0, Z). \end{array} \qquad (2)$$

Note that it makes sense to speak of 'the' position of x_i in Z, since M is linear (resp. affine), and thus x_i has exactly (resp. at most) one occurrence in Z.

We now investigate the effect on t_X and t_Y, when X reduces to Y by one application of the reduction rule for M. That is when Y is X where some subterm of the form $MX_0 \ldots X_n$ is substituted by $Z' = Z[X_0/x_0, \ldots, X_n/x_n]$. Then there is $\alpha \in \mathscr{S}$ such that $t_{MX_0 \ldots X_n}$ is a subterm of t_X at position α. Hence t_M is a subterm of t_X at position $\alpha\underbrace{0\ldots0}_{n+1}$ and for $0 \leq i \leq n$ each t_{X_i} is a subterm of t_X at position $\alpha\underbrace{0\ldots0}_{i}1$. On the other hand, each t_{X_i} is a subterm of $t_{Z'}$ at position α_i whenever $\alpha_i \neq \otimes$. Thus t_{X_i} is a subterm of t_Y at position $\alpha\alpha_i$ whenever $\alpha_i \neq \otimes$. We conclude that $t_Y : A_Y \longrightarrow \mathscr{V} \cup \mathscr{C}$ is given by

$$A_Y = (A_X \setminus A_X(\alpha)) \cup A_Y(\alpha)$$

where

$$A_Y(\alpha) = \{\alpha\alpha_i\beta : \alpha_i \neq \otimes \wedge \alpha\underbrace{0\ldots0}_{i}1\beta \in A_X \wedge 0 \leq i \leq n\}$$

and

$$t_Y(\gamma) = \begin{cases} t_X(\gamma) & \text{for } \gamma \in A_X \setminus A_X(\alpha) \\ t_X(\alpha\underbrace{0\ldots0}_{i}1\beta) & \text{for } \gamma = \alpha\alpha_i\beta \in A_Y(\alpha). \end{cases}$$

Conversely, a term Y expands to another term X, i.e. $Y \leq_1 X$, using (1), if and only if there is $\alpha \in \mathscr{S}$ such that for all $\alpha_i \neq \otimes$, $0 \leq i \leq n$, the set $A_Y(\alpha\alpha_i)$ is non-empty; hence an A-domain. Then one can obtain X from Y, or equivalently t_X from t_Y, substituting in A_Y every sequence of the form $\alpha\alpha_i\beta$ by $\alpha\underbrace{0\ldots0}_{i}1\beta$, for $\alpha_i \neq \otimes$, but maintaining the function value for these arguments. Furthermore, one has to add the sequence $\alpha\underbrace{0\ldots0}_{n+1}$ with image M to the domain, as well as for all $\alpha_i = \otimes$ the elements of some set $\{\alpha\underbrace{0\ldots0}_{i}1\beta : \beta \in B_i\}$, where B_i is any A-domain. The function values for these sequences may be chosen arbitrarily in $\mathscr{V} \cup \mathscr{C}$.

Thus, expanding terms corresponds essentially to the substitution of substrings by strings of the form $0\ldots01$. To stress this fact we will use the following representation

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for a linear (affine) proper combinator as in (1) and $\alpha_0, \ldots, \alpha_n$ defined in (2):

$$\psi(M) = \left\{ \begin{array}{lll} \alpha_0 & \Leftarrow_1 & 1 \\ & \vdots & \\ \alpha_n & \Leftarrow_1 & \underbrace{0 \ldots 0}_{n} 1. \end{array} \right.$$

This representation can be extended to sets of combinators:

$$\psi(\Sigma) = \{\psi(M) \mid M \in \Sigma\}.$$

Example
The set $\psi(\{B, C, I, K\})$ consists of

$$\psi(B) \quad = \quad \left\{ \begin{array}{lll} 11 & \Leftarrow_1 & 1 \\ 10 & \Leftarrow_1 & 01 \\ 0 & \Leftarrow_1 & 001 \end{array} \right. \qquad \psi(C) \quad = \quad \left\{ \begin{array}{lll} 01 & \Leftarrow_1 & 1 \\ 1 & \Leftarrow_1 & 01 \\ 00 & \Leftarrow_1 & 001 \end{array} \right.$$

$$\psi(I) = \{\epsilon \ \Leftarrow_1 \ 1 \quad \text{and} \quad \psi(K) = \left\{ \begin{array}{lll} \otimes & \Leftarrow_1 & 1 \\ \epsilon & \Leftarrow_1 & 01 \end{array} \right.$$

It is well known that to be a basis for one of the lambda (sub-)systems in Fig. 1, it is sufficient for a set Σ to define all proper combinators of that system (in fact, defining the elements of the bases in Fig. 1 would be enough). As an example, consider the combinator N with reduction rule $Nxyz \geq_1 y(zx)$. Then N can be defined by $B(CB)(CI)$ since:

	$y(zx)$	$\leq_1$	$y(Izx)$	$\leq_1$	$y(CIxz)$	$\leq_1$	$By(CIx)z$	$\leq_1$	$CB(CIx)yz$	$\leq_1$	$B(CB)(CI)xyz$
z:	10	$\Leftarrow_1$	101	$\Leftarrow_1$	11	$\Leftarrow_1$	1	$\Leftarrow_1$	1	$\Leftarrow_1$	1
y:	0	$\Leftarrow_1$	0	$\Leftarrow_1$	0	$\Leftarrow_1$	001	$\Leftarrow_1$	01	$\Leftarrow_1$	01
x:	11	$\Leftarrow_1$	11	$\Leftarrow_1$	101	$\Leftarrow_1$	011	$\Leftarrow_1$	0011	$\Leftarrow_1$	001
I:			100	$\Leftarrow_1$	1001	$\Leftarrow_1$	0101	$\Leftarrow_1$	00101	$\Leftarrow_1$	00011
C:					1000	$\Leftarrow_1$	0100	$\Leftarrow_1$	00100	$\Leftarrow_1$	00010
B:							000	$\Leftarrow_1$	0001	$\Leftarrow_1$	000011
C:									0000	$\Leftarrow_1$	000010
B:											00000

Note, for instance, that the last column has been obtained from the previous one by expansion with $\psi(B)$ at position 00. Indeed, all sequences of the form $00 \cdot 11 \cdot \beta$ have been substituted by $00 \cdot 1 \cdot \beta$, all sequences $00 \cdot 10 \cdot \beta$ by $00 \cdot 01 \cdot \beta$ and all $00 \cdot 0 \cdot \beta$ by $00 \cdot 001 \cdot \beta$. Furthermore, a sequence $00 \cdot 000$ with function value B has been created.

In general, a set of combinators Σ defines a linear (resp. affine) proper combinator N with reduction rule $Nx_0 \ldots x_n \geq_1 X$ if and only if there is a finite sequence

$$X \leq_1 Z_0 x_n \leq_1 Z_1 x_{n-1} x_n \leq_1 \ldots \leq_1 Z_n x_0 \ldots x_n$$

such that each Z_i for $0 \leq i \leq n$ is an applicative term containing only combinators in Σ and variables in $\{x_0, \ldots, x_{n-i-1}\}$. Hence it is easy to see that Σ is a basis for

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the linear (resp. affine) λ-calculus if and only if any applicative term X, with (at most) one occurrence of a variable x, can be expanded with combinators in Σ to another term of the form Z_0x such that x does not occur in Z_0. This means that $t_X : A_X \longrightarrow \mathscr{V} \cup \mathscr{C}$ is expandable by means of the rules in $\psi(\Sigma)$ to some $t_Z : A_Z \longrightarrow \mathscr{V} \cup \mathscr{C}$, such that $t_Z^{-1}(x) = \{1\}$, i.e. $Z = Z_0x$. In particular, one should be able to rewrite the position $\alpha = p(x, X)$ to 1, by means of the lines of the rules in $\psi(\Sigma)$, i.e. successively substituting subsequences in α, which are on the left-side of a line in a rule $\psi(M)$ by the right-side of the same line. This provides us with a necessary condition for a set of proper combinators to form a basis for the linear (resp. affine) lambda calculus. The condition is also sufficient, as will be shown in the appendix. In consequence of this property, and abusing the notation, from now on we will use the word 'rule' to denote the combinator rules $\psi(M)$ as well as the lines they contain. Furthermore, we write $\Sigma : \alpha \Leftarrow \beta$ if β is obtained by rewriting α a finite number of times with (lines of) rules in $\psi(\Sigma)$.

Lemma 2.5
Let Σ be a proper combinator basis for the linear (or affine) lambda calculus. Then $\Sigma : \alpha \Leftarrow 1$ for any $\alpha \in \mathscr{S}$ (resp. $\alpha \in \mathscr{S} \cup \{\otimes\}$).

3 Bases for the linear λ-calculus

It is well known that the set $\{I, C, B\}$ is a basis for the linear λ-calculus, i.e. every linear λ-term M can be defined by a combination of I, C and B. In the following we will give necessary and sufficient conditions for a (finite) set of proper combinators to be complete. We begin by recalling some useful results that are due to W. Craig and which can be found in Curry and Feys (1958).

Definition 3.1
Consider any proper combinator M and its reduction rule

$$Mx_1 \ldots x_n \geq_1 Z.$$

The combinator M has cancellative effect if at least one variable x_i does not occur in Z. If $Z = x_i$ with $1 \leq i \leq n$, then M is called a selector. We shall say that M has a compositive effect if and only if Z contains parentheses.

Theorem 3.2 (*Craig*)

1. Let X be a combination of proper combinators none of which is a selector. Then X is not a selector.
2. Let X be a combination of proper combinators none of which has any compositive effect, and let X be proper. Then X has no compositive effect.
3. Let X be a combination of proper combinators none of which has any cancellative effect, and let X be proper. Then X has no cancellative effect.

Since I is the only linear selector it becomes obvious that every complete set Σ of proper combinators for the linear λ-calculus must contain I as well as at least one combinator with compositive effect. On the other hand, no combinator with a cancellative effect is linear. Now we shall determine some necessary conditions on

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the rules in $\psi(\Sigma)$. From Lemma 2.5 we know that $\Sigma : 01 \Leftarrow 1$, i.e. there is a finite sequence

$$01 = \beta_1 \Leftarrow_1 \ldots \Leftarrow_1 \beta_n = 1 \tag{3}$$

where each step results from one application of a rule in $\psi(\Sigma)$. Since each β_{i+1} is obtained from β_i by substitution of a subsequence in β_i by a sequence of the form $0\ldots01$, it is easy to see that all β_i end with an 1 for $1 \leq i \leq n$. Thus, a rule of the form $\alpha 1 \Leftarrow_1 1$ with $\alpha \in \{0,1\}^+$ was applied at least once.

Lemma 3.3
If Σ is combinatorially complete for the linear λ-calculus, then $\exists \alpha \in \{0,1\}^+$ such that $\alpha 1 \Leftarrow_1 1 \;\in\; \psi(\Sigma)$.

For $\alpha \in \mathscr{S}$ let $\sharp_0(\alpha)$ and $\sharp_1(\alpha)$ denote respectively the number of 0's and the number of 1's in α. Then we conclude the following from $\Sigma : 01 \Leftarrow 1$.

Lemma 3.4
If Σ is combinatorially complete for the linear λ-calculus, then $\exists \gamma \in \{0,1\}^+$ such that $\sharp_0(\gamma) > 0$ and $\gamma \Leftarrow_1 1 \;\in\; \psi(\Sigma)$.

Definition 3.5
Let Σ be a finite set of linear proper combinators and let n be the number of lines of rules in $\psi(\Sigma)$. For every rule $\alpha_i \Leftarrow_1 \underbrace{0\ldots0}_{d_i}1$ in $\psi(\Sigma)$, $1 \leq i \leq n$ let $p_i = \sharp_0(\alpha_i) - d_i$. The diophantine equation of Σ is defined by

$$\sum_1^n p_i x_i = 1.$$

Example
The diophantine equation of $\Sigma = \{I, C, B\}$ is

$$x - y - z = 1$$

with solution $x = 1$ and $y = z = 0$. The coefficients of x and y result, respectively, from the first and the second rule in $\psi(C)$ while the coefficient of z results from the third rule in $\psi(B)$.

Note that there is a direct correspondence between the coefficients $p_1, \ldots, p_n$ and the rules in $\psi(\Sigma)$, which we will, for the moment, denote by $\varphi_1, \ldots, \varphi_n$. In fact, each p_i represents the number of 0's that are consumed by one application of φ_i. Consider once more the sequence (3) and for each φ_i let a_i be the number of times that this rule was applied. It is obvious that $x_i = a_i$, $1 \leq i \leq n$, is a solution of the diophantine equation of Σ:

Lemma 3.6
If Σ is combinatorially complete for the linear λ-calculus, then its diophantine equation

$$\sum_1^n p_i x_i = 1 \tag{4}$$

admits at least one solution $(a_1, \ldots, a_n) \in \mathbb{N}_0^n$.

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Lemmas 3.3, 3.4 and 3.6 together with Theorem 3.2 provide the basis for the "only-if" part of the following theorem. The proof for the fact that the enumerated conditions are also sufficient for a set Σ being a basis, can be found in the appendix.

Theorem 3.7

Let Σ be a finite set of linear proper combinators. Then Σ is a basis for the linear λ-calculus if and only if the following conditions are satisfied:

1. $I \in \Sigma$;
2. Σ contains at least one combinator with compositive effect;
3. $\exists \alpha \in \{0,1\}^+$ such that $\alpha 1 \Leftarrow_1 1 \;\in\; \psi(\Sigma)$;
4. $\exists \gamma \in \{0,1\}^+$ such that $\sharp_0(\gamma) > 0$ and $\gamma \Leftarrow_1 1 \;\in\; \psi(\Sigma)$;
5. the diophantine equation of Σ

$$\sum_1^n p_i x_i = 1 \tag{5}$$

is solvable.

Algorithms to solve (systems of) linear diophantine equations over the naturals are known since the beginning of this century (see Stanley (1983)), hence we conclude the following:

Corollary 3.8

Combinatory completeness for the linear λ-calculus is decidable for finite sets of proper combinators.

Example

Consider the set $\Sigma = \{I, A\}$, where A is defined by

$$Axyzu \geq_1 y(zu)x$$

Then $\psi(\Sigma)$ contains the following rules:

$$\psi(I) \quad = \quad \{\epsilon \;\Leftarrow_1\; 1$$

$$\psi(A) \quad = \quad \left\{ \begin{array}{lll} 011 & \Leftarrow_1 & 1 \\ 010 & \Leftarrow_1 & 01 \\ 00 & \Leftarrow_1 & 001 \\ 1 & \Leftarrow_1 & 0001 \end{array} \right.$$

and the diophantine equation of Σ is

$$x + y - 3z = 1$$

with solution $x = 1$ and $y = z = 0$. It is easy to verify that Σ satisfies all conditions of Theorem 3.7, and hence it is a basis for the linear λ-calculus.

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4 The affine λ-calculus

In this section we obtain a similar result to Theorem 3.7 for the affine λ-calculus. From now on let Σ be a finite set of affine proper combinators, then $\Sigma = \Sigma_0 \cup \Sigma_T$, where $\Sigma_T = \{T_1, \ldots, T_t\}$ contains all selectors in Σ.

Suppose none of the selectors in Σ_T has reduction rule $Tx_1 \ldots x_n \geq_1 x_n$. This implies that $\psi(\Sigma)$ contains no rule of the form $\epsilon \Leftarrow_1 1$. From Lemma 2.5 we know that $\Sigma : 01 \Leftarrow 1$, if Σ is a basis. Thus, at some point a sequence beginning with 0 must rewrite in one step to 1:

Lemma 4.1
If no selector in Σ has reduction rule $Tx_1 \ldots x_n \geq_1 x_n$, then $\exists \alpha \in \{0,1\}^*$ such that $0\alpha \Leftarrow_1 1 \ \in \ \psi(\Sigma)$.

Obviously, Lemmas 3.3, 3.4 and 3.6 in the previous section are also valid for any affine basis Σ, but since Lemma 3.6 is not strong enough to guarantee completeness, we define the diophantine equations of a set of affine proper combinators in a manner to control the consumption of both 0's and 1's during the application of the rules in $\psi(\Sigma)$.

Definition 4.2

- For each rule in $\psi(\Sigma)$ of the form $\underbrace{0\ldots0}_{u_i \geq 0}1 \Leftarrow_1 \underbrace{0\ldots0}_{v_i \geq 0}1$ let $t_i = u_i - v_i$ for $1 \leq i \leq m$ (here m denotes the number of rules of this form in $\psi(\Sigma)$). Then we define the equation $eq(\Sigma)$ by
$$\sum_i^m t_i z_i = 1.$$
- Consider all rules of the form $\alpha_i 1 \Leftarrow_1 \beta_i$ in $\psi(\Sigma_0)$ with $\sharp_1(\alpha_i) \geq 1$ for $1 \leq i \leq l$. Let $p_i^0 = \sharp_0(\alpha_i) - \sharp_0(\beta_i)$ and $p_i^1 = \sharp_1(\alpha_i)$. For all other rules of the form $\alpha_i \Leftarrow_1 \beta_i$ in $\psi(\Sigma_0)$ with $\alpha_i \neq \otimes$ and $1 \leq i \leq k$, take $q_i^0 = \sharp_0(\alpha_i) - \sharp_0(\beta_i)$ and $q_i^1 = \sharp_1(\alpha_i) - 1$. Moreover, suppose that $\psi(\Sigma_T)$ contains t rules of the form $\epsilon \Leftarrow_1 \underbrace{0\ldots0}_{d_i \geq 0}1$ for $1 \leq i \leq t$, corresponding, respectively, to the selectors $T_1, \ldots, T_t$. Then we define the system $Eq(\Sigma)$ as follows.
$$Eq(\Sigma) = \begin{cases} \sum_1^l p_i^0 x_i + \sum_1^k q_i^0 y_i - \sum_1^t d_i n_i & = 1, \\ \sum_1^l p_i^1 x_i + \sum_1^k q_i^1 y_i - \sum_1^t n_i & = 0. \end{cases}$$

Consider once more the sequence
$$\beta_0 = 01 \Leftarrow_1 \ldots \Leftarrow_1 1 = \beta_n, \tag{6}$$
and recall that the coefficients in $eq(\Sigma)$ and $Eq(\Sigma)$ correspond to rules in $\psi(\Sigma)$: the coefficients in $eq(\Sigma)$ and in the first equation of $Eq(\Sigma)$ represent the number of 0's that are consumed during one application of the corresponding rule, while the coefficients in the second equation of $Eq(\Sigma)$ stand for the number of 1's that are consumed. Suppose that in equation (6) only rules of the form $0\ldots01 \Leftarrow_1 0\ldots01$ were used. Let z_i be the number of times that each corresponding rule was applied. Then $(z_1, \ldots, z_m)$ is a solution of $eq(\Sigma)$, since in $01 \Leftarrow 1$ exactly one 0 was consumed.

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Otherwise, at least one β_i, with $1 \leq i \leq (n-1)$, contains more than one 1. Thus, a rule of the form $\alpha 1 \Leftarrow_1 \beta$, with $\sharp_1(\alpha) \geq 1$, was applied in (6) at least once. Now let x_i (resp. y_i and n_i) be the number of times that the corresponding rule was applied. Then $(x_1, \ldots, x_l, y_1, \ldots, y_k, n_1, \ldots, n_t) \in N_0^{l+k+t}$ is a solution of $Eq(\Sigma)$ with $x_i > 0$ for some $i \in \{1, \ldots l\}$, since 1 contains one 0 less than and as many 1's as 01. We conclude the following:

Lemma 4.3
If Σ is combinatorially complete for the affine λ-calculus, then

- $eq(\Sigma)$ admits a solution $(z_1, \ldots, z_m) \in \mathbb{N}_0^m$

or

- $Eq(\Sigma)$ admits a solution $(x_1, \ldots, x_l, y_1, \ldots, y_k, n_1, \ldots, n_t) \in \mathbb{N}_0^{l+k+t}$ with $x_i > 0$ for some $i \in \{1, \ldots, l\}$.

As in the previous section, Theorem 3.2 together with Lemmas 3.3, 3.4 4.1 and 4.3, provides us with necessary and sufficient conditions for a finite set of affine proper combinators to be complete for the affine lambda calculus.

Theorem 4.4
Let Σ be a finite set of affine proper combinators. Then Σ is a basis for the affine λ-calculus if and only if the following conditions are satisfied:

1. Σ contains at least one selector;
2. Σ contains at least one combinator with compositive effect;
3. Σ contains at least one combinator with cancellative effect;
4. if no selector in Σ has reduction rule $Tx_1 \ldots x_n \geq_1 x_n$, then $\exists \alpha \in \{0, 1\}^*$ such that $0\alpha \Leftarrow_1 1 \ \in \ \psi(\Sigma)$;
5. $\exists \beta \in \{0, 1\}^+$ such that $\beta 1 \Leftarrow_1 1 \ \in \ \psi(\Sigma)$;
6. $\exists \gamma \in \{0, 1\}^+$ such that $\sharp_0(\gamma) > 0$ and $\gamma \Leftarrow_1 1 \ \in \ \psi(\Sigma)$;
7. the equation $eq(\Sigma)$ has a solution $(z_1, \ldots, z_m) \in \mathbb{N}_0^m$ or $Eq(\Sigma)$ has a solution $(x_1, \ldots, x_l, y_1, \ldots, y_k, n_1, \ldots, n_t) \in \mathbb{N}_0^{l+k+t}$ with $x_i > 0$ for some $i \in \{1, \ldots, l\}$.

Corollary 4.5
Combinatory completeness for the affine λ-calculus is decidable for finite sets of proper combinators.

Example
Consider the set $\Sigma = \{K_2, A\}$, where K_2 and A are defined by

$$K_2 xyz \geq_1 x \quad \text{and} \quad Axyzuvw \geq_1 vw(ux)y.$$

Then $\psi(\Sigma)$ contains the rules

$$\psi(K_2) \;=\; \left\{ \begin{array}{lcl} \otimes & \Leftarrow_1 & 1 \\ \otimes & \Leftarrow_1 & 01 \\ \epsilon & \Leftarrow_1 & 001 \end{array} \right. \qquad \psi(A) \;=\; \left\{ \begin{array}{lcl} 001 & \Leftarrow_1 & 1 \\ 000 & \Leftarrow_1 & 01 \\ 010 & \Leftarrow_1 & 001 \\ \otimes & \Leftarrow_1 & 0001 \\ 1 & \Leftarrow_1 & 00001 \\ 011 & \Leftarrow_1 & 000001 \end{array} \right.$$

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which satisfy conditions 1–6 of Theorem 4.4. Nevertheless, $\{K_2, A\}$ is no basis for the affine λ-calculus, since neither

$$eq(\Sigma): \quad 2z_1 - 4z_2 = 1$$

nor

$$Eq(\Sigma): \left\{ \begin{array}{rcrcrcrcrcr} -4x & + & 2y_1 & + & 2y_2 & - & 4y_4 & - & 2n & = & 1 \\ x & & & - & y_2 & & & - & n & = & 0 \end{array} \right.$$

admit solutions over the naturals.

5 An extension to the λI-calculus

In the following we address the problem of extending sets of proper combinators that are complete for the linear λ-calculus to bases of the λI-calculus[1]. Again, a result from W. Craig (in Curry and Feys, 1958) provides us with a necessary property.

Definition 5.1
A proper combinator M with reduction rule $Mx_1 \dots x_n \geq_1 Z$ is called a duplicator if at least one variable x_i, $1 \leq i \leq n$, occurs more than once in Z. We say that M is a 2-duplicator if one variable occurs exactly twice in Z.

Theorem 5.2 (*Craig*)
Let X be a combination of proper combinators none of which has any duplicative effect, and let X be proper. Then X has no duplicative effect.

Thus, for a set of proper combinators to be complete for the λI-calculus, it has to contain at least one combinator with duplicative effect. In fact, it is easy to see that any basis for λI has to contain a 2-duplicator, since no combinator with cancellative effect is a λI-term.

Lemma 5.3
For $m, n, k \geq 0$ let p be the least positive integer such that

$$m + k + p + 3 \geq n + k + 2,$$

and consider the linear combinator

$$C^* = \lambda x_1 \dots x_{m+k+p+3}.(x_{m+2} \dots x_{m+k+p+2} x_2 \dots x_{m+1} x_{m+k+p+3} x_1).$$

Then there are $a, b \geq 0$ such that for any term X one has

$$C^* z \underbrace{I \dots I}_{m} (C^* z \underbrace{I \dots I}_{n}) \underbrace{I \dots I}_{k} X \underbrace{I \dots I}_{p} \;\geq\; X \underbrace{I \dots I}_{a} z \underbrace{I \dots I}_{b} z.$$

[1] A similar result (Theorem 1. in Statman, 1986) exists for extending bases of the affine λ-calculus to bases of the pure λ-calculus.

Proof
There is

$$C^* x_1 \dots x_{m+k+p+3} \ \geq_1 \ x_{m+2} \dots x_{m+k+p+2} x_2 \dots x_{m+1} x_{m+k+p+3} x_1, \tag{7}$$

thus

$$\begin{aligned} C^* z \underbrace{I \dots I}_{m} (C^* z \underbrace{I \dots I}_{n}) \underbrace{I \dots I}_{k} X \underbrace{I \dots I}_{p} \ & \geq_1 \ C^* z \underbrace{I \dots I}_{n} \underbrace{I \dots I}_{k} X \underbrace{I \dots I}_{p-1} \underbrace{I \dots I}_{m} I z \\ & \geq_1 \ I \dots I X \underbrace{I \dots I}_{a} z \underbrace{I \dots I}_{b} z \\ & \geq \ X \underbrace{I \dots I}_{a} z \underbrace{I \dots I}_{b} z. \end{aligned}$$

The last reduction steps use the reduction rule for I, whereas the first two reduction steps follow by application of equation (7). For the second step, note that

1. C^* has arity $m + k + p + 3$;
2. the first occurrence of z in $C^* z \underbrace{I \dots I}_{n} \underbrace{I \dots I}_{k} X \underbrace{I \dots I}_{p-1} \underbrace{I \dots I}_{m} I z$ is the first term after C^*, i.e. corresponds to x_1;
3. the subterm X corresponds to x_{n+k+2} and is an argument of C^*, since $m + k + p + 3 \geq n + k + 2$;
4. the second occurrence of z is the $(n + k + p + m + 3)$th term after C^*, hence it does not correspond to an argument of C^* unless $n = 0$. In this case this occurrence of z would correspond to the last argument of C^*.

Now it is easy to conclude that after the second reduction step one has X on the left of both occurrences of z, that one occurrence of z is in the last position of the term and all other subterms are occurrences of the identity combinator I. $\square$

Theorem 5.4
Let Σ be a basis for the linear λ-calculus. Then $\Sigma \cup \Gamma$ is complete for the λI-calculus if and only if Γ contains a 2-duplicator.

Proof
First suppose that $\Sigma \cup \Gamma$ is complete for the λI-calculus. Then the combinator $W_* x \geq xx$ should be definable by $\Sigma \cup \Gamma$, which, on the other hand, must not contain any combinator with cancellative effect. Thus, there must be at least one 2-duplicator in Γ, since there is no duplicator in Σ.

Now let $D \in \Gamma$ be a 2-duplicator with reduction rule $D x_1 \dots z \dots x_d \ \geq_1 \ Z$, such that the variable z occurs exactly twice in Z. Then there are integers $m, n, k \geq 0$ such that

$$D \underbrace{I \dots I z I \dots I}_{d} \geq z \underbrace{I \dots I}_{m} (z \underbrace{I \dots I}_{n}) \underbrace{I \dots I}_{k} \tag{8}$$

Consider a proper λI-combinator M with reduction rule

$$M x_1 \dots x_N \geq_1 X,$$

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where X is a combination of $x_1, \ldots, x_N$ in which each x_i, $1 \leq i \leq N$, occurs at least once. Then, M is definable by $\Sigma \cup \Gamma$ if and only if there is a sequence

$$X = X_0 \leq X_1 x_N \leq \ldots \leq X_N x_1 \ldots x_N$$

such that each X_j, $1 \leq j \leq N$, is a combination over $\{x_1, \ldots, x_{N-j}\} \cup \Sigma \cup \Gamma$.

If x_N occurs exactly once in $X = X_0$, then the existence of X_1 is guaranteed, since Σ is a basis for the linear λ-calculus. Otherwise, suppose that x_N occurs $t \geq 2$ times in X. We will show that there is a combination X^1 over $\{x_1, \ldots, x_N\} \cup \Sigma \cup \Gamma$ such that $X \leq X^1 x_N$, where x_N occurs $t-2$ times in X^1. Then, repeating the process to $X^1 x_N$, i.e. $X^1 x_N \leq X^2 x_N$ with $t-3$ occurrences of x_N in X^2, one eventually obtains

$$X \leq X^1 x_N \leq X^2 x_N \leq \ldots \leq X^{t-1} x_N = X_1 x_N$$

where X_1 is a combination over $\{x_1, \ldots, x_{N-1}\} \cup \Sigma \cup \Gamma$.

Now, let x_N have $t \geq 2$ occurrences in X. Then, one can expand X with combinators in Σ such that

$$\begin{aligned} X &\leq \underbrace{I \ldots I}_{a+b} X \\ &\leq X' \underbrace{I \ldots I}_{a} x_N \underbrace{I \ldots I}_{b} x_N. \end{aligned}$$

Furthermore, one has by equation (8), and since C^* is definable by Σ,

$$\begin{aligned} X' \underbrace{I \ldots I}_{a} x_N \underbrace{I \ldots I}_{b} x_N &\leq C^* x_N \underbrace{I \ldots I}_{m} (C^* x_N \underbrace{I \ldots I}_{n}) \underbrace{I \ldots I}_{k} X' \underbrace{I \ldots I}_{p} \\ &\leq DI \ldots I (C^* x_N) I \ldots I X \underbrace{I \ldots I}_{p} \\ &\leq X^1 x_N. \end{aligned}$$

It remains to repeat the process to $x_{N-1}, \ldots, x_1$. $\square$

6 Appendix

6.1 Preliminaries

As noted in section 2, a finite set of linear (resp. affine) proper combinators Σ is complete for the linear (resp. affine) lambda calculus if for any term X and any variable x the function $t_X : A_X \longrightarrow \mathscr{V} \cup \mathscr{C}$ is expandable with rules $\psi(M) \in \psi(\Sigma)$ to $t_Z : A_Z \longrightarrow \mathscr{V} \cup \mathscr{C}$ such that $t_Z^{-1}(x) = \{1\}$. In particular, $\alpha = p(x, X)$ can be rewritten to 1 by means of the lines of the rules in $\psi(\Sigma)$, i.e. $\Sigma : \alpha \Leftarrow 1$. We will show now that, due to the obligatory presence of selectors in Σ, this condition is sufficient, i.e. if $\Sigma : \alpha \Leftarrow 1$, then t_X expands with rules in $\psi(\Sigma)$ to some function t_Z such that $t_Z^{-1}(x) = \{1\}$.

Lemma 6.1
Let T be a selector, M a linear or affine proper combinator with

$$\psi(M) = \begin{cases} \alpha_0 & \Leftarrow_1 \quad 1 \\ & \vdots \\ \alpha_n & \Leftarrow_1 \quad \underbrace{0 \ldots 0}_{n} 1 \end{cases}$$

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and consider an applicative term X such that $p(x, X) = \alpha \in \mathscr{S} \cup \{\otimes\}$. If some line of $\psi(M)$ applies to α at position β, then it is possible to expand $t_X : A_X \longrightarrow \mathscr{V} \cup \mathscr{C}$ a finite number of times with $\psi(T)$ to $t_{\tilde{X}} : A_{\tilde{X}} \longrightarrow \mathscr{V} \cup \mathscr{C}$ such that $p(x, \tilde{X}) = \alpha$ and $t_{\tilde{X}}$ is expandable with $\psi(M)$ at position β.

Proof
We show the result for $\alpha = \beta\alpha_i\gamma$, since the proof for $\alpha = \otimes$ is very similar. The result is trivial for $n = 0$. Thus consider $n \geq 1$.

For every selector T one has

$$\psi(T) = \begin{cases} \otimes & \Leftarrow_1 \quad 1 \\ & \vdots \\ \epsilon & \Leftarrow_1 \quad \underbrace{0\ldots0}_{d\geq 0}1 \\ & \vdots \\ \otimes & \Leftarrow_1 \quad \underbrace{0\ldots0}_{N}1 \end{cases}$$

and whenever $\epsilon \Leftarrow_1 0\ldots01 \in \psi(T')$, then T' is itself a selector.

Hence suppose that $\alpha_i = a_1 \ldots a_s$, with $s \geq 1$ and $a_1, \ldots, a_s \in \{0, 1\}$. Consider any $\alpha_j \neq \alpha_i, \otimes$. If $\beta\alpha_j\gamma_j \in A_X$, we are done. Otherwise $\alpha_j = a_1 \ldots a_{q-1}\overline{a}_q b_1 \ldots b_t$, where $q \in \{1, \ldots, s\}$ and $t \geq 0$. On the other hand, there is at least one position $\beta a_1 \ldots a_{q-1}\overline{a}_q b_1 \ldots b_{t'} \in A_X$, with $0 \leq t' < t$. Thus, expanding t_X at position $\beta a_1 \ldots a_{q-1}\overline{a}_q b_1 \ldots b_{t'}$ with $\psi(T)$, we obtain positions $\beta a_1 \ldots a_{q-1}\overline{a}_q b_1 \ldots b_{t'}\underbrace{0\ldots0}_{u\geq 0}1$, for $u \in \{0, \ldots, N\}$, as well as one position $\beta a_1 \ldots a_{q-1}\overline{a}_q b_1 \ldots b_{t'}\underbrace{0\ldots0}_{N+1}$. Obviously, either one of these positions is of the form $\beta\alpha_j\gamma_j$, for some $\gamma_j \in \{0, 1\}^*$, and we are done, or otherwise one of these positions is a prefix of $\beta\alpha_j$ and repeating the process to it, one eventually obtains an argument with position $\beta\alpha_j\gamma_j$. Note that the previous method can be applied to $\alpha_j \neq \alpha_i$ in arbitrary order without destroying the existence of positions that have been created previously. $\square$

Corollary 6.2
A finite set of linear (resp. affine) proper combinators Σ is a basis for the linear (resp. affine) λ-calculus if and only if $\Sigma : \alpha \Leftarrow 1$ for any $\alpha \in \mathscr{S}$ (resp. $\alpha \in \mathscr{S} \cup \{\otimes\}$).

6.2 Proof of Theorem 3.7

Proof
Due to Theorem 3.2 and Lemmas 3.3, 3.4 and 3.6, it remains to show that conditions 1–5 guarantee the completeness of Σ.

By Corollary 6.2, and since $\epsilon \Leftarrow_1 1 \in \psi(\Sigma)$, it is sufficient to prove that $\Sigma : \alpha \Leftarrow 1$ for all $\alpha \in \{0, 1\}^+$. We begin to show that

$$\exists k_0 \geq 0 : k \geq k_0 \text{ implies } \underbrace{0\ldots0}_{k}0 \Leftarrow \underbrace{0\ldots0}_{k}\underbrace{1\ldots1}_{\sum a_i}$$

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where $(a_1, \ldots, a_n)$ is a solution of $\sum_1^n p_i x_i = 1$. This follows from the fact that for each $p \in \{p_1, \ldots, p_n\}$ $\psi(\Sigma)$ contains a rule of the form $\gamma_p \Leftarrow_1 \underbrace{0\ldots0}_{d}1$ with $\sharp_0(\gamma_p) = d + p$. Thus, choosing k sufficiently big, and using whenever necessary the rule $\epsilon \Leftarrow_1 1$, one has

$$\underbrace{0\ldots0}_{k}0 \Leftarrow \underbrace{0\ldots0}_{k-(d+p)+1}\gamma_p \Leftarrow \underbrace{0\ldots0}_{k-p}01 \Leftarrow \ldots \Leftarrow \underbrace{0\ldots0}_{k-pa}0\underbrace{1\ldots1}_{a}$$

and applying a_i times each rule $\gamma_{p_i} \Leftarrow_1 \underbrace{0\ldots0}_{d_i}1$ the result follows from $\sum p_i a_i = 1$.

For the rest of the proof we will distinguish between two cases:

(1) $\exists n \geq 2 \ : \ \underbrace{1\ldots1}_{n} \Leftarrow_1 1 \in \psi(\Sigma)$.

Since $\epsilon \Leftarrow_1 1 \in \psi(\Sigma)$ implies $\Sigma : \alpha\beta \Leftarrow_1 \alpha 1 \beta$ for all $\alpha, \beta \in \mathscr{S}$, it is easy to conclude that any sequence of 1's rewrites to 1. Now, it is sufficient to show that 0 rewrites to a finite sequence of 1's: Due to the fact that the left-sides of the rules in any $\psi(M)$ form an A-domain and since $\underbrace{1\ldots1}_{n\geq2} \Leftarrow_1 1 \ \in \psi(\Sigma)$, there are

$$\begin{array}{lcl} \alpha_1 & \in & \{0, 01, 011, \ldots\} \\ \alpha_2 & \in & \{10, 101, 1011, \ldots\} \\ & \vdots & \\ \alpha_n & \in & \{\underbrace{1\ldots1}_{n-1}0, \underbrace{1\ldots1}_{n-1}01, \underbrace{1\ldots1}_{n-1}011, \ldots\} \end{array}$$

such that

$$\alpha_i \Leftarrow_1 \underbrace{0\ldots0}_{d_i\geq1}1 \ \in \psi(\Sigma)$$

and $d_i \neq d_j$, for $i \neq j$. Since $n \geq 2$, there is at least one rule $\alpha \Leftarrow_1 \underbrace{0\ldots0}_{d}1$, $\alpha \in \{\alpha_1, \ldots, \alpha_n\}$ in $\psi(\Sigma)$ such that $\sharp_0(\alpha) = 1$ and $d \geq 2$. Using $\epsilon \Leftarrow_1 1$ and $\alpha \Leftarrow_1 \underbrace{0\ldots0}_{d}1$ one obtains

$$0 \Leftarrow \alpha \Leftarrow \underbrace{0\ldots0}_{d-1}01 \Leftarrow \underbrace{0\ldots0}_{d-1}\alpha 1 \Leftarrow \underbrace{0\ldots0}_{d-1}\underbrace{0\ldots0}_{d-1}011 \Leftarrow \ldots \Leftarrow \underbrace{0\ldots0}_{k}01\ldots1$$

where k is the least multiple of $(d-1)$ and $p = \sharp_0(\gamma)$ greater than or equal to k_0, where γ is as in condition 4. Then,

$$\underbrace{0\ldots0}_{k}01\ldots1 \Leftarrow \underbrace{0\ldots0}_{k}1\ldots1 \Leftarrow \underbrace{0\ldots0}_{k-p}1\ldots1 \Leftarrow \ldots \Leftarrow 1\ldots1.$$

(2) $\forall n \geq 2 \ \ \underbrace{1\ldots1}_{n} \Leftarrow_1 1 \notin \psi(\Sigma)$.

Obviously, this implies that $\sharp_0(\alpha) = s \geq 1$ for α as in condition 3. Hence, there are $\tilde{n}, d \geq 1$ such that

$$\underbrace{1\ldots1}_{\tilde{n}} \Leftarrow_1 \underbrace{0\ldots0}_{d}1 \ \in \psi(\Sigma) \qquad (9)$$

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Thus,

$$0 \Leftarrow 0\underbrace{1\ldots1}_{\tilde{n}} \Leftarrow \underbrace{0\ldots0}_{d}01 \Leftarrow \underbrace{0\ldots0}_{k}01\ldots1 \Leftarrow \underbrace{0\ldots0}_{k}1\ldots1 \Leftarrow 1\ldots1$$

where k is the least multiple of d and s that is greater than or equal to k_0. It remains to show that $\underbrace{1\ldots1}_{y} \Leftarrow 1, \forall y \geq 1$. If $\tilde{n} \geq 2$, then

$$\begin{aligned} 1\ldots1 &\Leftarrow \underbrace{1\ldots1}_{m(\tilde{n}-1)s}1 \Leftarrow \underbrace{0\ldots0}_{mds}1 \Leftarrow \underbrace{0\ldots0}_{(md-1)s}\alpha 1 \\ &\Leftarrow \underbrace{0\ldots0}_{(md-1)s}1 \Leftarrow 1 \end{aligned}$$

where $m(\tilde{n}-1)s \geq y-1$ is a multiple of $\tilde{n}-1$ and s. Finally, suppose $\tilde{n} = 1$ for all rules as in (9). Then we conclude from condition 3 that there is a rule of the form $1 \Leftarrow_1 \underbrace{0\ldots0}_{d\geq 1}1 \ \in \psi(\Sigma)$. On the other hand, Σ contains a compositor. Hence there is at least one rule of the form $\beta 11 \Leftarrow_1 \underbrace{0\ldots0}_{c\geq 0}1$ in $\psi(\Sigma)$, and consequently $\sharp_0(\beta) = e \geq 1$. Let $f = e - c$. Then,

$$\begin{aligned} \underbrace{1\ldots1}_{y} &\Leftarrow \underbrace{1\ldots1}_{m_1 s+1} \Leftarrow \underbrace{0\ldots0}_{m_2 sd}\underbrace{1\ldots1}_{m_1 s+1} \Leftarrow \underbrace{0\ldots0}_{m_2 sd-e}\beta 11\underbrace{1\ldots1}_{m_1 s-1} \\ &\Leftarrow \underbrace{0\ldots0}_{m_2 sd-f}\underbrace{1\ldots1}_{m_1 s} \Leftarrow \underbrace{0\ldots0}_{(m_2 d-m_1 f)s}1 \Leftarrow 1 \end{aligned}$$

where $m_1 s \geq y-1$ is a multiple of s and where $m_2 sd$ is a multiple of s and d such that $m_2 sd \geq e$ and $m_2 d \geq m_1 f$.

□

6.3 Proof of Theorem 4.4

Proof

The 'only-if' part of this proof is provided by Theorem 3.2, together with Lemmas 3.3, 3.4, 4.1 and 4.3. It remains to prove that any sequence in $\mathscr{S} \cup \{\otimes\}$ rewrites to 1. Since Σ contains a combinator with cancellative effect, there is at least one rule of the form $\otimes \Leftarrow_1 0\ldots01$ in $\psi(\Sigma)$. Thus, it really suffices to show that any sequence in $\mathscr{S}$ rewrites to 1. We will divide the proof into two parts:

$\boxed{\Sigma : \epsilon \Leftarrow 1}$

If Σ contains a selector T with reduction rule $Tx_1\ldots x_n \geq_1 x_n$, then $\epsilon \Leftarrow_1 1 \ \in \psi(\Sigma)$ and we are done. Otherwise we begin by proving some properties of Σ.

P1 $\exists m \geq 1$ such that $1 \Leftarrow \underbrace{0\ldots0}_{m}1$:

From $0\alpha \Leftarrow_1 1 \ \in \psi(\Sigma)$ (Condition 4) we conclude that $\underbrace{1\ldots1}_{p\geq 1} \Leftarrow_1 \underbrace{0\ldots0}_{q\geq 1}1 \ \in \psi(\Sigma)$. Condition 1 guarantees that there is some selector in $\psi(\Sigma)$. Since we already excluded selectors with reduction rules $Tx_1 \dots x_n \geq x_n$, $\psi(\Sigma)$ contains the rule $\epsilon \Leftarrow_1 \underbrace{0\dots0}_{\geq 1}1$. From now on let $d \geq 1$ denote the number of 0's in the right-side of that rule. Then,

$$1 \Leftarrow_1 \underbrace{0\dots0}_{d}11 \Leftarrow_1 \dots \Leftarrow_1 \underbrace{0\dots0}_{d(p-1)}\underbrace{1\dots1}_{p} \Leftarrow_1 \underbrace{0\dots0}_{d(p-1)+q}1.$$

From $q \geq 1$ we conclude that there is $m \geq 1$ such that $\Sigma : 1 \Leftarrow \underbrace{0\dots0}_{m}1$.

P2 $\forall \alpha \in \mathscr{S}\ \exists p \geq 0$ such that $1 \Leftarrow \underbrace{0\dots0}_{p}\alpha 1$:

This property follows from $\epsilon \Leftarrow_1 \underbrace{0\dots0}_{d\geq 1}1$ and $1 \Leftarrow \underbrace{0\dots0}_{m\geq 1}1$.

P3 $\exists k_0 \geq 0 : k \geq k_0$ implies $\underbrace{0\dots0}_{k+1}1 \Leftarrow \underbrace{0\dots0}_{k}1$:

If $eq(\Sigma)$ has a solution $(z_1, \dots, z_m)$, then the result is trivial, since for a sufficiently big number of 0's on the left we can apply, one after another each rule $\underbrace{0\dots0}_{u_i}1 \Leftarrow_1 \underbrace{0\dots0}_{v_i}1$ exactly z_i times to $\underbrace{0\dots0}_{k+1}1$ obtaining $\underbrace{0\dots0}_{k}1$.

Otherwise let $(x_1, \dots, x_l, y_1, \dots, y_k, n_1, \dots, n_t) \in \mathbb{N}_0^{l+k+t}$ be a solution of $Eq(\Sigma)$ with at least one coefficient $x > 0$ which corresponds to a rule of the form $\alpha 1 \Leftarrow_1 \beta$ where $\sharp_1(\alpha) \geq 1$. Now let Γ be the multi-set containing for every $i \in \{1, \dots, l\}$, x_i occurrences of the corresponding rule $\alpha_i 1 \Leftarrow_1 \beta$, for every $i \in \{1, \dots, k\}$, y_i occurrences of the corresponding rule $\alpha_i \Leftarrow_1 \beta_i$ as well as for $i \in \{1, \dots, t\}$ n_i occurrences of $\epsilon \Leftarrow_1 \underbrace{0\dots0}_{d_i\geq 1}1$.

Note that we have the result above, if we can apply the rules in Γ (each exactly one time) to a sequence γ_1 of the form $0\dots01$, containing a sufficient number of 0's, and obtain another sequence γ_2, which has necessarily an 1 in the last position. Then γ_2 is of the form $0\dots01$ and contains exactly one 0 less than γ_1. In fact, we do not have to be concerned about other occurrences of 1's in γ_2 or about the number of 0's, since that is guaranteed by $(x_1, \dots, x_l, y_1, \dots, y_k, n_1, \dots, n_t)$ being a solution of $Eq(\Sigma)$.

Now let Γ^+ be a copy of Γ where some occurrences of ϵ, 0 and 1 in the left-side of rules are negatively or positively marked: every 1 that does not occur in the last position is marked positively, while every 0 in the last position as well as every ϵ is marked negatively (Ex.: $\epsilon^- \Leftarrow_1 0\dots01$, $01^+1^+1^+00^- \Leftarrow_1 0\dots01$ and $1^+01 \Leftarrow_1 0\dots01$). It is important to note that Γ^+ contains as many positively as negatively marked symbols: for every occurrence of a rule $\delta \Leftarrow 0\dots01$ in Γ^+ the number of positive symbols minus the number of negative symbols equals exactly the corresponding coefficient for that rule in the second equation of $Eq(\Sigma)$. Thus the result follows from $\sum_1^l p_i^1 x_i + \sum_1^k q_i^1 y_i - \sum_1^t n_i = 0$.

We now show that, given a sequence γ_2 of the form $0\dots01$ with a sufficient number of 0's, one can obtain another sequence γ_1 with an 1 in the last position, substituting one after another right-sides of rules in Γ^+ by left-sides, and using each rule exactly once. As $(x_1, \dots, x_l, y_1, \dots, y_k, n_1, \dots, n_t)$ is a solution of $Eq(\Sigma)$

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we conclude then that γ_1 is of the form $0\ldots01$ with exactly one 0 more than γ_2, and obviously $\gamma_1 \Leftarrow \gamma_2$. Thus, consider $\gamma_2 = 0\ldots01$ with a sufficient number of 0's on the left. First expand γ_2 with all unmarked rules in Γ^+, i.e. rules of the form $0\ldots01 \Leftarrow_1 0\ldots01$. Next substitute in the result $0\ldots0\underbrace{0\ldots0}_{c}1$ by $0\ldots0\alpha1$, where $\alpha1 \Leftarrow_1 \underbrace{0\ldots0}_{c}1$, with $\sharp_1(\alpha) \geq 1$, is a rule in Γ^+ (the existence of this rule follows from $x > 0$). The resulting sequence contains at least one positively marked 1, i.e. it is of the form $0\ldots01^+\beta_11$. Now apply, one-by-one, and always to the most-left 1^+ the remaining rules of the form $\beta_21 \Leftarrow_1 0\ldots01$, substituting $0\ldots01^+\beta_11$ by $0\ldots0\beta_21^+\beta_11$. Next do the same with the rules of the form $\gamma_20^- \Leftarrow_1 0\ldots01$ and $\sharp_1(\gamma_2) \geq 1$. Here a sequence of the form $0\ldots01^+\gamma_11$ is extended to $0\ldots0\gamma_20\gamma_11$ (i.e. a positive symbol is destroyed by a negative one). Finally, repeat the process with the remaining rules which are now of the form $0\ldots00^- \Leftarrow_1 0\ldots01$ or $\epsilon^- \Leftarrow_1 0\ldots01$. Since Γ^+ contained in the beginning as much positively as negatively marked symbols and since we applied the rules of Γ^+ in such an order that after the first application of $\alpha1 \Leftarrow_1 \underbrace{0\ldots0}_{c}1$ and until there are only negatively marked rules left, the remaining rules contain always more negative than positive symbols, the process stops exactly when all rules are used and produces a sequence of the form $0\ldots01$.

Using the properties above, we now show $\Sigma : \epsilon \Leftarrow 1$:

- First suppose that $0\alpha1 \Leftarrow_1 1 \in \psi(\Sigma)$ for some $\alpha \in \mathscr{S}$. If $0\alpha1 = \underbrace{0\ldots0}_{q}1$, let vq be the least multiple of q greater than or equal to k_0. Then,

$$\begin{array}{lllllll} \epsilon & \Leftarrow_1 & \underbrace{0\ldots0}_{d\geq1}1 & \overset{\text{P1}}{\Leftarrow} & \underbrace{0\ldots0}_{t\geq vq}1 & \overset{\text{P3}}{\Leftarrow} & \underbrace{0\ldots0}_{vq}1 \quad \Leftarrow_1 \quad \underbrace{0\ldots0}_{(v-1)q}1 \\ & \Leftarrow_1 & \ldots & \Leftarrow_1 & 1. & & \end{array}$$

  Otherwise $\sharp_1(\alpha) \geq 1$, i.e. $0\alpha1 = \underbrace{0\ldots0}_{q\geq1}1\beta1$. Let vq be the smallest multiple of q greater or equal to k_0. Then,

$$\begin{array}{lllll} \epsilon \quad \Leftarrow_1 & \underbrace{0\ldots0}_{d\geq1}1 & \overset{\text{P2}}{\Leftarrow} & 0\ldots01\underbrace{\beta1\beta1\ldots\beta1}_{v} \\ \overset{\text{P1}}{\Leftarrow} & \underbrace{0\ldots0}_{t\geq vq}1\underbrace{\beta1\beta1\ldots\beta1}_{v} & \overset{\text{P3}}{\Leftarrow} & \underbrace{0\ldots0}_{vq}1\underbrace{\beta1\beta1\ldots\beta1}_{v} \\ \Leftarrow_1 & \underbrace{0\ldots0}_{(v-1)q}1\underbrace{\beta1\beta1\ldots\beta1}_{v-1} & \Leftarrow_1 & \ldots \Leftarrow_1 1. \end{array}$$

- Now suppose that there is no rule of the form $0\alpha1 \Leftarrow_1 1$ in $\psi(\Sigma)$. Then we conclude from conditions 4 and 5 that $\psi(\Sigma)$ contains rules of the form $1\beta1 \Leftarrow_1 1$ and $0\alpha0 \Leftarrow_1 1$ (including $0 \Leftarrow_1 1$) with $\alpha, \beta \in \mathscr{S}$. If α contains at least one 1, i.e. $0\alpha0 = \underbrace{0\ldots0}_{q\geq1}1\gamma0$, let vq be the least multiple of q which is

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greater or equal to k_0. Then,

$$\begin{array}{llll}
\epsilon & \Leftarrow_1 \underbrace{0\ldots0}_{d\geq1}1 & \overset{\text{P2}}{\Leftarrow} & 0\ldots01\underbrace{\gamma0\ldots\gamma0}_{v}\beta1 \\
& \overset{\text{P1}}{\Leftarrow} \underbrace{0\ldots0}_{t\geq vq}1\underbrace{\gamma0\ldots\gamma0}_{v}\beta1 & \overset{\text{P3}}{\Leftarrow} & \underbrace{0\ldots0}_{vq}1\underbrace{\gamma0\ldots\gamma0}_{v}\beta1 \\
& \Leftarrow_1 \underbrace{0\ldots0}_{(v-1)q}1\underbrace{\gamma0\ldots\gamma0}_{v-1}\beta1 & \Leftarrow_1 & \ldots \Leftarrow_1 1\beta1 \Leftarrow_1 1.
\end{array}$$

- Finally, consider $1\beta1 \Leftarrow_1 1$ and $\underbrace{0\ldots0}_{n\geq1} \Leftarrow_1 1$ in $\psi(\Sigma)$. Let $q = (\sharp_1(\beta)+1)\cdot n + \sharp_0(\beta)$. Applying $\underbrace{0\ldots0}_{n\geq1} \Leftarrow_1 1$ in the right positions $\sharp_1(\beta)+1$ times, one has

$$\underbrace{0\ldots0}_{q}1 \Leftarrow_1 1\underbrace{0\ldots0}_{q-n}1 \Leftarrow_1 \ldots \Leftarrow_1 1\beta1 \Leftarrow_1 1.$$

Hence, we proceed as in the first case.

$\boxed{\forall\alpha \in \{0,1\}^+ \ \Sigma : \alpha \Leftarrow 1}$

This part is identical to the proof of Theorem 3.7, but uses P3 instead of $\exists k_0 \geq 0 : k \geq k_0$ implies $\underbrace{0\ldots0}_{k}0 \ \Leftarrow \ \underbrace{0\ldots0}_{k}\underbrace{1\ldots1}_{\sum a_i}$. $\square$

References

Abdali, S. K. (1976) An abstraction algorithm for combinatory logic. *J. Symbolic Logic*, **41**, 222–224.

Anderson, A. R. and Belnap, N. D. (1975) *Entailment. The Logic of Relevance and Necessity, Vol. 1.* Princeton University Press.

Barendregt, H. (1981) *The Lambda Calculus – its syntax and semantics.* North-Holland, Amsterdam.

Curry, H. B. (1929) An analysis of logical substitution. *Am.J.Math.*, **51**, 363–384.

Curry, H. B. and Feys, R. (1958) *Combinatory Logic, vol. 1.* North-Holland.

Fitch, F. B. (1936) A system of formal logic without an analogue to the Curry W operator. *J. Symbolic Logic*, **1**, 92–100.

Guessarian, I. (1981) *Algebraic Semantics. Lecture Notes in Computer Science 99.* Springer-Verlag.

Hindley, J. R. (1989) BCK-combinators and linear λ-terms have types. *Theoretical Computer Science*, **64**, 97–105.

Meredith, C. A. and Prior, A. N. (1963) Notes on the axiomatics of the propositional calculus. *Notre Dame J. Formal Logic*, **4**, 171–187.

Schönfinkel, M. (1924) Über die Bausteine der mathematischen Logik. *Mathematische Annalen*, **92**, 305–316.

Stanley, R. P. (1983) *Combinatorics and Commutative Algebra.* Vol. 41. Birkhäuser.

Statman, R. (1986) On Translating Lambda Terms into Combinators; The Basis Problem. *Proc. First Annual IEEE Symposium on Logic in Computer Science*, pp. 378–382. IEEE.

Trigg, P., Hindley, J. R. and Bunder, M. W. (1994) Combinatory abstraction using B, B' and friends. *Theoretical Computer Science*, **135**, 405–422.

Turner, D. A. (1979) Another algorithm for bracket abstraction. *J. Symbolic Logic*, **44**, 267–270.