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ON EXCEPTIONAL SETS: THE SOLUTION OF A PROBLEM POSED BY K. MAHLER

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Abstract

In this paper, we shall prove that any subset of $\overline{\mathbb{Q}}$, which is closed under complex conjugation, is the exceptional set of uncountably many transcendental entire functions with rational coefficients. This solves an old question proposed by Mahler [*Lectures on Transcendental Numbers*, Lecture Notes in Mathematics, 546 (Springer, Berlin, 1976)].

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1. Introduction

A *transcendental function* is a function $f(x)$ such that the only complex polynomial P satisfying $P(x, f(x)) = 0$, for all x in its domain, is the null polynomial. For instance, the trigonometric functions, the exponential function and their inverses are transcendental functions.

Weierstrass initiated the investigation of the set of algebraic numbers where a given transcendental entire function f takes algebraic values. Denote by $\overline{\mathbb{Q}}$ the field of algebraic numbers. For an entire function f, we define the exceptional set S_f of f as

$$S_f = \{\alpha \in \overline{\mathbb{Q}} \ : \ f(\alpha) \in \overline{\mathbb{Q}}\}.$$

For instance, the Hermite–Lindemann theorem implies that if $S \subseteq \overline{\mathbb{Q}}$ is finite, then the exceptional set of $\exp(\prod_{\alpha \in S}(z - \alpha))$ is S. The exceptional sets of the functions 2^z and $e^{z\pi+1}$ are $\mathbb{Q}$ and $\emptyset$, respectively, as shown by the Gelfond–Schneider theorem and Baker's theorem. Assuming Schanuel's conjecture, we see that the exceptional sets of 2^{2^z} and $2^{2^{2^{z-1}}}$ are $\mathbb{Z}$ and $\mathbb{Z}_{>0}$, respectively.

The study of exceptional sets started in 1886 with a letter from Weierstrass to Strauss. In this letter, Weierstrass conjectured the existence of a transcendental entire function whose exceptional set is $\overline{\mathbb{Q}}$. This assertion was proved in 1895 by Stäckel [4],

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who established a much more general result: *for each countable subset* $\Sigma \subseteq \mathbb{C}$ *and each dense subset* $T \subseteq \mathbb{C}$*, there exists a transcendental entire function* f *such that* $f(\Sigma) \subseteq T$ (Weierstrass' assertion is obtained by choosing $\Sigma = T = \overline{\mathbb{Q}}$).

The question of the possible sets S_f has been solved in [1]: *any subset of algebraic numbers is the exceptional set of some transcendental entire function*. However, no information about the arithmetic nature of the coefficients of the Taylor series of f is obtained in this construction.

In 1965, Mahler [2] investigated the possible exceptional sets of entire functions having rational coefficients in their Taylor series (the set of these functions is denoted by T_∞). In particular, he proved that if S *is closed relative to* $\overline{\mathbb{Q}}$ (that is, if $\alpha \in S$, then any algebraic conjugate of α also lies in S), then there is a function $f \in T_\infty$ such that $S_f = S$. We remark that the function constructed by Mahler is of the form $f(z) = \sum_{k\geq 1} a_k P_1(z) \cdots P_k(z)$, where $\{P_1(z), P_2(z), \ldots\}$ is an enumeration of the minimal polynomials (over $\mathbb{Q}$) of the elements of S. In order to prove that $S_f = S$ (in particular, that $f(\overline{\mathbb{Q}} \backslash S) \cap \overline{\mathbb{Q}} = \emptyset$), Mahler used the observation that $P_1(\beta) \cdots P_k(\beta) \neq 0$ for all $k \geq 1$ and $\beta \in \overline{\mathbb{Q}} \backslash S$, in the case in which S is closed relative to $\overline{\mathbb{Q}}$. We refer the reader to [3, 5] (and references therein) for more results about the arithmetic behaviour of transcendental functions.

Based on his earlier result, in 1976, Mahler [3, page 58] suggested the following question.

Question 1.1. Does there exist for any choice of S (closed under complex conjugation) a series f in T_∞ for which $S_f = S$?

The phrase in parentheses does not appear in Mahler's original question. However, it is necessary because the exceptional set of a function $f \in T_\infty$ must be closed under complex conjugation (since $\overline{f(\alpha)} = f(\overline{\alpha})$). We interpret 'any choice' in Mahler's statement to mean 'any admissible choice'.

In this paper, we give an affirmative answer to Mahler's Question 1.1. More precisely, we prove the following theorem.

Theorem 1.2. *Every subset of* $\overline{\mathbb{Q}}$*, closed under complex conjugation, is the exceptional set of uncountably many transcendental functions in* T_∞.

In order to prove this theorem, we shall prove a stronger result about the behaviour of some functions in $\mathbb{K}[[z]]$ for a given dense set $\mathbb{K}$.

Theorem 1.3. *Let* A *be a countable set and let* $\mathbb{K}$ *be a dense subset of* $\mathbb{C}$*. For each* $\alpha \in A$*, fix a dense subset* $E_\alpha \subseteq \mathbb{C}$*. Then there exist uncountably many transcendental entire functions* $f \in \mathbb{K}[[z]]$ *such that* $f(\alpha) \in E_\alpha$ *for all* $\alpha \in A$.

We remark on some differences between our construction in Theorem 1.3 and the one in [1]. The functions in [1] have the form $f(z) = \sum_{k\geq 0} a_k P_k(z)$ and in each inductive step the coefficient a_n is chosen to ensure that $f(\beta) \in E_\beta$. Unfortunately, in that case, it is not possible to specify the arithmetic nature of a_n. In our present construction, in each inductive step, we shall apply an intermediate step (the construction of the functions $f_{m,1}(z)$ below) in order to ensure that a_n has the desired properties.

2. The proofs

PROOF THAT THEOREM 1.3 IMPLIES THEOREM 1.2. In the statement of Theorem 1.3, choose $A = \overline{\mathbb{Q}}$ and $\mathbb{K} = \mathbb{Q}^* + i\mathbb{Q}$. Write $S = \{\alpha_1, \alpha_2, \ldots\}$ and $\overline{\mathbb{Q}} \backslash S = \{\beta_1, \beta_2, \ldots\}$ (one of them may be finite). Now, we define

$$E_\alpha = \begin{cases} \overline{\mathbb{Q}} & \text{if } \alpha \in S, \\ \mathbb{K} \cdot \pi^n & \text{if } \alpha = \beta_n. \end{cases}$$

By Theorem 1.3, there exist uncountably many transcendental entire functions $f(z) = \sum_{k \geq 0} a_k z^k \in \mathbb{K}[[z]]$ such that $f(\alpha) \in E_\alpha$ for all $\alpha \in \overline{\mathbb{Q}}$. Now, define the function $\psi : \mathbb{C} \to \mathbb{C}$ by

$$\psi(z) = \frac{f(z) + \overline{f(\overline{z})}}{2}.$$

Note that $\psi(z) = \sum_{k \geq 0} \Re(a_k) z^k \in T_\infty$ is transcendental (since $\Re(a_k) \neq 0$ for all $k \geq 0$). Thus, it suffices to prove that $S_\psi = S$. In fact, if $\alpha \in S$, then $\overline{\alpha} \in S$ and thus $f(\alpha)$ and $f(\overline{\alpha})$ are algebraic numbers and so is $\psi(\alpha)$. In the case of $\alpha = \beta_n$, we must distinguish two cases: when $\beta_n \in \mathbb{R}$, then $\psi(\alpha) = \Re(f(\beta_n))$ is transcendental, since $f(\beta_n) \in \mathbb{K} \cdot \pi^n$. When $\beta_n \notin \mathbb{R}$, then $\overline{\beta_n} = \beta_m$ for some $m \neq n$. Thus, there exist nonzero algebraic numbers γ_1, γ_2 such that

$$\psi(\beta_n) = \frac{\gamma_1 \pi^n + \gamma_2 \pi^m}{2},$$

which is transcendental, since $\overline{\mathbb{Q}}$ is algebraically closed and π is transcendental. In conclusion, $\psi \in T_\infty$ is a transcendental function with $S_\psi = S$. □

PROOF OF THEOREM 1.3. Let $\{\alpha_1, \alpha_2, \alpha_3, \ldots\}$ be an enumeration of A. (Without loss of generality, we assume that $0 \notin A$.)

We shall construct the function $f(z) = \sum_{n \geqslant 0} \epsilon_n P_n(z)$, where $P_n(z) \in \overline{\mathbb{Q}}[z]$ has degree m_n. The polynomials P_n and the constants ϵ_n will be chosen conveniently so that f will satisfy the desired conditions.

Our first condition is $0 < |\epsilon_n| < (L(P_n) m_n!)^{-1} =: t_n$ for all $n \geq 0$. (Here $L(P)$ denotes the length of P.) Since $|P_n(z)| \leq L(P_n) \max\{1, |z|\}^{m_n}$ for all z belonging to the open ball $B(0, R)$,

$$|\epsilon_n P_n(z)| < \frac{1}{L(P_n) m_n!} L(P_n) \max\{1, R\}^{m_n} = \frac{\max\{1, R\}^{m_n}}{m_n!}.$$

Thus, f is an entire function, since the series $\sum_{n=0}^{\infty} \epsilon_n P_n(z)$, which defines f, converges uniformly in any of these balls.

Define $f_1(z) = \epsilon_0 + \epsilon_1(z - \alpha_1)$ for some nonzero $\epsilon_0 \in E_{\alpha_1} \cap B(0, 1)$ and choose $\epsilon_1 \in B(0, t_1)$ (where $P_1(z) = z - \alpha_1$) such that $a_0 := \epsilon_0 - \epsilon_1 \alpha_1 \in \mathbb{K}^*$. Thus, $f_1(\alpha_1) \in E_{\alpha_1}$ and the constant term of f_1 lies in $\mathbb{K}^*$.

Let $f_{2,1}$ be the function defined as $f_{2,1}(z) = f_1(z) + \epsilon_2 P_2(z)$, where $P_2(z) = z(z - \alpha_1)$. Then $f_{2,1}(\alpha_1) = f_1(\alpha_1) \in E_{\alpha_1}$. By the density of E_{α_2}, we can choose $\epsilon_2 \in B(0, t_2) \backslash \{0\}$ such that $f_{2,1}(\alpha_2) = f_1(\alpha_2) + \epsilon_2 \alpha_2(\alpha_2 - \alpha_1) \in E_{\alpha_2}$.

Now, consider the function $f_2(z) = f_{2,1}(z) + \epsilon_3 P_3(z)$, where $P_3(z) = P_2(z)(z - \alpha_2)$. Our goal is to choose ϵ_3 such that the coefficient of z in f_2 lies in $\mathbb{K}^*$. Observe that this coefficient is $a_1 := \epsilon_3\alpha_1\alpha_2 - \epsilon_2\alpha_1$. Since $\alpha_1\alpha_2 \neq 0$, we can choose $\epsilon_3 \in B(0, t_3)\backslash\{0\}$ such that $a_1 \in \mathbb{K}^*$. Note that $f_2(\alpha_i) \in E_{\alpha_i}$ for $i \in \{1, 2\}$ and the first two coefficients of f_2 (a_0 and a_1) belong to $\mathbb{K}^*$.

Suppose, by induction, that the function $f_n(z) = \sum_{k=0}^{n-1} a_k z^k + \sum_{k=n}^{2n-1} b_k z^k$ has been constructed such that $a_0, \ldots, a_{n-1} \in \mathbb{K}^*$ and $f_n(\alpha_i) \in E_{\alpha_i}$ for all $1 \leq i \leq n$. Now, let us construct f_{n+1} with the desired properties.

Define $f_{n+1,1}$ by

$$f_{n+1,1}(z) = f_n(z) + \epsilon_{2n} z^n (z - \alpha_1) \cdots (z - \alpha_n).$$

Note that $f_{n+1,1}(\alpha_i) \in E_{\alpha_i}$ for all $1 \leq i \leq n$. Also, the first n coefficients of $f_{n+1,1}$ and f_n are equal (because of the factor z^n in the right-hand side above) and they belong to $\mathbb{K}^*$. Setting $P_{2n}(z) = z^n(z - \alpha_1) \cdots (z - \alpha_n)$, we can choose $\epsilon_{2n} \in B(0, t_{2n})\backslash\{0\}$ such that $f_{n+1,1}(\alpha_{n+1}) \in E_{\alpha_{n+1}}$.

The next step is to perturb the previous function to force the coefficient of z^n (in this new function) to be in $\mathbb{K}$. For that, define

$$f_{n+1}(z) = f_{n+1,1}(z) + \epsilon_{2n+1} P_{2n+1}(z),$$

where $P_{2n+1}(z) = P_{2n}(z)(z - \alpha_{n+1})$. Since $a_n := b_n + (-1)^{n+1}\epsilon_{2n+1}\alpha_1 \cdots \alpha_{n+1}$ is the coefficient of z^n in f_{n+1}, by the density of $\mathbb{K}$, we can choose $\epsilon_{2n+1} \in B(0, t_{2n+1})\backslash\{0\}$ such that $a_n \in \mathbb{K}^*$.

In conclusion, our desired function $f(z) = \sum_{n\geq 0} \epsilon_n P_n(z) = \sum_{n\geq 0} a_n z^n \in \mathbb{K}[[z]]$ maps α into E_α for all $\alpha \in A$ and its coefficients belong to $\mathbb{K}^*$. This function is transcendental, since it is not a polynomial, because $a_n \neq 0$ for all $n \geq 0$. Also, there is an ∞-ary tree of different possibilities for f (because in each step we have infinitely many possible choices for ϵ). Thus, we have constructed uncountably many possible functions. The proof is complete. □

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References

[1] J. Huang, D. Marques and M. Mereb, 'Algebraic values of transcendental functions at algebraic points', *Bull. Aust. Math. Soc.* **82** (2010), 322–327.

[2] K. Mahler, 'Arithmetic properties of lacunary power series with integral coefficients', *J. Aust. Math. Soc.* **5** (1965), 56–64.

[3] K. Mahler, *Lectures on Transcendental Numbers*, Lecture Notes in Mathematics, 546 (Springer, Berlin, 1976).

[4] P. Stäckel, 'Ueber arithmetische Eingenschaften analytischer Functionen', *Math. Ann.* **46** (1895), 513–520.

[5] M. Waldschmidt, 'Algebraic values of analytic functions', *J. Comput. Appl. Math.* **160** (2003), 323–333.

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