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Measures of maximal entropy of bounded density shifts

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Abstract. We find sufficient conditions for bounded density shifts to have a unique measure of maximal entropy. We also prove that every measure of maximal entropy of a bounded density shift is fully supported. As a consequence of this, we obtain that bounded density shifts are surjunctive.

Key words: bounded density shifts, intrinsic ergodicity, measure of maximal entropy, entropy minimality
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1. *Introduction*

The concept of entropy is of particular interest when trying to define formally how a system behaves at equilibrium. Given a dynamical system, we say that an invariant measure is a uniform equilibrium state if it achieves the maximal possible entropy. It has been of interest to physicists and mathematicians to determine whether a system has a unique equilibrium state or not. When this happens, some mathematicians say the system is intrinsically ergodic and some physicists say the system does not have phase transition.

In this paper, we are interested in trying to determine if bounded density shifts are intrinsically ergodic. Bounded density shifts were introduced by Stanley in [22]. These subshifts are defined somewhat similarly to the classical β-shifts in that they both are hereditary [14], meaning that membership in the shift is preserved under coordinatewise reduction of letters. Whereas β-shifts are 'bounded from above' by a specific sequence coming from a β-expansion, bounded density shifts are restricted by length-dependent bounds on the sums of letters in subwords.

Stanley proved characterizations of when bounded density shifts are shifts of finite type, sofic, or specified, which are remarkably similar to those proved in [21] for β-shifts.

A very effective way of proving that a transitive β-shift is intrinsically ergodic is using the Climenhaga–Thompson decomposition [9] (see §2.3), which uses specification of a sub-language. Using this powerful result, one can prove that β-shifts (and their factors) are intrinsically ergodic in a few lines (see [9, §3.1]).

Proving that bounded density shifts are intrinsically ergodic seems more mysterious. In this paper, we also use the Climenhaga–Thompson theorem to prove a fairly general sufficient condition (Theorem 3.5). The main application of the result is the following.

THEOREM 1.1. *Let $X_f \subset \{0, \ldots, m\}^{\mathbb{Z}}$ be a bounded density shift and α_f its limiting gradient. If $\alpha_f > \sum_{i=1}^{m} i/(i+1)$, then X_f is intrinsically ergodic.*

It is not difficult to find examples with this property. For a binary shift, all we need is $\alpha_f > 1/2$. Furthermore, we do not know if any bounded density subshift fails to satisfy the conditions of Theorem 3.5 (Question 3.7). We conjecture that the answer of this question is positive at least for binary subshifts and that every bounded density shift is intrinsically ergodic.

This is not the first paper to study intrinsic ergodicity of bounded density shifts. This has been done in [8, 19]. Our hypotheses are also much simpler and provide proofs of intrinsic ergodicity for new classes of bounded density shifts.

Furthermore, we prove that every measure of maximal entropy of a bounded density shift (with positive entropy) is fully supported. This property is sometimes known as entropy minimality because it is equivalent to having lower topological entropy on every proper subshift. As a consequence of this, we prove that synchronized bounded density shifts are always intrinsically ergodic and we also obtain surjunctivity of bounded density shifts. In the last section of the paper, we prove that these shifts possess universality properties.

2. *Definitions and preliminary results*

2.1. *Subshifts.* We devote this section to collect some basic definitions in symbolic dynamics. For a broader introduction to subshifts, languages, and their properties, see [15]. Let $\mathcal{A}$ be a finite set of symbols. We say that w is a *word* if there exists $n \in \mathbb{N}$ such that $w \in \mathcal{A}^n$ and we denote the *length* of w by $|w|$. Let ε denote the empty word, that is, the word with no symbols.

A word u is a *subword* of w if $u = w_k w_{k+1} \cdots w_l$ for some $1 \leq k \leq l \leq |w|$. For words $w^{(1)}, \ldots, w^{(n)}$, we use $w^{(1)} \cdots w^{(n)}$ to represent their concatenation. We say that a word u is a *prefix* of w if $u = w_1 \cdots w_k$ for some $1 \leq k \leq |w|$ and a suffix if $u = w_k \cdots w_{|w|}$ for some $1 \leq k \leq |w|$, denote by $\mathrm{Suf}(w)$ and $\mathrm{Pre}(w)$ the sets of non-empty suffixes and prefixes, respectively, for w.

We endow $\mathcal{A}^{\mathbb{Z}}$ with the product topology. When describing a point $x \in \mathcal{A}^{\mathbb{Z}}$ as a sequence, we use a dot to indicate the central position as follows: $x = \cdots x_{-1}.x_0 x_1 \cdots$, where x_i to represent the ith coordinate of x. We represent intervals of integers with $[i, j]$, and $x_{[i,j]} = x_i x_{i+1} \cdots x_j$.

The shift map $\sigma : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is defined by $\sigma(x) = \cdots x_{-1} x_0 . x_1 x_2 \cdots$. We say that a set $X \subseteq \mathcal{A}^{\mathbb{Z}}$ is a *subshift* if it is closed and invariant under σ.

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For any subshift X, let

$$\mathcal{L}_n(X) = \{w \in \mathcal{A}^n : \text{ there exists } x \in X \text{ and } i, j \in \mathbb{Z} \text{ such that } x_{[i,j]} = w\}.$$

We define $\mathcal{L}(X) = \bigcup_{i=0}^{\infty} \mathcal{L}_n(X)$ as the *language* of the subshift X. Given a word w and $k \in \mathbb{Z}$, we define its *cylinder set* as $[w]_k = \{x \in X : x_{[k,k+|w|-1]} = w\}$. The cylinder sets form a basis of the topology of $\mathcal{A}^{\mathbb{Z}}$.

2.2. *Specification properties.* A subshift X is *specified* if there exists $M \in \mathbb{N}$ such that for all $u, w \in \mathcal{L}(X)$, there is a $v \in \mathcal{L}_M(X)$ such that $uvw \in \mathcal{L}(X)$. Following [9], we also define specification for subsets of the language.

Let X be a subshift, $\mathcal{G} \subset \mathcal{L}(X)$, and $t \in \mathbb{N}_0$. We say that $\mathcal{G}$ has *specification (with gap size t)* if for all $m \in \mathbb{N}$ and $w^{(1)}, \ldots, w^{(m)} \in \mathcal{G}$, there exists $v^{(1)}, \ldots, v^{(m-1)} \in \mathcal{L}_t(X)$ such that

$$w = w^{(1)}v^{(1)}w^{(2)}v^{(2)} \cdots v^{(m-1)}w^{(m)} \in \mathcal{L}(X).$$

Moreover, if the cylinder $[w]_0$ contains a periodic point of period exactly $|w| + t$, then we say that $\mathcal{G}$ has *periodic specification.*

2.3. *Measures of maximal entropy.* For any subshift X, we denote by $M(X)$ the set of Borel probability measures on X. Equipped with the weak* topology, $M(X)$ is a compact topological space.

For any $\mu \in M(X)$ and any finite measurable partition ξ of X, the *entropy of* ξ (with respect to μ), denoted by $H_\mu(\xi)$, is defined by

$$H_\mu(\xi) = -\sum_{A \in \xi} \mu(A) \log \mu(A),$$

where terms with $\mu(A) = 0$ are omitted.

Given a subshift X, we denote the σ-invariant Borel probability measures with $M(X, \sigma)$. For $\mu \in M(X, \sigma)$, the *entropy of* μ (for the shift map σ) is defined by

$$h_\mu(X) = \lim_{n \to \infty} \frac{-1}{n} \sum_{w \in \mathcal{L}_n(X)} \mu([w]_0) \log \mu([w]_0) = \lim_{n \to \infty} \frac{-1}{n} H_\mu(\xi^{(n)}), \tag{1}$$

where $\xi^{(n)}$ represents the partition of X into cylinder sets from the first n letters, that is, $\xi^{(n)} = \{[w]_0 \ : \ w \in \mathcal{A}^n\}$.

We note for future reference that $\xi^{(n)} = \bigvee_{i=0}^{n-1} \sigma^{-i}\xi^{(1)}$, where $\xi^{(1)}$ is the partition based on x_0 and $\vee$ is the join of partitions. We will later need to make use of the following basic facts about entropy; for proofs and general introduction to entropy theory, see [24].

THEOREM 2.1. [24, Theorem 4.3] *For any subshift X, $\mu \in M(X)$, and ξ, η finite partitions of X, $H_\mu(\xi \vee \eta) \leq H_\mu(\xi) + H_\mu(\eta)$.*

THEOREM 2.2. [24, Corollary 4.2.1] *For any subshift X and $\mu \in M(X)$, if ξ is a finite measurable partition of X with k sets, then $H_\mu(\xi) \leq \log(k)$, with equality only when $\mu(A) = k^{-1}$ for all $A \in \xi$.*

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THEOREM 2.3. [24, p. 184] *For any subshift X, finite measurable partition* ξ *of X, measures* $\mu_i \in M(X)$*, and* $p_i \geq 0$ $(1 \leq i \leq n)$ *with* $\sum_{i=1}^{n} p_i = 1$*,* $H_{\sum_{i=1}^{n} p_i\mu_i}(\xi) \geq \sum_{i=1}^{n} p_i H_{\mu_i}(\xi)$.

By the well-known variational principle, the supremum of $h_\mu(X)$ over all $\mu \in M(X, \sigma)$ is the *topological entropy* $h_{\text{top}}(X)$ of X. For any subshift X, we have that

$$h_{\text{top}}(X) = \lim_{n\to\infty} \frac{1}{n} \log |\mathcal{L}_n(X)|. \tag{2}$$

For general topological dynamical systems (TDSs), the supremum above may not be achieved. However, every subshift has at least one measure of maximal entropy, that is, $\nu \in M(X, \sigma)$ achieving the supremum above, meaning that $h_\nu(X) = h_{\text{top}}(X)$ (e.g. see [24, Remark (2), p. 192]).

We say a subshift is *intrinsically ergodic* if there is only one (probability) measure of maximal entropy.

Every specified subshift is intrinsically ergodic [1]. This result was generalized in several works, including [9, 18]. Before stating the result, we need some extra definitions.

Given a collection of words $\mathcal{D} \subseteq \mathcal{L}(X)$ and $n \geq 1$, we define $\mathcal{D}_n = \mathcal{D} \cap \mathcal{L}_n(X)$. We denote the *growth rate* of $\mathcal{D}$ by

$$h(\mathcal{D}) = \limsup_{n\to\infty} \frac{1}{n} \log |\mathcal{D}_n|. \tag{3}$$

Note that $h(\mathcal{L}(X)) = h_{\text{top}}(X)$.

Following [9], we say that $\mathcal{L}(X)$ *admits a decomposition* $\mathcal{C}^p\mathcal{G}\mathcal{C}^s$ for $\mathcal{C}^p, \mathcal{G}, \mathcal{C}^s \subset \mathcal{L}(X)$ if every $w \in \mathcal{L}(X)$ can be written as uvw for some $u \in \mathcal{C}^p$, $v \in \mathcal{G}$, $w \in \mathcal{C}^s$. For such a decomposition, we define the collection of words $\mathcal{G}(M)$ for each $M \in \mathbb{N}$ by

$$\mathcal{G}(M) = \{uvw \ : \ u \in \mathcal{C}^p, v \in \mathcal{G}, w \in \mathcal{C}^s, |u| \leq M, |w| \leq M\}. \tag{4}$$

Recall that Per(n) denotes the set of points with period at most n under σ.

THEOREM 2.4. (Climenhaga and Thompson [9]) *Let X be a subshift whose language* $\mathcal{L}(X)$ *admits a decomposition* $\mathcal{L}(X) = \mathcal{C}^p\mathcal{G}\mathcal{C}^s$ *and suppose that the following conditions are satisfied:*

(1) $\mathcal{G}$ *has specification;*
(2) $h(\mathcal{C}^p \cup \mathcal{C}^s) < h_{\text{top}}(X)$*;*
(3) *for every* $M \in \mathbb{N}$*, there exists* τ *such that given* $v \in \mathcal{G}(M)$*, there exists words* u, w *with* $|u| \leq \tau$, $|w| \leq \tau$ *for which* $uvw \in \mathcal{G}$.

Then X is intrinsically ergodic. Furthermore, if $\mathcal{G}$ *has periodic specification, then*

$$\mu_n = \frac{1}{|\text{Per}(n)|} \sum_{x\in\text{Per}(n)} \delta_x \tag{5}$$

converges to the measure of maximal entropy in the weak topology.*

Remark. Using results from [17], Climenhaga explained in a blog post [7] that condition (3) is actually not required to prove uniqueness of the measure of maximal entropy.

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However, this condition is not difficult to check for bounded density shifts with positive entropy (Lemma 3.4) and so we verify it regardless.

2.4. *Bounded density shifts.* Bounded density shifts were introduced in [22] (see also [2, Ch. 3.4]). Let $f : \mathbb{N}_0 \to [0, \infty)$ be a function. We say f is *canonical* if:

- $f(0) = 0$;
- $f(m+1) \geq f(m)$ for all $m \geq 0$; and
- $f(m+n) \leq f(m) + f(n)$ for all $n, m \in \mathbb{N}$.

The *bounded density shift* associated to a canonical function, f, is defined as follows:

$$X_f = \left\{ x \in (\mathbb{N}_0)^{\mathbb{Z}} : \text{ for all } p \in \mathbb{N} \text{ and } \text{ for all } i \in \mathbb{Z} \sum_{r=i}^{i+p-1} x_r \leq f(p) \right\}. \tag{6}$$

Note that X_f is a subshift on the alphabet $\mathcal{A} = \{0, 1, \ldots, \lfloor f(1) \rfloor\}$.

Actually, bounded density shifts can be defined for any function $f : \mathbb{N}_0 \to [0, \infty)$, but it was shown in [22] that every bounded density shift can be defined by some canonical f.

Definition. Let X_f be a bounded density shift, the limit

$$\lim_{n \to \infty} \frac{f(n)}{n} \tag{7}$$

is called the *limiting gradient* and is denoted by α_f.

The existence of the limit is given by Fekete's lemma and the definition of canonical function; furthermore, the limit is an infimum and so $f(n) \geq \alpha_f n$ for all n.

There exist bounded density shifts with $\alpha_f = 0$, but they are fairly trivial systems where the upper density of non-zero coordinates is always 0. A bounded density shift has positive topological entropy if and only if $\alpha_f > 0$ (see [14, Theorem 12]) if and only if it is coded (determined by a labeled irreducible graph with possibly countably many vertices) [22, Theorem 3.1].

As we mentioned in the previous section, the specification property guarantees intrinsic ergodicity. For bounded density shifts, X_f is specified with specification constant M if and only if 0^M is intrinsically synchronizing [22, Theorem 5.1]. Bounded density shifts with positive topological entropy without specification can easily be constructed [22].

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A subshift X with alphabet $A = \{0, 1, \ldots, n\}$ is *hereditary* if every time there is $x \in X$ and $y \in A^{\mathbb{Z}}$ with $y_i \leq x_i$ for all $i \in \mathbb{Z}$, then $y \in X$. It is not difficult to check that bounded density shifts are hereditary.

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3. *Intrinsic ergodicity*

In this section, we fix a binary bounded density shift X_f. We define

$$\mathcal{G} = \left\{ w \in \mathcal{L}(X_f) : \text{ if } u \in \text{Pre}(w) \cup \text{Suf}(w), \text{ then } \frac{1}{|u|} \sum_{i=1}^{|u|} u_i < \alpha_f \right\} \quad \text{and}$$

$$\mathcal{B} = \mathcal{C}^p = \mathcal{C}^s = \left\{ v \in \mathcal{L}(X_f) : \frac{1}{|v|} \sum_{i=1}^{|v|} v_i \geq \alpha_f \right\} \cup \{\epsilon\},$$

where ϵ denotes the empty word.

LEMMA 3.1. *The language $\mathcal{L}(X_f)$ admits a decomposition $\mathcal{BGB}$.*

Proof. Let $z \in \mathcal{L}(X_f)$. Define u to be the prefix of z in $\mathcal{B}$ of maximal length (which may be the empty word ϵ) and denote its length by $M \geq 0$. Let z' be the maximal proper subword of z that does not overlap with u, that is, $z' = z_{[M+1,|z|]}$. Similarly, define w to be the suffix of z' in $\mathcal{B}$ of maximal length (which may be the empty word ϵ) and denote its length by $N \geq 0$.

We write $y = z_{[M+1,|z|-N]}$ and assume for a contradiction that $y \notin \mathcal{G}$. Then by definition, there exists a word $v \in \text{Pre}(y) \cup \text{Suf}(y)$ with

$$\frac{1}{|v|} \sum_{i=1}^{|v|} v_i \geq \alpha_f.$$

If $v \in \text{Pre}(y)$, then uv would be a prefix of z in $\mathcal{B}$ longer than u, contradicting minimality of u. Similarly, if $v \in \text{Suf}(y)$, then vw would be a suffix of z' in $\mathcal{B}$ longer than w, contradicting minimality of w. Therefore, we have a contradiction and $y \in \mathcal{G}$, and so $z = uyw \in \mathcal{BGB}$. □

LEMMA 3.2. *The set $\mathcal{G}$ has specification.*

Proof. We will show that $\mathcal{G}$ has periodic specification with gap size $t = 0$. Let $m \in \mathbb{N}$, $w^{(1)}, \ldots, w^{(m)} \in \mathcal{G}$, $v^{(a)} \in \text{Suf}(w^{(m)})$, $v^{(b)} \in \text{Pre}(w^{(1)})$ and $z = v^{(a)} w^{(1)} \cdots w^{(m)} v^{(b)}$. We compute

$$\begin{aligned}
\sum_{i=1}^{|z|} z_i &= \sum_{i=1}^{|v^{(a)}|} v_i^{(a)} + \sum_{i=1}^{|w^{(1)}|} w_i^{(1)} + \cdots + \sum_{i=1}^{|w^{(m)}|} w_i^{(m)} + \sum_{i=1}^{|v^{(b)}|} v_i^{(b)} \\
&< |v^{(b)}|\alpha_f + |w^{(1)}|\alpha_f + \cdots + |w^{(m)}|\alpha_f + |v^{(b)}|\alpha_f \\
&= \alpha_f \left(|v^{(a)}| + \sum_{i=1}^{m} |w^{(i)}| + |v^{(b)}| \right) \\
&= \alpha_f |z| \\
&\leq f(|z|).
\end{aligned}$$

This implies that any periodic point made from concatenations of words from $\mathcal{G}$ is in X_f. We conclude that $\mathcal{G}$ has periodic specification. □

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In the second part of the following proposition, we use techniques from Misiurewicz's proof of the variational principle [16] to build measures with entropy higher or equal than that of a sub-language. These applications of the tools from [16] have already been noted in [4, Proposition 5.1] and [17, Lemma 6.8].

PROPOSITION 3.3. *There exists $\mu \in M(X_f, \sigma)$ with $\sum_{i=0}^{\lfloor f(1)\rfloor} i\mu([i]_0) \geq \alpha_f$ and $h(\mathcal{B}) \leq h_\mu(X_f)$.*

Proof. For each $n \in \mathbb{N}$ and $w \in \mathcal{L}_n(X_f) \cap \mathcal{B}$, consider the set:

$$K_n = \{{}^\infty 0.w0^\infty : w \in \mathcal{L}_n(X_f) \cap \mathcal{B}\}.$$

By construction, $|K_n| = |\mathcal{L}_n(X_f) \cap \mathcal{B}|$. Let $\nu_n \in M(X_f)$ be the atomic measure concentrated uniformly on the points of K_n, that is,

$$\nu_n = \frac{1}{|K_n|} \sum_{x \in K_n} \delta_x.$$

Let $\mu_n \in M(X_f)$ be defined by

$$\mu_n = \frac{1}{n} \sum_{j=0}^{n-1} \nu_n \circ \sigma^{-j}.$$

Note that

$$\begin{aligned}
\sum_{i=0}^{\lfloor f(1)\rfloor} i\mu_n([i]_0) &= \sum_{i=0}^{\lfloor f(1)\rfloor} \frac{i}{n} \sum_{j=0}^{n-1} \nu_n \circ \sigma^{-j}([i]_0) \\
&= \sum_{i=0}^{\lfloor f(1)\rfloor} \frac{i}{n} \sum_{j=1}^{n} \frac{|\{w \in \mathcal{L}_n(X_f) \cap \mathcal{B} : w_i = i\}|}{|K_n|} \\
&= \frac{1}{|K_n|} \sum_{w \in \mathcal{L}_n(X_f) \cap \mathcal{B}} \left(\frac{1}{n} \sum_{j=1}^{n} w_j\right) \\
&\geq \alpha_f.
\end{aligned}$$

Since $M(X_f)$ is compact (in the weak* topology), we can choose a subsequence such that

$$\lim_{j\to\infty} \frac{1}{n_j} \log |\mathcal{L}_{n_j}(X_f) \cap \mathcal{B}| = \limsup_{n\to\infty} \frac{1}{n} \log |\mathcal{L}_n(X_f) \cap \mathcal{B}| = h(\mathcal{B}), \tag{8}$$

and $\mu_{n_j} \to \mu \in M(X_f)$. By the definition of μ_n, it is routine to check that $\mu \in M(X_f, \sigma)$, that is, μ is σ-invariant.

We will use techniques from the proof of the variational principle in [16] to prove that

$$h_\mu(X_f) \geq \limsup_{n\to\infty} \frac{1}{n} \log |\mathcal{L}_n(X_f) \cap \mathcal{B}| = h(\mathcal{B}). \tag{9}$$

First, since $\sum_{i=0}^{\lfloor f(1)\rfloor} i\mu_{n_j}([i]_0) \geq \alpha_f$ and $\mu_{n_j} \to \mu$, we also have that $\sum_{i=0}^{\lfloor f(1)\rfloor} i\mu([i]_0) \geq \alpha_f$. Consider the partition given by the alphabet $\xi = \{[0]_0, \ldots, [\lfloor f(1)\rfloor]_0\}$. Since all

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$w \in \mathcal{L}_{n_j}(X_f) \cap \mathcal{B}$ have equal measure $\nu_{n_j}([w]_0) = |K_{n_j}|^{-1}$ and all other $w \in \mathcal{A}_j^n$ have $\nu_{n_j}([w]_0) = 0$, by Theorem 2.2,

$$H_{\nu_{n_j}}\bigg(\bigvee_{i=0}^{n_j-1} \sigma^{-i}\xi\bigg) = -\sum_{w\in\mathcal{L}_{n_j}(X_f)\cap\mathcal{B}} \nu_{n_j}([w]_0)\log\nu_{n_j}([w]_0) = \log|\mathcal{L}_{n_j}(X_f)\cap\mathcal{B}|. \quad (10)$$

Let $q, n \in \mathbb{N}$ with $1 < q < n$ and define $a(t) = \lfloor (n-t)/q \rfloor$ for $0 \le t < q$. Note that $a(0) \ge a(1) \ge \cdots \ge a(q-1)$. For every $0 \le t \le q-1$, we define

$$S_t = \{0, 1, \ldots, t-1, t + a(t)q, t + a(t)q + 1, \ldots, n-1\}.$$

So, for any such t, we can rewrite $\{0, 1, \ldots, n-1\}$ as follows:

$$\{0, 1, \ldots, n-1\} = \{t + rq + i | 0 \le r < a(t), 0 \le i < q\} \cup S_t. \quad (11)$$

Observe that

$$t + a(t)q = t + \bigg\lfloor \frac{n-t}{q} \bigg\rfloor q \ge t + \bigg(\frac{n-t}{q} - 1\bigg)q = t + n - t - q = n - q.$$

Thus, the cardinality of S_t is at most $2q$.

Using equation (11), we get

$$\bigvee_{i=0}^{n_j-1} \sigma^{-i}\xi = \bigg(\bigvee_{r=0}^{a(t)-1} \sigma^{-(rq+t)} \bigvee_{i=0}^{q-1} \sigma^{-i}\xi\bigg) \vee \bigvee_{l\in S_t} \sigma^{-l}\xi. \quad (12)$$

Combining equations (10), (12), and Theorem 2.1, we obtain

$$\begin{aligned}\log|\mathcal{L}_{n_j}(X_f)\cap\mathcal{B}| &= H_{\nu_{n_j}}\bigg(\bigvee_{i=0}^{n_j-1} \sigma^{-i}\xi\bigg) \\ &\le \sum_{r=0}^{a(t)-1} H_{\nu_{n_j}}\bigg(\sigma^{-(rq+t)} \bigvee_{i=0}^{q-1} \sigma^{-i}\xi\bigg) + \sum_{l\in S_t} H_{\nu_{n_j}}(\sigma^{-l}\xi) \\ &\le \sum_{r=0}^{a(t)-1} H_{\nu_{n_j}\circ\sigma^{-(rq+t)}}\bigg(\bigvee_{i=0}^{q-1} \sigma^{-i}\xi\bigg) + 2q\log(l). \quad (13)\end{aligned}$$

For the inequality $\sum_{l\in S_t} H_{\nu_{n_j}}(\sigma^{-l}\xi) \le 2q\log(l)$, we apply Theorem 2.2. We note that for each $0 \le t \le q-1$, we have

$$(a(t)-1)q + t \le \bigg\lfloor \frac{n-t}{q} - 1 \bigg\rfloor q + t = n - q. \quad (14)$$

Summing the first term in the last line of equation (13) over t from 0 to $q-1$, and using that the numbers $\{t + rq : 0 \le t \le q-1, 0 \le r \le a(t)-1\}$ are all distinct and are all no greater than $n-q$, yields

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$$\sum_{t=0}^{q-1}\bigg(\sum_{r=0}^{a(t)-1} H_{\nu_{n_j}\circ\sigma^{-(rq+t)}}\bigg(\bigvee_{i=0}^{q-1}\sigma^{-i}\xi\bigg)\bigg) = \sum_{r=0}^{a(0)-1} H_{\nu_{n_j}\circ\sigma^{-(rq)}}\bigg(\bigvee_{i=0}^{q-1}\sigma^{-i}\xi\bigg) + \cdots$$
$$\cdots + \sum_{r=0}^{a(q-1)-1} H_{\nu_{n_j}\circ\sigma^{-(rq+q-1)}}\bigg(\bigvee_{i=0}^{q-1}\sigma^{-i}\xi\bigg)$$
$$= \sum_{p=0}^{n_j-1} H_{\nu_{n_j}\circ\sigma^{-p}}\bigg(\bigvee_{i=0}^{q-1}\sigma^{-i}\xi\bigg). \tag{15}$$

Using equations (13) and (15), we get

$$q\,\log|\mathcal{L}_{n_j}(X_f)\cap\mathcal{B}| \le \sum_{p=0}^{n_j-1} H_{\nu_{n_j}\circ\sigma^{-p}}\bigg(\bigvee_{i=0}^{q-1}\sigma^{-i}\xi\bigg) + \frac{2q^2}{n_j}\log(l).$$

Now, we divide by n_j and apply Theorem 2.3 (with $p_i = 1/n_j$) to obtain

$$\frac{q}{n_j}\log|\mathcal{L}_{n_j}(X_f)\cap\mathcal{B}| \le H_{\mu_{n_j}}\bigg(\bigvee_{i=0}^{q-1}\sigma^{-i}\xi\bigg) + \frac{2q^2}{n_j^2}\log(l). \tag{16}$$

We will also use that

$$\lim_{k\to\infty} H_{\mu_{n_{j_k}}}\bigg(\bigvee_{i=0}^{q-1}\sigma^{-i}\xi\bigg) = H_{\mu}\bigg(\bigvee_{i=0}^{q-1}\sigma^{-i}\xi\bigg), \tag{17}$$

which is obtained using the definition of weak* convergence. Then, combining equations (16) and (17) yields

$$\begin{aligned} qh(\mathcal{B}) &= \lim_{k\to\infty}\frac{q}{n_{j_k}}\log|\mathcal{L}_{n_{j_k}}(X_f)\cap\mathcal{B}| \\ &\le \lim_{k\to\infty} H_{\mu_{n_{j_k}}}\bigg(\bigvee_{i=0}^{q-1}\sigma^{-i}\xi\bigg) + \lim_{k\to\infty}\frac{2q^2}{n_{j_k}}\log(l) \\ &= H_{\mu}\bigg(\bigvee_{i=0}^{q-1}\sigma^{-i}\xi\bigg). \end{aligned}$$

Now, by definition of $h_\mu(X_f)$,

$$h(B) \le \lim_{q\to\infty}\frac{1}{q}H_\mu\bigg(\bigvee_{i=0}^{q-1}\sigma^{-i}\xi\bigg) = h_\mu(X_f).$$ □

LEMMA 3.4. *For every $M\in\mathbb{N}$, there exists τ such that given $v\in\mathcal{G}(M)$, there exist words u, w with $|u|\le\tau$, $|w|\le\tau$ for which $uvw\in\mathcal{G}$.*

Proof. Let $M\in\mathbb{N}$ and $v\in\mathcal{G}(M)$. This implies that there exist $u', w'\in\mathcal{B}$, $v'\in\mathcal{G}$ such that $v = u'v'w'$ and $|u'|\le M$, $|w'|\le M$. Choose $u = w = 0^\tau$, with $\tau = \lceil 2M\lfloor f(1)\rfloor/\alpha_f\rceil$.

Let $z\in \mathrm{Pre}(0^\tau u'v'w'0^\tau)$. Consider the following sets, $N_1 = [1,\tau]$, $N_2 = [\tau+1, \tau+|u'|]\cup[\tau+|u'v'|+1, \tau+|u'v'w'|]$, and $N_3 = [\tau+|u'|+1, \tau+|u'v'|]$. Note that N_2

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corresponds to the section where u' and w' appear and N_3 where v' appears. Also, we can assume that $|z| \geq \tau$ (otherwise we are considering that $z \in \mathrm{Pre}(0^\tau)$), then

$$\begin{aligned}
\frac{1}{|z|}\sum_{i=1}^{|z|} z_i &= \frac{1}{|z|}\Bigg(\sum_{i\in N_1\cap[1,|z|]} z_i + \sum_{i\in N_2\cap[1,|z|]} z_i + \sum_{i\in N_3\cap[1,|z|]} z_i\Bigg)\\
&= \frac{1}{|z|}\Bigg(\frac{|N_1\cap[1,|z|]|}{|N_1\cap[1,|z|]|}\sum_{i\in N_2\cap[1,|z|]} z_i \;+\; \frac{|N_3\cap[1,|z|]|}{|N_3\cap[1,|z|]|}\sum_{i\in N_3\cap[1,|z|]} z_i\Bigg)\\
&\leq \frac{1}{|z|}\Bigg(\frac{|N_1\cap[1,|z|]|}{|N_1\cap[1,|z|]|}2M\lfloor f(1)\rfloor + \alpha_f|N_3\cap[1,|z|]|\Bigg)\\
&= \frac{1}{|z|}\Bigg(|N_1\cap[1,|z|]|\frac{2M\lfloor f(1)\rfloor}{\tau} + \alpha_f|N_3\cap[1,|z|]|\Bigg)\\
&\leq \frac{1}{|z|}(\alpha_f|N_1\cap[1,|z|]| + \alpha_f|N_3\cap[1,|z|]|)\\
&= \alpha_f\Bigg(\frac{|N_1\cap[1,|z|]| + |N_3\cap[1,|z|]|}{|z|}\Bigg)\\
&\leq \alpha_f.
\end{aligned}$$

Here, the first inequality holds since $v' \in \mathcal{G}$, the second equality holds because $|N_1 \cap [1, |z|]| = \tau$ (using $|z| \geq \tau$), and the second inequality holds since $\tau \geq 2M\lfloor f(1)\rfloor/\alpha_f$.

The proof for $z \in \mathrm{Suf}(0^\tau u'v'w'0^\tau)$ is similar. □

THEOREM 3.5. *Let X_f be a bounded density shift. If every measure of maximal entropy μ has the property that $\sum_i^{\lfloor f(1)\rfloor} i\mu([i]_0) < \alpha_f$, then X_f is intrinsically ergodic and*

$$\mu_n = \frac{1}{|\mathrm{Per}(n)|}\sum_{x\in\mathrm{Per}(n)} \delta_x \tag{18}$$

converges to the measure of maximal entropy in the weak topology.*

Proof. If $\alpha_f = 0$, then since all sequences have frequency 0 of non-0 symbols, the unique invariant measure is the delta measure of ${}^\infty 0^\infty$.

If $\alpha_f > 0$, we will obtain the result using Theorem 2.4. First note that $\mathcal{B} = \mathcal{C}^p = \mathcal{C}^s$. Using Lemma 3.1, we obtain $\mathcal{L}(X) = \mathcal{C}^p\mathcal{G}\mathcal{C}^s$. Now we will check the numbered hypotheses of Theorem 2.4.

(1) Lemma 3.2 gives us that $\mathcal{G}$ has specification.

(2) Let μ' be the measure constructed in Lemma 3.3. By hypothesis, it cannot be a measure of maximal entropy. Thus, $h(\mathcal{C}^p \cup \mathcal{C}^s) = h(\mathcal{B}) \leq h_{\mu'}(X_f) < h_{\mathrm{top}}(X_f)$.

(3) We obtain this property using Lemma 3.4. □

The main application of the previous result that we have is the following.

COROLLARY 3.6. *Let X_f be a bounded density shift. If $\alpha_f > \sum_{i=1}^{\lfloor f(1)\rfloor}(i/(i+1))$, then $\sum_i^{\lfloor f(1)\rfloor} i\mu([i]_0) < \alpha_f$ for every measure of maximal entropy μ. This implies that X_f is*

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intrinsically ergodic and

$$\mu_n = \frac{1}{|\mathrm{Per}(n)|} \sum_{x \in \mathrm{Per}(n)} \delta_x \tag{19}$$

converges to the measure of maximal entropy in the weak topology.*

Proof. Using [**12**, Corollary 4.6] and the fact that bounded density shifts are hereditary, we have that for any measure of maximal entropy,

$$\mu([i]_0) \leq \mu([i-1]_0).$$

Since μ is a probability measure, this implies that $\mu([i]_0) \leq 1/(i+1)$. Thus,

$$\sum_{i=1}^{\lfloor f(1) \rfloor} i \cdot \mu([i]_0) \leq \sum_{i=1}^{\lfloor f(1) \rfloor} i/i + 1.$$

We obtain the result using Theorem 3.5. □

Remark. In particular, every binary bounded density shift with $\alpha_f > 1/2$ is intrinsically ergodic.

Furthermore, we suspect that the hypothesis of Theorem 3.5 may always be satisfied, at least for binary subshifts, leading to the following questions.

Question 3.7. Let X be a hereditary binary subshift with positive topological entropy. Is it true that for any measure of maximal entropy μ, we have that $\mu([1]_0) < \sup_{\nu \in M(X)} \nu([1]_0)$?

A reason to suspect Question 3.7 is true is that if X is hereditary and $\mu([1]_0)$ achieves its (positive) supremum, then it should be possible to increase the entropy of μ by allowing a small proportion of randomly chosen 1 symbols to change to 0s. Some circumstantial evidence is given by the class of $\mathcal{B}$-free shifts, for which it is known that maximal entropy is achieved by such a procedure (cf. [**13**, Theorem 2.1.8]). We also ask the corresponding question for bounded density shifts on larger alphabets.

Question 3.8. Is it true that for every bounded density shift, we have that

$$\sum_{i}^{\lfloor f(1) \rfloor} i\mu([i]_0) < \alpha_f$$

for every measure of maximal entropy?

One more natural question is whether we can prove stronger properties on the unique measure of maximal entropy via arguments such as those in [**4**, **19**].

Question 3.9. Let X_f be an intrinsically ergodic bounded density shift. Does the measure of maximal entropy have the K-property? Is it Bernoulli?

We do not know how to approach this question with current techniques. All arguments we are aware of which prove Bernoulli require connection to countable-state Markov shifts,

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which do not seem clear for bounded density shifts. Additionally, the usual argument to prove K-property (without Bernoulli) is to show that the product of (X_f, σ) with itself has a unique measure of maximal entropy, but in general, Climenhaga–Thompson decompositions are not preserved under products and we do not see any reason that bounded density structure improves the situation. We note that purely being hereditary does not necessarily imply either property, as in [13], it was shown that for $\mathcal{B}$-free shifts, the unique measure of maximal entropy factors onto the so-called Mirsky measure, which is of zero entropy; this precludes the K-property.

4. *Entropy minimality and surjunctivity*

We will now prove a property called entropy minimality for all bounded density shifts for $\alpha_f > 0$ using results from [12]. We first need some definitions.

A subshift X is *entropy minimal* if every subshift strictly contained in X has lower topological entropy. Equivalently, X is entropy minimal if every measure of maximal entropy on X is fully supported.

Let X be a subshift and $v \in \mathcal{L}(X)$. The *extender set* of v is defined by

$$E_{X_f}(v) = \{y \in \{0, 1, \ldots, \lfloor f(1) \rfloor\}^{\mathbb{Z}} : y_{(-\infty,0]} v y_{[1,\infty)} \in X_f\}.$$

THEOREM 4.1. (García-Ramos and Pavlov [12]) *Let X be a subshift with $h_{\mathrm{top}}(X) > 0$, μ a measure of maximal entropy, and $v, w \in \mathcal{L}(X)$. If $E_X(v) \subseteq E_X(w)$, then*

$$\mu(v) \leq \mu(w) e^{h_{\mathrm{top}}(X)(|w|-|v|)}.$$

THEOREM 4.2. *Every bounded density shift (with $\alpha_f > 0$) is entropy minimal.*

Proof. Let X_f be a bounded density shift, $\mu \in M(X_f, \sigma)$ a measure of maximal entropy, and $w \in \mathcal{L}(X_f)$. Since the topological entropy of X_f is positive, then $1 \in \mathcal{L}(X_f)$ and $\mu([1]_0) > 0$ (otherwise $\mu([0]_0) = 1$ and the entropy cannot be positive). By Poincaré's recurrence theorem, there exists $v' \in \mathcal{L}(X_f)$ for which $\mu([v']_0) > 0$ and

$$\sum_{i=1}^{|v'|} v'_i > \sum_{i=1}^{|w|} w_i.$$

We can then define v which is coordinatewise less than or equal to w with

$$\sum_{i=1}^{|v|} v_i = \sum_{i=1}^{|w|} w_i.$$

By the fact that X_f is hereditary, $E_{X_f}(v') \subset E_{X_f}(v)$ and so by Theorem 4.1, $\mu([v]) \geq \mu([v']) > 0$.

We want to prove that $E_{X_f}(v) \subseteq E_{X_f}(0^{|v|} w 0^{|v|})$. Let $y \in E_{X_f}(v)$, with $x = y_{(-\infty,0]}.v y_{[1,\infty)} \in X_f$ and $x' = y_{(-\infty,0]}.0^{|v|} w 0^{|v|} y_{[1,\infty)}$. Let $n < m \in \mathbb{Z}$. We consider two cases, when $x'_{[n,m]}$ is a subword of $0^{|v|} w 0^{|v|}$ and when it is not. If $x'_{[n,m]}$ is a subword of $0^{|v|} w 0^{|v|}$, then $x'_{[n,m]} \in \mathcal{L}(X_f)$ since $w \in \mathcal{L}(X_f)$ [22, Lemma 2.3]. Otherwise, there

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exists $p \in \mathbb{Z}$ such that

$$\sum_{i=n}^{m} x'_i \leq \sum_{i=n+p}^{m+p} x_i \leq f(m-n).$$

This implies that $x'_{[n,m]} \in \mathcal{L}(X_f)$. Thus, $x' \in X_f$ and so $y \in E_{X_f}(0^{|v|}w0^{|v|})$. Since y was arbitrary, $E_{X_f}(v) \subseteq E_{X_f}(0^{|v|}w0^{|v|})$. Using Theorem 4.1, we conclude that

$$\mu([w]_0) \geq \mu([0^{|v|}w0^{|v|}]_0) \geq \mu([v]_0)e^{-h_{\text{top}}(X)(|w|-|v|)} > 0.$$

Therefore, μ is fully supported. □

Let X be a subshift. A word $v \in \mathcal{L}(X)$ is *intrinsically synchronizing* if $uv, vw \in \mathcal{L}(X)$, then $uvw \in \mathcal{L}(X)$.

A subshift is *synchronized* if there exists $v \in \mathcal{L}(X)$ such that v is an intrinsically synchronizing word.

Every entropy minimal synchronized subshift is intrinsically ergodic [12, 23] and every synchronized subshift is coded [11]. Hence, we obtain the following corollary.

COROLLARY 4.3. *Every synchronized bounded density shift is intrinsically ergodic.*

Another application of entropy minimality is surjunctivity. Given a subshift X, we say $\phi : X \to X$ is a *shift-endomorphism* if it is continuous and it commutes with the shift. If a shift-endomorphism is bijective, we say it is a *shift-automorphism*.

A subshift X is said to be *surjunctive* if every injective shift-endomorphism of X is a shift-automorphism. Every full shift is surjunctive [10, Ch. 3]. The following result is known (e.g. see [5]) but it is not explicitly stated. We write the proof since the argument is simple.

LEMMA 4.4. *Every entropy minimal subshift is surjunctive.*

Proof. Let X be a subshift and $\phi : X \to X$ an injective shift-endomorphism. This implies that $\phi(X)$ is a subshift which is topologically conjugate to X. Since topological entropy is conjugacy-invariant, $\phi(X)$ has the same topological entropy as X. If X is entropy minimal, then $\phi(X) = X$. □

Using this and Theorem 4.2, we obtain the following.

COROLLARY 4.5. *Every bounded density shift with positive topological entropy is surjunctive.*

5. *Universality*

A dynamical system is said to be universal if every system with smaller entropy can be embedded in the original system (this can be studied either in the topological or measure-theoretic category). For instance, measure-theoretic universality of the full shift follows from Krieger's generator theorem. Results about both types (topological and measure-theoretical) of universality have been proved for systems with specification in

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[3, 6, 20], and we can prove a topological universality result for bounded density subshifts as well. We first need some basic definitions about topological dynamical systems.

A *topological dynamical system* is a pair (X, T), where X is a compact metrizable space and $T : X \to X$ is a continuous function. Let (X, T) and (X', T') be two topological dynamical systems. We say X and X' are *conjugated* if there exists a homeomorphism $f : X \to X'$ such that $T' \circ f = f \circ T$.

For any TDS (X, T), one can assign a *topological entropy* $h_{\text{top}}(X, T)$. When the system is a subshift, the notion coincides with the definition in §2.3. For the definition, see [24, Ch. 7].

Let $\gamma \in \mathbb{R}_+$. We say a subshift X is γ*-universal* if for any TDS with $h_{\text{top}}(X_1, T_1) < \gamma$, there is a subshift $X' \subset X$ such that (X_1, T_1) is conjugated to (X', σ).

THEOREM 5.1. (Burguet [3]) *Every subshift X with specification is $h_{\text{top}}(X)$-universal.*

Let $\alpha \in \mathbb{R}_+$. We define X_α as the bounded density shift obtained with the function $f(n) = \lfloor n\alpha \rfloor$. Using [22, Theorem 1.3], we have that X_α has specification.

Given a bounded density shift X_f, one can check that $X_{\alpha_f} \subset X_f$. Let $x \in X_{\alpha_f}$, then for every $i \in \mathbb{Z}$ and for every $p \in \mathbb{N}$, we have

$$\sum_{r=i}^{i+p-1} x_r \leq \lfloor p\alpha_f \rfloor \leq p\alpha_f \leq p\frac{f(p)}{p} \leq f(p).$$

Therefore, $x \in X_f$ and $X_{\alpha_f} \subset X_f$.

COROLLARY 5.2. *Let X_f be a bounded density shift. We have that X_f is $h_{\text{top}}(X_{\alpha_f})$-universal.*

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