Can. J. Math. Vol. **46** (6), 1994 pp. 1121–1134

ALGEBRAIC CYCLES IN FAMILIES OF ABELIAN VARIETIES

SALMAN ABDULALI

ABSTRACT. If the Hodge $*$-operator on the L_2-cohomology of Kuga fiber varieties is algebraic, then the Hodge conjecture is true for all abelian varieties.

1. **Introduction.** The Hodge conjecture predicts that the **Q**-linear span of the classes of algebraic subvarieties in the cohomology of a smooth complex projective variety X is given by the *Hodge ring*

$$\mathrm{Hdg}(X) := \bigoplus_p H^{2p}(X, \mathbf{Q}) \cap H^{p,p}(X, \mathbf{C}),$$

the elements of this ring being called *Hodge cycles*. (We abuse notation and use the same symbol to denote a complex algebraic variety and its associated complex analytic space.) For elements of $H^2(X, \mathbf{Q})$ this is a result of Lefschetz. It follows that if the Hodge ring of X is generated by its elements of degree 2, then the Hodge conjecture is true for X, and all algebraic classes represent intersections of divisors. There are many abelian varieties whose Hodge ring is not generated in degree 2; some examples given by Weil [W] are widely believed to be likely counterexamples to the Hodge conjecture. The only abelian varieties, whose Hodge rings are not generated in degree 2, but for which the Hodge conjecture *is* known, are due to Shioda [So] and Schoen [Sc1, Sc2].

Grothendieck's Standard Conjecture A [Gk2, p. 196] states that for a smooth, projective variety X over **C**, the operator Λ of Hodge theory takes algebraic cycles to algebraic cycles. Equivalently, the space of algebraic cycles is invariant under the Hodge $*$-operator, or, numerical and homological equivalence are the same [L, Theorem 1, pp. 367–368]. In contrast to the Hodge conjecture, this conjecture is known for many classes of varieties including all abelian varieties [L, Theorem 3, p. 372].

In this paper we show that the Hodge conjecture is true for all abelian varieties if the analog of Standard Conjecture A is true for the L_2-cohomology of Kuga fiber varieties. By this we mean that the Hodge $*$-operator on the L_2-cohomology takes algebraic cycles to algebraic cycles. (We actually need only a weaker statement which we formulate later as Conjecture 5.3.) The link between the Hodge conjecture and the standard conjectures is the following conjecture of Grothendieck [Gk1, footnote 13, p. 103], which we shall refer to as the *invariant cycles conjecture*.

Received by the editors March 29, 1993; revised May 26, 1994.

AMS subject classification: 14C30.

https://doi.org/10.4153/CJM-1994-063-0 Published online by Cambridge University Press

CONJECTURE 1.1. *Let $f: A \longrightarrow V$ be a smooth and proper morphism of smooth quasiprojective varieties over* $\mathbf{C}$. *Let $P \in V$, and $\Gamma := \pi_1(V, P)$. The space of all $s \in H^0(V, R^b f_* \mathbf{Q}) \cong H^b(A_P, \mathbf{Q})^\Gamma$, which represent algebraic cycles in $H^b(A_P, \mathbf{Q})^\Gamma$ is independent of P.*

This conjecture was the principal motivation for much of Griffiths' work on variation of Hodge structures [Gf, Remark, p. 146]. He proved the analogous statement for Hodge cycles, assuming V to be compact [Gf, Corollary 7.3, p. 146]. The compactness assumption was removed by Deligne ("Théorème de la partie fixe" [D1, Corollaire 4.1.2, p. 42]). Thus the Hodge conjecture implies the invariant cycles conjecture.

In §6 we show that the invariant cycles conjecture for Kuga fiber varieties of Hodge type implies the Hodge conjecture for all abelian varieties. (The definition of Kuga fiber varieties is reviewed in §2.) The proof is similar to Deligne's proof of his absolute Hodge cycles theorem, where "Principle B" [D2, Theorem 2.12, p. 37] plays the role of the invariant cycles conjecture.

In §4 and §5 we show that the invariant cycles conjecture is true for Kuga fiber varieties if the Hodge $*$-operator on the L_2-cohomology is algebraic. In order to explain the idea of the proof, let us *assume for the time being that A is compact*. Then the Leray spectral sequence for $A \longrightarrow V$ degenerates and we have

$$H^r(A, \mathbf{C}) = \bigoplus_{a+b=r} H^{\langle a,b \rangle}(A, \mathbf{C}),$$

where $H^{\langle a,b\rangle}(A, \mathbf{C}) \cong H^a(V, R^b f_* \mathbf{C})$. For $P \in V$, let j_P be the inclusion of the fiber A_P into A. Then the pullback j_P^* induces an isomorphism of $H^{\langle 0,2b\rangle}(A, \mathbf{C})$ with $H^{2b}(A_P, \mathbf{C})^\Gamma$. Since the pullback of an algebraic cycle is algebraic, we have

$$\dim AH^{2b}(A_P, \mathbf{C})^\Gamma \geq \dim AH^{\langle 0,2b\rangle}(A, \mathbf{C}), \tag{1.2}$$

where AH denotes the subspace spanned by algebraic classes.

Next we consider the Gysin homomorphism

$$j_{P*}: H^{2m-2b}(A_P, \mathbf{C}) \longrightarrow H^{2m-2b+2d}(A, \mathbf{C}),$$

where m is the dimension of a fiber, and d is the dimension of V. It induces an isomorphism

$$H^{2m-2b}(A_P, \mathbf{C})^\Gamma \longrightarrow H^{\langle 2d,2m-2b\rangle}(A, \mathbf{C}).$$

Since j_{P*} takes algebraic cycles to algebraic cycles, we have

$$\dim AH^{2m-2b}(A_P, \mathbf{C})^\Gamma \leq \dim AH^{\langle 2d,2m-2b\rangle}(A, \mathbf{C})$$

for any point P. Since Grothendieck's standard conjectures are known for abelian varieties [L] we also have $\dim AH^{2b}(A_P, \mathbf{C})^\Gamma = \dim AH^{2m-2b}(A_P, \mathbf{C})^\Gamma$. The spaces $H^{\langle 0,2b\rangle}(A, \mathbf{C})$ and $H^{\langle 2d,2m-2b\rangle}(A, \mathbf{C})$ are dual under the Hodge $*$-operator. *Assume now* *that Grothendieck's standard conjectures are true for A*. Then $\dim AH^{\langle 2d, 2m-2b\rangle}(A, \mathbf{C}) = \dim AH^{\langle 0,2b\rangle}(A, \mathbf{C})$. Combining all of this, we have

$$\dim AH^{2b}(A_P, \mathbf{C})^{\Gamma} \leq \dim AH^{\langle 0,2b\rangle}(A, \mathbf{C}). \tag{1.3}$$

(1.2) and (1.3) together imply that $\dim AH^{2b}(A_P, \mathbf{C})^{\Gamma} = \dim AH^{\langle 0,2b\rangle}(A, \mathbf{C})$ is independent of P. Thus every Γ-invariant algebraic cycle on a fiber A_P is the pullback by j_P^* of an algebraic cycle in $AH^{\langle 0,*\rangle}(A, \mathbf{C})$. This completes the proof (in outline) for compact A. If A is not compact, then we modify the above argument by using L_2-cohomology.

Though we have assumed that $A \longrightarrow V$ is a Kuga fiber variety, the argument is valid in far greater generality. We will not pursue this here; see however [An2], part of which was motivated by an earlier version of this paper.

I would like to thank the University of Toronto, and especially Professor Kumar Murty, for inviting me to Toronto for an extended visit, and providing an atmosphere which made this paper possible.

2. **Kuga fiber varieties.** In this section we recall Kuga's construction of families of abelian varieties [K1]. Details may also be found in [Sa2]; [K3] is a brief, but useful, survey.

Let G be a connected, semisimple, linear algebraic group over $\mathbf{Q}$ of hermitian type with no nontrivial normal $\mathbf{Q}$-subgroup H such that $H(\mathbf{R})$ is compact. Then $X := G(\mathbf{R})^0/K$ is a bounded symmetric domain, where K is a maximal compact subgroup of $G(\mathbf{R})^0$. Let $\mathfrak{g} := \operatorname{Lie} G(\mathbf{R})$, $\mathfrak{k} := \operatorname{Lie} K$, and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition. Let x be the unique fixed point of K in X. Differentiating the natural map $G(\mathbf{R})^0 \longrightarrow X$ gives an isomorphism of $\mathfrak{p}$ with $T_x(X)$, the tangent space of X at x, and there exists a unique $H_0 \in Z(\mathfrak{k})$, called the *H-element* at x, such that $\operatorname{ad} H_0|\mathfrak{p}$ is the complex structure on $T_x(X)$. Since $\mathfrak{g}$ is semisimple, the Killing form is a nondegenerate bilinear form; its restriction to $\mathfrak{p}$ is symmetric and positive definite. There is a unique $G(\mathbf{R})^0$-invariant Riemannian metric on X, which agrees with the Killing form on $\mathfrak{p} = T_x(X)$. Call this metric ds_0^2.

Let Γ be a torsion-free arithmetic subgroup of G. Then $V := \Gamma \setminus X$ is a smooth quasiprojective algebraic variety. The metric ds_0^2 induces a metric on V, which we again denote by ds_0^2. With this metric, V is a complete Kähler manifold.

Let β be a nondegenerate alternating form on a finite-dimensional rational vector space F. The symplectic group $\operatorname{Sp}(F, \beta)$ is a $\mathbf{Q}$-algebraic group of hermitian type; the associated symmetric domain is the *Siegel space*

$$\mathfrak{S}(F, \beta) := \Big\{ J \in \operatorname{GL}\big(F(\mathbf{R})\big) \mid J^2 = -I \text{ and } \beta(x, Jy) \text{ is symmetric, positive definite} \Big\}.$$

$\operatorname{Sp}(F, \beta)$ acts on $\mathfrak{S}(F, \beta)$ by conjugation. The H-element at a point $J \in \mathfrak{S}(F, \beta)$ is $J/2$.

Let $\rho\colon G \longrightarrow \operatorname{Sp}(F, \beta)$ be a representation defined over $\mathbf{Q}$, and $\tau\colon X \longrightarrow \mathfrak{S}(F, \beta)$ a holomorphic map such that $\tau(g \cdot x) = \rho(g) \cdot \tau(x)$ for all $g \in G(\mathbf{R})^0$, $x \in X$. Let H_0 be the

H-element at a base point $x \in X$, and H_0' the H-element at $\tau(x)$. We say that the pair (ρ, τ) satisfies the H_1-*condition*, or is *strongly equivariant* if

$$[d\rho(H_0) - H_0', d\rho(g)] = 0 \quad \text{for all } g \in \mathfrak{g},$$

and we say that the H_2-*condition* is satisfied if

$$d\rho(H_0) = H_0'.$$

These conditions are independent of the choice of the base point x.

Assume that the H_1-condition is satisfied. Let L be a lattice in F such that $\rho(\Gamma)L = L$. The quotient

$$A := (\Gamma \ltimes_\rho L) \setminus \big(X \times F(\mathbf{R})\big)$$

is a torus bundle over $V = \Gamma \setminus X$. Kuga showed that there is a complex structure on A such that the fiber A_P over any point $P \in V$ is an abelian variety isomorphic to the torus $F(\mathbf{R})/L$ with the complex structure $\tau(x)$, where x is a point in X lying over P [K1, Theorem II-6-3, p. 114]. A has a structure of quasiprojective algebraic variety such that the projection $f: A \longrightarrow V$ is a morphism of algebraic varieties [D2, p. 74]. This fiber variety is called a *Kuga fiber variety*.

The tangent space at a point $(x, u) \in X \times F(\mathbf{R})$ is $T_x(X) \times F(\mathbf{R})$. The form

$$S_x(u, v) := \beta\big(u, \tau(x)v\big) \tag{2.1}$$

is symmetric and positive definite on $F(\mathbf{R})$. We can therefore define a Riemannian metric ds^2 on $X \times F(\mathbf{R})$ such that ds^2 agrees with ds_0^2 on the first factor, and equals S_x on $\{x\} \times F(\mathbf{R})$. Then ds^2 is $\Gamma \ltimes_\rho L$-invariant. Therefore it induces a metric on A, which we again denote by ds^2. With this metric, A is a complete Kähler manifold.

We have a vector bundle $X \times F(\mathbf{R}) \longrightarrow X$, with a metric given by (2.1). This extends to a hermitian metric on the complex vector bundle $E := X \times F(\mathbf{C}) \longrightarrow X$. Taking the quotient by Γ gives a hermitian vector bundle over V, which we again denote by E.

Of special interest are Kuga fiber varieties of *Hodge type*. These are constructed using the inclusion of the Hodge group of a complex abelian variety into the symplectic group of a Riemann form; see [Mm1] or [Mm2] for details.

PROPOSITION 2.2. *A Kuga fiber variety $A \longrightarrow V$ constructed from a symplectic representation $\rho: G \longrightarrow \mathrm{Sp}(F, \beta)$ satisfying the H_2-condition is of Hodge type, the Hodge group of every fiber is contained in $\rho(G)$, and the Hodge group of a general fiber equals $\rho(G)$.*

PROOF. Any H_2-homomorphism is *rigid* in the sense that the equivariant holomorphic map τ is uniquely determined by ρ (see [Sa2, p. 183]). Therefore, Theorem 3.4 of [Ab3] implies that $A \longrightarrow V$ is of Hodge type. The definition of a Hodge family shows that $\rho(G)$ is the semisimple part of the Hodge group of a general fiber. But, according to [Ab2, Proposition 1.3, p. 335], $\rho(G)$ contains the Hodge group of every fiber. Therefore $\rho(G)$ must be the Hodge group of a general fiber. ∎

3. L_2**-cohomology.** In this section we review the definition and summarize the basic properties of L_2-cohomology. [CGM] is an excellent survey article. [BC], [Ca], [Z1], and [Z2] are primarily concerned with arithmetic varieties; for other papers see the list of references in [Z2].

Let M be a complete Riemannian manifold, and E a hermitian vector bundle on M defined by a finite dimensional unitary representation of the fundamental group of M. We denote by $\Omega^r(M,E)$ the space of E-valued C^∞ r-forms on M. A form $\omega \in \Omega^r(M,E)$ is said to be *square integrable* if $\int_M \omega \wedge *\bar{\omega}$ is finite. Let $\Omega^r_{(2)}(M,E)$ be the set of $\omega \in \Omega^r(M,E)$ such that ω and $d\omega$ are both square integrable. An inner product on $\Omega^r_{(2)}(M,E)$ is given by

$$(\omega,\eta) := \int_M \omega \wedge *\bar{\eta}.$$

The *L_2-cohomology* of M, denoted $H^\bullet_{(2)}(M,E)$, is the cohomology of the complex $\Omega^\bullet_{(2)}(M,E)$.

Alternatively, the L_2-cohomology may be defined as the cohomology of the complex $L^\bullet_{(2)}(M,E)$, consisting of measurable E-valued differential forms ω on M such that ω and $d\omega$ are square integrable. Cheeger [Ch, p. 94] has shown that the inclusion of $\Omega^\bullet_{(2)}(M,E)$ into $L^\bullet_{(2)}(M,E)$ induces an isomorphism on cohomology, so the two definitions are equivalent.

The *reduced L_2-cohomology*, denoted $\bar{H}^r_{(2)}(M,E)$, is defined to be the quotient of the closed forms in $L^r_{(2)}(M,E)$ by the closure of the exact forms. There is a natural surjection of $H^r_{(2)}(M,E)$ onto $\bar{H}^r_{(2)}(M,E)$.

Let δ be the formal adjoint of d. A differential form ω is called *harmonic* if $d\omega = \delta\omega = 0$. Denote by $\mathcal{H}^r_{(2)}(M,E)$ the space of L_2 harmonic r-forms. Since M is complete, the natural map of $\mathcal{H}^r_{(2)}(M,E)$ into $H^r_{(2)}(M,E)$ is injective (*cf.* [CGM, p. 317]), and the natural map of $\mathcal{H}^r_{(2)}(M,E)$ into $\bar{H}^r_{(2)}(M,E)$ is an isomorphism (*cf.* [CGM, p. 318]). If $H^r_{(2)}(M,E)$ is finite dimensional, then $H^r_{(2)}(M,E) = \bar{H}^r_{(2)}(M,E) \cong \mathcal{H}^r_{(2)}(M,E)$ (*cf.* [Z1, Proposition 1.11, p. 174]).

Assume now that M and E are as above, and $H^\bullet_{(2)}(M,E)$ is finite dimensional. Then Poincaré duality is satisfied, *i.e.*, $H^r_{(2)}(M,E)$ and $H^s_{(2)}(M,E)$ are dual if $r+s = \dim M$; the duality is induced by the Hodge $*$-operator (*cf.* [Ca, pp. 72–74]). In the infinite dimensional case, the reduced L_2-cohomology satisfies Poincaré duality, as long as M is a complete Kähler manifold (*cf.* [CGM, p. 318]).

A complete manifold has *negligible boundary* in the sense that $\int_M d\omega = 0$ for all $\omega \in \Omega^r_{(2)}(M,E)$ [Ga, Theorem, p. 141]. Hence $\int_M \omega$ is well-defined for $\omega \in H^\bullet_{(2)}(M,E)$.

4. L_2**-cohomology of Kuga fiber varieties.** The cohomology of compact Kuga fiber varieties was described by Kuga in [K1, pp. 71–95]; a summary appears in [K3, pp. 340–342]. In this section, we generalize some of these results to the L_2-cohomology of noncompact families of abelian varieties. The principal difference in approach is that group cohomology can no longer be used. Unfortunately, I do not know how to show that the L_2-cohomology of a Kuga fiber variety is finite dimensional, so I work with the reduced L_2-cohomology instead.

4.1 *Differential forms.* Let $f\colon A \longrightarrow V$ be a Kuga fiber variety, $V = \Gamma \backslash X$, $d := \dim V = \dim X$, m the dimension of a fiber.

Since X is a bounded symmetric domain, there is a global real coordinate system $\{x^1, \ldots, x^{2d}\}$ on X. Let $\{u^1, \ldots, u^{2m}\}$ be coordinate functions on F, with respect to some basis, *i.e.*, a basis of F^*. Then $\{x^1, \ldots, x^{2d}, u^1, \ldots, u^{2m}\}$ is a global coordinate system on $X \times F(\mathbf{R})$. Such a coordinate system will be called *admissible*.

We identify a differential form on V with a Γ-invariant form on X. We also identify a differential form on A with a $\Gamma \ltimes_\rho L$-invariant form on $X \times F(\mathbf{R})$. Note that any Γ-invariant form is G-invariant because Γ is Zariski-dense in G [B, Theorem 1, p. 78], and ρ is an algebraic representation of G.

Let $\Omega^{\langle a,b\rangle}\big(X \times F(\mathbf{R}), \mathbf{C}\big)$ be the space of complex valued differential forms of the form

$$\sum_{\substack{|C|=a \\ |D|=b}} \varphi_{C,D}(x,u)\, dx^C \wedge du^D.$$

An element of this space will be said to be of *type* $\langle a,b\rangle$. The type does not depend on the choice of admissible coordinate system. Then

$$\Omega^r\big(X \times F(\mathbf{R}), \mathbf{C}\big) = \bigoplus_{a+b=r} \Omega^{\langle a,b\rangle}\big(X \times F(\mathbf{R}), \mathbf{C}\big).$$

Furthermore, as in [K1, pp. 81–82], we have an orthogonal direct sum

$$\Omega^r_{(2)}(A, \mathbf{C}) = \bigoplus_{a+b=r} \Omega^{\langle a,b\rangle}_{(2)}(A, \mathbf{C}),$$

where

$$\Omega^{\langle a,b\rangle}_{(2)}(A, \mathbf{C}) := \Omega^{\langle a,b\rangle}\big(X \times F(\mathbf{R}), \mathbf{C}\big)^{\Gamma \ltimes_\rho L} \cap \Omega^{a+b}_{(2)}(A, \mathbf{C}).$$

Now let $\omega \in \Omega^{\langle a,b\rangle}\big(X \times F(\mathbf{R}), \mathbf{C}\big)$. We can write

$$\omega = \sum_{|C|=a} dx^C \wedge \eta_C,$$

where

$$\eta_C(x,u) := \sum_{|D|=b} \varphi_{C,D}(x,u)\, du^D.$$

For each $x \in X$, $\eta_C(x,u)$ is a b-form on $F(\mathbf{R})$. Denote by d the exterior derivative on $X \times F(\mathbf{R})$, by d_1 the exterior derivative on $F(\mathbf{R})$, and by d_V the exterior derivative on X. Then

$$dw = d_V\omega + \sum_{|C|=a} dx^C \wedge d_1\eta_C. \tag{4.1.1}$$

Thus $d\omega = 0$ if and only if $d_V\omega = 0$ and $d_1\eta_C = 0$ for each C [K1, Corollary II-3-3, p. 86].

Next we shall examine the action of the Hodge $*$-operator on ω. Let $*_0$ denote the Hodge $*$-operator on $\Omega^r(X, \mathbf{C})$ with respect to the metric ds_0^2, $*$ the Hodge $*$-operator on $\Omega^r\big(X \times F(\mathbf{R}), \mathbf{C}\big)$ with respect to the metric ds^2, and $*_x$ the Hodge $*$-operator on $\Omega^r\big(F(\mathbf{R}), \mathbf{C}\big)$ with respect to the metric S_x. Then [K1, Proposition II-3-1, p. 85]

(4.1.2) $$*\omega = (-1)^{ab} \sum_{|C|=a} (*_0\, dx^C) \wedge (*_x \eta_C) \in \Omega^{\langle 2d-a, 2m-b\rangle}\big(X \times F(\mathbf{R}), \mathbf{C}\big).$$

4.2 *Harmonic forms.* We will now concentrate on those differential forms on A which can be interpreted as automorphic forms on V. Let $\mathcal{K}^r\big(X \times F(\mathbf{R}), \mathbf{C}\big)$ be the space of differential forms of the form

$$\omega = \sum_{C,D} \varphi_{C,D}(x)\, dx^C \wedge du^D,$$

so that each $\varphi_{C,D}$ is a function of x alone. Then

$$\mathcal{K}^r\big(X \times F(\mathbf{R}), \mathbf{C}\big) = \bigoplus_{a+b=r} \mathcal{K}^{\langle a,b\rangle}\big(X \times F(\mathbf{R}), \mathbf{C}\big),$$

where

$$\mathcal{K}^{\langle a,b\rangle}\big(X \times F(\mathbf{R}), \mathbf{C}\big) := \mathcal{K}^r\big(X \times F(\mathbf{R}), \mathbf{C}\big) \cap \Omega^{\langle a,b\rangle}\big(X \times F(\mathbf{R}), \mathbf{C}\big).$$

Next, let

$$\mathcal{K}^r_{(2)}(A, \mathbf{C}) := \mathcal{K}^r\big(X \times F(\mathbf{R}), \mathbf{C}\big)^{\Gamma \ltimes_\rho L} \cap \Omega^r_{(2)}(A, \mathbf{C}),$$
$$\mathcal{K}^{\langle a,b\rangle}_{(2)}(A, \mathbf{C}) := \mathcal{K}^r_{(2)}(A, \mathbf{C}) \cap \Omega^{a+b}_{(2)}(A, \mathbf{C}),$$

and

$$\mathcal{H}^{\langle a,b\rangle}_{(2)}(A, \mathbf{C}) := \mathcal{H}^{a+b}_{(2)}(A, \mathbf{C}) \cap \Omega^{\langle a,b\rangle}_{(2)}(A, \mathbf{C}).$$

Then we have

(4.2.1) $$\mathcal{K}^r_{(2)}(A, \mathbf{C}) = \bigoplus_{a+b=r} \mathcal{K}^{\langle a,b\rangle}_{(2)}(A, \mathbf{C}).$$

We can identify $\mathcal{K}^{\langle a,b\rangle}\big(X \times F(\mathbf{R}), \mathbf{C}\big)$ with the space of differential forms $\Omega^a(X, \Lambda^b E^*)$, where E is the vector bundle constructed in §2 [K1, Proposition II-3-6, p. 89]. With this identification we have

(4.2.2) $$\mathcal{K}^{\langle a,b\rangle}_{(2)}(A, \mathbf{C}) \cong \Omega^a_{(2)}(V, \Lambda^b E^*).$$

LEMMA 4.2.3 ([K1, Lemma II-3-5, p. 87], in compact case).

$$\mathcal{H}^r_{(2)}(A, \mathbf{C}) \subset \mathcal{K}^r_{(2)}(A, \mathbf{C})$$

and

$$\mathcal{H}^{\langle a,b\rangle}_{(2)}(A, \mathbf{C}) \subset \mathcal{K}^{\langle a,b\rangle}_{(2)}(A, \mathbf{C}).$$

PROOF. The $\Gamma \ltimes_\rho L$-invariant of ω implies the L-invariance of η_C. Hence we may considei η_C as a form on the torus $T = F(\mathbf{R})/L$. We have seen that it is a closed form. Let $\delta = *d*$ be the adjoint of d. $\delta\omega = 0$ implies, by (4.1.1) and (4.1.2), that $\delta_x \eta_C = 0$ for every x, where δ_x is the adjoint of the exterior derivative on T with respect to the metric S_x. Thus η_C is a harmonic form on T for each x. But any harmonic form on a torus has constant coefficients. Therefore each $\varphi_{C,D}$ is a function of x alone. ■

https://doi.org/10.4153/CJM-1994-063-0 Published online by Cambridge University Press

LEMMA 4.2.4 ([K1, Theorem II-3-12, p. 94], in compact case).

$$\mathcal{H}^r_{(2)}(A, \mathbf{C}) = \bigoplus_{a+b=r} \mathcal{H}^{\langle a,b\rangle}_{(2)}(A, \mathbf{C}).$$

PROOF. Let $\omega \in \mathcal{H}^r_{(2)}(A, \mathbf{C})$. Use (4.2.1) and Lemma 4.2.3 to write $\omega = \sum \omega^{\langle a,b\rangle}$ with $\omega^{\langle a,b\rangle} \in \mathcal{K}^{\langle a,b\rangle}_{(2)}(A, \mathbf{C})$. Since $d\omega^{\langle a,b\rangle} \in \mathcal{K}^{\langle a+1,b\rangle}_{(2)}(A, \mathbf{C})$, $d\omega = 0$ implies that $d\omega^{\langle a,b\rangle} = 0$ for all a, b. Similarly, since $\delta\omega^{\langle a,b\rangle} \in \mathcal{K}^{\langle a-1,b\rangle}_{(2)}(A, \mathbf{C})$, $\delta\omega = 0$ implies that $\delta\omega^{\langle a,b\rangle} = 0$ for all a, b. Thus each $\omega^{\langle a,b\rangle}$ is a harmonic form. ∎

LEMMA 4.2.5. *$\mathcal{H}^r_{(2)}(A, \mathbf{C})$ is finite dimensional.*

PROOF. The isomorphism (4.2.2) induces an isomorphism

$$\mathcal{H}^{\langle a,b\rangle}_{(2)}(A, \mathbf{C}) \cong \mathcal{H}^a_{(2)}(V, \Lambda^b E^*).$$

The L_2-cohomology groups of arithmetic varieties are finite dimensional ([BC, Theorem A, p. 625]; see also [Z2, (1.6), p. 380]); hence the previous lemma shows that $\mathcal{H}^r_{(2)}(A, \mathbf{C})$ is finite dimensional. ∎

4.3 *Restriction to a fiber.* Denote by j_P the inclusion of a fiber A_P into A.

LEMMA 4.3.1 ([K4, 1-3 [D], p. 17], in compact case). *The restriction of j_P^* to $\mathcal{H}^{\langle 0,r\rangle}_{(2)}(A, \mathbf{C})$ induces an isomomorphism*

$$\mathcal{H}^{\langle 0,r\rangle}_{(2)}(A, \mathbf{C}) \longrightarrow H^r(A_P, \mathbf{C})^\Gamma.$$

PROOF. Let $\omega \in \mathcal{H}^{\langle 0,r\rangle}_{(2)}(A, \mathbf{C})$. Lemma 4.2.3 implies that we can write ω in the form

$$\omega = \sum_{|D|=r} \varphi_D(x)\, du^D.$$

Then

$$j_P^*\omega = \sum_{|D|=r} \varphi_D(P)\, du^D.$$

Since ω is a closed form, each φ_D is a constant function. The $\Gamma \ltimes_\rho L$-invariance of ω then implies the Γ-invariance of $j_P^*\omega$.

Conversely, we will show that any Γ-invariant ω of this form, with constant coefficients, is an L_2-form on A. Let $x \in X$, $g \in G(\mathbf{R})^0$, and $y = gx$. Then

$$S_x(u, v) = \beta\big(u, \tau(x)v\big) = \beta\big(u, \rho(g)^{-1}\tau(y)\rho(g)v\big) = S_y\big(\rho(g)u, \rho(g)v\big)$$

since $\rho(g)$ belongs to the symplectic group of β. It follows that $*_x\omega$ independent of x. Therefore the L_2-norm of ω is given by $\|\omega\|_2^2 = (\operatorname{vol} V)\|j_P^*(\omega)\|_2^2$. Since V has finite volume, the norm is finite. ∎

LEMMA 4.3.2. *If $\omega_P \in H^{2r}(A_P, \mathbf{C})$ is Γ-invariant, then so is $*\omega_P$.*

PROOF. Lemma 4.3.1 shows that $\omega_P = j_P^*\omega$ for a unique $\omega \in \mathcal{H}_{(2)}^{\langle 0,2r\rangle}(A, \mathbf{C})$. (4.1.2) shows that $*\omega = *_0(1) \wedge *_x\omega \in \mathcal{H}_{(2)}^{\langle 2d,2m-2r\rangle}(A, \mathbf{C})$. In particular, $*_0(1) \wedge *_x\omega$ is a $\Gamma \ltimes_\rho L$-invariant form, which implies that $*\omega_P$ is Γ-invariant. ■

4.4 *The Gysin map.* Let $j_{P*}: H^{2r}(A_P, \mathbf{C}) \longrightarrow \mathcal{H}_{(2)}^{2d+2r}(A, \mathbf{C})$ be the Gysin map, *i.e.*, the Poincaré dual of $j_P^*: \mathcal{H}_{(2)}^{2m-2r}(A, \mathbf{C}) \longrightarrow H^{2m-2r}(A_P, \mathbf{C})$. We may describe j_{P*} on harmonic forms as follows: $j_{P*}(\omega)$ is the unique element of $\mathcal{H}_{(2)}^{2d+2r}(A, \mathbf{C})$ such that

$$\int_A j_{P*}\omega \wedge \zeta = \int_{A_P} \omega \wedge j_P^*\zeta \quad \text{for all } \zeta \in \mathcal{H}_{(2)}^{2m-2r}(A, \mathbf{C}).$$

LEMMA 4.4.1 ([Ab1, Lemma 3.3.2, p. 45], in compact case). *Let $\Omega_V := *(1)$ be the volume form on V. Then*

$$j_{P*}j_P^*(\omega) = \mathrm{vol}(V)^{-1} f^*(\Omega_V) \wedge \omega \quad \textit{for all } \omega \in \mathcal{H}_{(2)}^{\langle 0,2r\rangle}(A, \mathbf{C}).$$

The image of j_{P*} is contained in $\mathcal{H}_{(2)}^{\langle 2d,2r\rangle}(A, \mathbf{C})$.

PROOF. Observe that $f^*(\Omega_V) \wedge \omega \in \mathcal{H}_{(2)}^{\langle 2d,2r\rangle}(A, \mathbf{C})$.

We have to show that

$$\int_A f^*(\Omega_V) \wedge \omega \wedge \zeta = \mathrm{vol}(V) \int_{A_P} j_P^*\omega \wedge j_P^*\zeta \quad \text{for all } \zeta \in \mathcal{H}_{(2)}^{2m-2r}(A, \mathbf{C}). \tag{4.4.2}$$

Because of Lemma 4.2.4 it is enough to prove this for $\zeta \in \mathcal{H}_{(2)}^{\langle a,b\rangle}(A, \mathbf{C})$ with $a + b = 2m - 2r$. Then both sides of (4.4.2) are zero unless $a = 0$ and $b = 2m - 2r$, which we assume. Let D_Γ be a fundamental domain for the action of Γ on X, and D_L a fundamental domain for the lattice L. Then

$$\begin{aligned}
\int_A f^*(\Omega_V) \wedge \omega \wedge \zeta &= \int_{D_\Gamma \times D_L} f^*(\Omega_V) \wedge \omega \wedge \zeta \\
&= \int_{D_\Gamma} \Omega_V \int_{D_L} j_P^*(\omega \wedge \zeta) \\
&= \mathrm{vol}(V) \int_{A_P} j_P^*\omega \wedge j_P^*\zeta.
\end{aligned}$$

■

5. **Algebraic cycles.** Let X be a smooth algebraic variety with a complete Kähler metric. Then we have seen that the harmonic forms on X satisfy Poincaré duality. Let Z be an algebraic cycle of codimension r on X. The *class* of Z is an element $c(Z) \in \mathcal{H}_{(2)}^{2r}(X, \mathbf{C})$ such that

$$\int_Z \omega = \int_X c(Z) \wedge \omega \quad \text{for all } \omega \in \mathcal{H}_{(2)}^{\bullet}(X, \mathbf{C}).$$

Because of Poincaré duality, the class of Z is unique whenever it exists. We denote by $AH_{(2)}^{2r}(X, \mathbf{C})$ the subspace of $H_{(2)}^{2r}(X, \mathbf{C})$ spanned by the classes of algebraic cycles, and by $AH_{(2)}^{\langle a,b\rangle}(A, \mathbf{C})$ the subspace of $\mathcal{H}_{(2)}^{\langle a,b\rangle}(A, \mathbf{C})$ spanned by algebraic cycles.

We have seen (Lemma 4.3.1) that

$$j_P^*: \mathcal{H}_{(2)}^{\langle 0,2r\rangle}(A, \mathbf{C}) \longrightarrow H^{2r}(A_P, \mathbf{C})^\Gamma$$

is an isomorphism. Since j_P^* takes algebraic cycles to algebraic cycles, we have an injective homomorphism

$$\varphi_P\colon AH_{(2)}^{\langle 0,2r\rangle}(A,\mathbf{C}) \longrightarrow AH^{2r}(A_P,\mathbf{C})^\Gamma.$$

Lemma 4.4.1 shows that

$$j_{P*}\colon H^{2r}(A_P,\mathbf{C})^\Gamma \longrightarrow \mathcal{H}_{(2)}^{\langle 2d,2r\rangle}(A,\mathbf{C})$$

is an isomorphism. Since j_{P*} takes algebraic classes to algebraic classes (the class of Z in A_P goes to the class of Z in A), we have an injective homomorphism

$$\psi_P\colon AH^{2r}(A_P,\mathbf{C})^\Gamma \longrightarrow AH_{(2)}^{\langle 2d,2r\rangle}(A,\mathbf{C}).$$

The Hodge $*$-operator gives an isomorphism

$$\mathcal{H}_{(2)}^{\langle 0,2r\rangle}(A,\mathbf{C}) \longrightarrow \mathcal{H}_{(2)}^{\langle 2d,2m-r\rangle}(A,\mathbf{C}).$$

Lemma 4.4.1 shows that

$$(5.1)\qquad *\omega = *_0(1)\wedge *_P\omega = \mathrm{vol}(V)j_{P*}(*_P j_P^*\omega) \quad \text{for } \omega\in\mathcal{H}_{(2)}^{\langle 0,2r\rangle}(A,\mathbf{C}).$$

Since the Hodge $*$-operator on abelian varieties is algebraic [L, Theorem 3, p. 372, and Theorem 1, p. 367], we have an injective linear map

$$(5.2)\qquad *\colon AH_{(2)}^{\langle 0,2r\rangle}(A,\mathbf{C}) \longrightarrow AH_{(2)}^{\langle 2d,2m-2r\rangle}(A,\mathbf{C}).$$

CONJECTURE 5.3. *The map (5.2) is an isomorphism.*

REMARK 5.4. As remarked in the introduction, for compact A, Conjecture 5.3 is a special case of Grothendieck's standard conjectures. In the noncompact case, it may still be true that Conjecture 5.3 follows from the standard conjecture for a smooth compactification of A.

THEOREM 5.5. *If Conjecture 5.3 is true then*

$$\varphi_P\colon AH_{(2)}^{\langle 0,2r\rangle}(A,\mathbf{C}) \longrightarrow AH^{2r}(A_P,\mathbf{C})^\Gamma$$

is an isomorphism for every $P\in V$. In particular, the space of Γ-invariant algebraic cycles on A_p is independent of P, and the invariant cycles conjecture (Conjecture 1.1) is true.

PROOF. Let $\omega_P \in AH^{2r}(A_P,\mathbf{C})^\Gamma$. Then $\omega_P = j_P^*(\omega)$ for a unique $\omega\in\mathcal{H}_{(2)}^{\langle 0,2r\rangle}(A,\mathbf{C})$ (Lemma 4.3.1). (5.1) shows that $*\omega = \mathrm{vol}(V)j_{P*}(*\omega_P)$ is algebraic. Conjecture 5.3 implies that ω is algebraic, so $\omega\in AH_{(2)}^{\langle 0,2r\rangle}(A,\mathbf{C})$, and $\varphi_P(\omega)=\omega_P$. Thus φ_P is surjective, and the proof is complete. ■

REMARK 5.6. Suppose P is a generic point of V in the sense of Weil (over some field of definition). Then Kuga has shown (without assuming the standard conjectures) that every algebraic cycle on A_P is Γ-invariant [K2, Theorem, p. 76], and φ_P is an isomorphism [K4, 1.4.8, p. 18] (see also [K2, p. 77], [HK, p. 5], [T, p. 108]). His proofs assume that A is compact, but we can remove this assumption by using L_2-cohomology.

https://doi.org/10.4153/CJM-1994-063-0 Published online by Cambridge University Press

6. **The Hodge conjecture.**

MAIN THEOREM 6.1. *(a) If the invariant cycles conjecture (Conjecture 1.1) is true for all Kuga fiber varieties of Hodge type, then the Hodge conjecture is true for all abelian varieties.*

(b) If Conjecture 5.3 is true for all Kuga fiber varieties of Hodge type, then the Hodge conjecture is true for all abelian varieties.

The proof of Theorem 6.1 appears at the end of this section.

LEMMA 6.2. *If Conjecture 1.1 is true for all Kuga fiber varieties of Hodge type, then the Hodge conjecture for abelian varieties of* CM*-type implies the Hodge conjecture for all abelian varieties.*

PROOF. The proof is similar to the proof of the corresponding fact for absolute Hodge cycles [D2, p. 71]. Let A_0 be an abelian variety. The inclusion of the Hodge group of A_0 into the symplectic group defines a Hodge family, $A \longrightarrow V$, which has a fiber A_1 of CM-type [Mm2, p. 348]. The fundamental group, Γ, of V is contained in the Hodge group of A_0; hence any Hodge cycle ω_0 on A_0 is Γ-invariant. Then, the Theorem of the Fixed Part [D1, Corollaire 4.1.2, p. 42] implies that ω_0 determines a Hodge cycle ω_1 on A_1, which is algebraic by hypothesis. Thus Conjecture 1.1 implies that ω_0 is an algebraic cycle. ∎

6.3 *Weil cycles.* Let E be a CM-field, k the totally real subfield of E, and T a skew-hermitian form on $F := E^{2p}$. We assume that T is *split, i.e.*, there exists a totally isotropic subspace of F of dimension p. Let G be the restriction from k to $\mathbf{Q}$ of the special unitary group of T; then

$$G(\mathbf{Q}) = \{\alpha \in \mathrm{SL}_{2p}(E) \mid T(\alpha x, \alpha y) = T(x, y)\}.$$

G is a semisimple group of hermitian type; our assumption on T implies that $G(\mathbf{R}) \cong \mathrm{SU}(p,p)^g$, where $2g = [E : \mathbf{Q}]$ [D2, Corollary 4.2, p. 51]. The associated symmetric domain is given by

$$X := \{z \in M_p(\mathbf{C}) \mid I - z^t\bar{z} > 0\}^g. \tag{6.3.1}$$

Let $\beta := \mathrm{tr}_{E/\mathbf{Q}} T$. There is a unique holomorphic map $\tau: X \longrightarrow \mathfrak{S}(F, \beta)$ which is strongly equivariant with the inclusion $\rho: G \longrightarrow \mathrm{Sp}(F, \beta)$ [Sal, §1.5, pp. 432–433]. Furthermore, the pair (ρ, τ) satisfies the H_2-condition (see [Sa2, pp. 182–183, especially (4.10) and (4.14)]). Therefore, choosing a torsion-free arithmetic subgroup Γ of G, and a Γ-lattice L in F, we have a Kuga fiber variety $A \longrightarrow V$. If Γ is a principal congruence subgroup of G, then A is a PEL-family as defined by Shimura [Sm].

Since the H_2-condition is satisfied by this family of abelian varieties, every G-invariant cycle in the cohomology of a fiber A_P is a Hodge cycle (Proposition 2.2). The subspace $\Lambda_E^{2p} F^*$ of $\Lambda_{\mathbf{Q}}^{2p} F^* = H^{2p}(A_P, \mathbf{Q})$ is a 1-dimensional vector space over E, generated by the determinant; hence it consists of Hodge cycles. These cycles are called *Weil cycles*. Weil [W] considered them for E an imaginary quadratic field.

PROPOSITION 6.4 (ANDRÉ [AN1]). *If all Weil cycles are algebraic, then the Hodge conjecture is true for all abelian varieties of* CM-*type.* ∎

PROOF OF THEOREM 6.1. Note that (b) follows from (a) and Theorem 5.5. Lemma 6.2 and Proposition 6.4 show that the Hodge conjecture for all abelian varieties is a consequence of the algebraicity of Weil cycles, assuming Conjecture 1.1. To prove that a Weil cycle on a fiber A_P is algebraic (modulo Conjecture 1.1) it suffices to show that it is algebraic on any one member of the family. In the special case where E is imaginary quadratic, Weil pointed out [W, p. 425] that these cycles do become algebraic in special fibers. In fact, this observation of Weil motivated this entire paper.

Let $F_0 := \mathbf{Q}^{2p}$, so that $F = F_0 \otimes E$, and let β_0 be any nondegenerate alternating form on F_0. Define a skew-hermitian form T' on F by

$$T'(ax, by) := a\bar{b}\beta_0(x,y), \quad \text{for } x, y \in F_0,\ a, b \in E.$$

Let I_0 be a maximal totally isotropic subspace for β_0 on F_0; its dimension is p. Then $I := I_0 \otimes E$ is a p-dimensional totally isotropic subspace for T' on F, so T' is split, and $(F, T') \cong (F, T)$ [D2, Corollary 4.2, p. 51]. We identify them for simplicity. The Siegel space $\mathfrak{S}(F_0, \beta_0)$ may be identified (*cf.* [Sa2, Theorem 7.5, p. 81]), via the Harish-Chandra embedding, with the bounded domain

$$X_0 := \{z \in M_p(\mathbf{C}) \mid {}^t z = z, I - z{}^t\bar{z} > 0\}.$$

There is therefore a map τ_0 of X_0 into the symmetric domain X (6.3.1), whose projection to each factor is the identity. This is strongly equivariant with the inclusion $\mathrm{Sp}(F_0, \beta_0) \subset G$; in fact the H_2-condition is satisfied. We thus get a subfamily of $A \longrightarrow V$, whose general member has Hodge group $\mathrm{Sp}(F_0, \beta_0)$ by Proposition 2.2. The invariant theory of the symplectic group shows that the Hodge ring of such an abelian variety is generated in degree 2. (This is stated in [K2, Proposition, p. 80]; a complete proof may be found in [R, pp. 528–530]). Thus all Weil cycles become algebraic in this subfamily. ∎

REFERENCES

[Ab1] S. Abdulali, *Absolute Hodge Cycles in Kuga Fiber Varieties*, Thesis, State University of New York, Stony Brook, 1985.

[Ab2] ———, *Zeta functions of Kuga fiber varieties*, Duke Math. J. **57**(1988), 333–345.

[Ab3] ———, *Conjugates of strongly equivariant maps*, Pacific J. Math., to appear.

[An1] Y. André, *Une remarque à propos des cycles de Hodge de type* CM. In: Séminaire de Théorie des Nombres, Paris, 1989–90, (ed. S. David), Progr. Math. **102**, Birkhäuser, Boston, 1992, 1–7.

[An2] ———, *Pour une théorie inconditionelle des motifs*, preprint.

[B] A. Borel, *Density and maximality of arithmetic subgroups*, J. Reine Angew. Math. **224**(1966), 78–89.

[BC] A. Borel and W. Casselman, *L^2-cohomology of locally symmetric manifolds of finite volume*, Duke Math. J. **50**(1983), 625–647.

[Ca] W. Casselman, *Introduction to the L^2-cohomology of arithmetic quotients of bounded symmetric domains*. In: Complex Analytic Singularities, (eds. T. Suwa and P. Wagreich), Adv. Stud. Pure Math. **8**, Kinokuniya, Tokyo, 1986, 69–93.

[Ch] J. Cheeger, *On the Hodge theory of Riemannian pseudomanifolds*. In: Geometry of the Laplace Operator, (eds. R. Osserman and A. Weinstein), Proc. Sympos. Pure Math. **36**, Amer. Math. Soc., Providence, 1980, 91–146.

[CGM] J. Cheeger, M. Goresky and R. MacPherson, *L^2-cohomology and intersection homology of singular algebraic varieties*. In: Seminar on Differential Geometry, (ed. S.-T. Yau), Ann. of Math. Stud. **102**, Princeton Univ. Press, Princeton, 1982, 303–340.

[D1] P. Deligne, *Théorie de Hodge II*, Inst. Hautes Études Sci. Publ. Math. **40**(1971), 5–57.

[D2] P. Deligne (notes by J. S. Milne), *Hodge cycles on abelian varieties*. In: P. Deligne, J. S. Milne, A. Ogus and K.-y. Shih, *Hodge cycles, Motives, and Shimura Varieties*, Lecture Notes in Math. **900**, Springer-Verlag, Berlin, Heidelberg, New York, 1982, 9–100.

[Ga] M. Gaffney, *A special Stokes's theorem for complete Riemannian manifolds*, Ann. of Math. (2) **60**(1954), 140–145.

[Gf] P. Griffiths, *Periods of integrals on algebraic manifolds, III (some global differential-geometric properties of the period mapping)*, Inst. Hautes Études Sci. Publ. Math. **38**(1970), 125–180.

[Gk1] A. Grothendieck, *On the de Rham cohomology of algebraic varieties*, Inst. Hautes Études Sci. Publ. Math. **29**(1966), 96–103.

[Gk2] ———, *Standard conjectures on algebraic cycles*. In: Algebraic Geometry (Papers presented at the Bombay Colloquium, 1968), Tata Inst. Fund. Res. Studies in Math. **4**, Oxford University Press, Bombay-London, 1969, 193–199.

[HK] R. Hall and M. Kuga, *Algebraic cycles in a fiber variety*, Sci. Papers College Gen. Ed. Univ. Tokyo **25**(1975), 1–6.

[K1] M. Kuga, *Fiber Varieties over a Symmetric Space whose Fibers are Abelian Varieties*, Vol. I, Lecture Notes, Univ. Chicago, Chicago, 1964.

[K2] ———, *Fibred variety over symmetric space whose fibres are abelian varieties*. In: Proceedings of the U.S.-Japan Seminar in Differential Geometry, Kyoto, 1965, Nippon Hyoronsha, Tokyo, 1966, 72–81.

[K3] ———, *Fiber varieties over a symmetric space whose fibers are abelian varieties*. In: Algebraic Groups and Discontinuous Subgroups, (eds. A. Borel and G. D. Mostow), Proc. Sympos. Pure Math. **9**, Amer. Math. Soc., Providence, 1966, 338–346.

[K4] ———, *Algebraic cycles in gtfabv*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **29**(1982), 13–29.

[L] D. Lieberman, *Numerical and homological equivalence of algebraic cycles on Hodge manifolds*, Amer. J. Math. **90**(1968), 366–374.

[Mm1] D. Mumford, *Families of abelian varieties*. In: Algebraic Groups and Discontinuous Subgroups, (eds. A. Borel and G. D. Mostow), Proc. Sympos. Pure. Math. **9**, Amer. Math. Soc., Providence, 1966, 347–351.

[Mm2] ———, *A note of Shimura's paper "Discontinuous groups and abelian varieties"*, Math. Ann. **181** (1969), 345–351.

[R] K. Ribet, *Hodge classes on certain types of abelian varieties*, Amer. J. Math. **105**(1983), 523–538.

[Sa1] I. Satake, *Holomorphic imbeddings of symmetric domains into a Siegel space*, Amer. J. Math. **87**(1965), 425–461.

[Sa2] ———, *Algebraic Structures of Symmetric Domains*, Publ. Math. Soc. Japan **14**, Kanô Mem. Lect. **4**, Iwanami Shoten, Japan and Princeton Univ. Press, Princeton, 1980.

[Sc1] C. Schoen, *Hodge classes on self-products of a variety with an automorphism*, Compositio Math. **65** (1988), 3–32.

[Sc2] ———, *Cyclic covers of P^v branched along $v+2$ hyperplanes and the generalized Hodge conjecture for certain abelian varieties*. In: Arithmetic of Complex Manifolds, (eds. W.-P. Barth and H. Lange), Lecture Notes in Math. **1399**, Springer-Verlag, Berlin, Heidelberg, New York, 1989, 137–154.

[Sm] G. Shimura, *Moduli of abelian varieties and number theory*. In: Algebraic Groups and Discontinuous Subgroups, (eds. A. Borel and G. D. Mostow), Proc. Sympos. Pure Math. **9**, Amer. Math. Soc., Providence, 1966, 312–332.

[So] T. Shioda, *Algebraic cycles on abelian varieties of Fermat type*, Math. Ann. **258**(1981), 65–80.

[T] M.-c. Tjiok, *Algebraic Cycles in a Certain Fiber Variety*, Thesis, State University of New York, Stony Brook, 1980.

[W] A. Weil, *Abelian varieties and the Hodge ring*. In: Œuvres Scientifiques Collected Papers, Vol. III, 1964–1978, Springer-Verlag, New York, 1979, Corrected second printing, 1980, 421–429.

[Z1] S. Zucker, *Cohomology of warped products and arithmetic groups*, Invent. Math. **70**(1982), 169–218.
[Z2] ———, *L^2-cohomology of Shimura varieties*. In: Automorphic Forms, Shimura Varieties, and L-functions, (eds. L. Clozel and J. S. Milne), Vol. 2, Perspect. Math. **11**, Academic Press, San Diego, 1990, 377–391.

Department of Mathematics
University of Toronto
Toronto, Ontario
M5S 1A1

Current address:
Department of Mathematics and Computer Science
Clark University
Worcester, Massachusetts 01610
U.S.A.

https://doi.org/10.4153/CJM-1994-063-0 Published online by Cambridge University Press