Canad. J. Math. Vol. **56** (5), 2004 pp. 926–944

K-Homology of the Rotation Algebras A_θ

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Abstract. We study the K-homology of the rotation algebras A_θ using the six-term cyclic sequence for the K-homology of a crossed product by $\mathbf{Z}$. In the case that θ is irrational, we use Pimsner and Voiculescu's work on AF-embeddings of the A_θ to search for the missing generator of the even K-homology.

1 Introduction

In this paper we are concerned with the two-dimensional noncommutative tori, the rotation algebras A_θ. For $\theta \in [0, 1)$, we define A_θ to be the universal C*-algebra generated by unitaries U and V satisfying the relation $VU = \lambda UV$, where $\lambda = e^{2\pi i\theta}$. These algebras have been extensively studied from many different viewpoints. A thorough overview of the literature appears in Rieffel's survey article [9].

We study the K-homology of the rotation algebras, by which we mean the Kasparov groups $KK^i(A_\theta, \mathbf{C})$ $(i = 0, 1)$, and in particular we are interested in exhibiting the generating Fredholm modules. We make extensive use of the six-term cyclic sequence for K-homology of a crossed product by $\mathbf{Z}$, dual to the Pimsner-Voiculescu sequence [5] on K-theory. In the commutative situation $\theta = 0$, all four generators of the K-homology can be exhibited concretely. Three of these Fredholm modules generalize immediately to the case where $\theta \neq 0$; however the canonical "zero dimensional" Fredholm module $\mathbf{z}_0$ vanishes. In the final section of this paper we attempt to describe this missing generator, via Pimsner and Voiculescu's work on embedding the A_θ in AF-algebras [6].

We note that the K-homology of the rotation algebras was previously studied by Popa and Rieffel in [7], who calculated the Ext groups. This approach predated the formalism of Fredholm modules.

2 Fredholm Modules as K-Homology

Recall that a Fredholm module over a $*$-algebra A is a triple (H, π, F), where π is a $*$-representation of A as bounded operators on the Hilbert space H. The operator F is a selfadjoint element of $\mathbf{B}(H)$, satisfying $F^2 = 1$, such that the commutators $[F, \pi(a)]$ are compact operators for all $a \in A$. Such a Fredholm module is called odd.

An even Fredholm module is the above data, together with a $\mathbf{Z}_2$-grading of the Hilbert space H, given by a grading operator $\gamma \in \mathbf{B}(H)$ with $\gamma = \gamma^*$, $\gamma^2 = 1$, $[\gamma, \pi(a)] = 0$ for all $a \in A$, and $F\gamma = -\gamma F$. In general the $*$-algebra A will be a

Received by the editors January 7,2002; revised February 27, 2003.
AMS subject classification: Primary 58B34; Secondary 19K33, 46L.

dense subalgebra of a C^*-algebra, closed under the holomorphic functional calculus. Fredholm modules should be thought of as abstract elliptic operators, since they are motivated by axiomatizing the important properties of elliptic pseudodifferential operators on closed manifolds.

This definition is due to Connes [3, p. 288]. In Kasparov's framework the K-homology groups are given by specializing the second variable in the KK-functor to be the complex numbers **C**. Equivalence classes of even Fredholm modules make up the even K-homology group $KK^0(A, \mathbf{C})$. Odd Fredholm modules make up the odd K-homology $KK^1(A, \mathbf{C})$. A Fredholm module is said to be degenerate if $[F, \pi(a)] = 0$ for all $a \in A$. Degenerate Fredholm modules represent the identity element of the corresponding K-homology group.

Two simple examples of an even and an odd Fredholm module that we will use extensively in the sequel, are as follows. Let A be a C^*-algebra, H a finite-dimensional Hilbert space, and $\varphi\colon A \to \mathbf{B}(H)$ a $*$-homomorphism.

Example 1 We construct a canonical even Fredholm module $\mathbf{z}_0 \in KK^0(A, \mathbf{C})$:

$$\mathbf{z}_0 = \left(H_0 = H \oplus H, \pi_0 = \varphi \oplus 0, F_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right).$$

In general $\mathbf{z}_0$ may well represent a trivial element of the even K-homology of A (for example, if φ is the zero homomorphism). However, if A is unital, and φ is a nonzero $*$-homomorphism, the Chern character of $\mathbf{z}_0$ pairs nontrivially with $[1] \in K_0(A)$, showing that $\mathbf{z}_0$ is a nontrivial element of K-homology, and also that $[1] \neq 0 \in K_0(A)$. More precisely:

Lemma 1 *Suppose that $e = (e_{ij}) \in M_q(A)$ is a projection, $e = e^* = e^2$. Then*

$$\langle ch_*(\mathbf{z}_0), [e]\rangle = \sum_{k=1}^{q} \operatorname{Tr}(\varphi(e_{kk})).$$

In particular, if A is unital then $\langle ch_*(\mathbf{z}_0), [1]\rangle = \operatorname{Tr}(\varphi(1))$.

Proof Here, $ch_*\colon KK^0(A, \mathbf{C}) \to HC^{even}(A)$, is the even Chern character as defined in [3, p. 295], mapping the even K-homology of A into even periodic cyclic cohomology, and $\langle \cdot, \cdot \rangle$ denotes the pairing between K-theory and periodic cyclic cohomology defined in [3, p. 224]. We give all the details of this calculation, to avoid later repetition. We have

$$\langle ch_*(\mathbf{z}_0), [e]\rangle = \lim_{n\to\infty} (n!)^{-1} \sum_{i_0, i_1, \dots, i_{2n}=1}^{q} \psi_{2n}(e_{i_0,i_1}, e_{i_1,i_2}, \dots, e_{i_{2n},i_0}),$$

where (for each n) ψ_{2n} is the cyclic $2n$-cocycle defined by

$$\psi_{2n}(a_0, a_1, \dots, a_{2n}) = (-1)^{n(2n-1)} \Gamma(n+1) \operatorname{Tr}(\gamma\pi_0(a_0)[F_0, \pi_0(a_1)] \cdots [F_0, \pi_0(a_{2n})]).$$

Since $\Gamma(n+1) = n!$ it follows that

$$\langle ch_*(\mathbf{z}_0), [e]\rangle =$$
$$\lim_{n\to\infty} (-1)^n \sum_{i_0,i_1,\dots,i_{2n}=1}^{q} \operatorname{Tr}\left(\gamma\pi_0(e_{i_0,i_1})[F_0,\pi_0(e_{i_1,i_2})]\cdots[F_0,\pi_0(e_{i_{2n},i_0})]\right).$$

Now, for any $a \in A$, $[F_0, \pi_0(a)] = \left(\begin{smallmatrix} 0 & -\varphi(a) \\ \varphi(a) & 0 \end{smallmatrix}\right)$, and hence

$$(1) \quad \sum_{i_0,i_1,\dots,i_{2n}=1}^{q} \operatorname{Tr}(\gamma\pi_0(e_{i_0,i_1})[F_0,\pi_0(e_{i_1,i_2})]\cdots[F_0,\pi_0(e_{i_{2n},i_0})])$$
$$= \sum_{i_0=1}^{q} (-1)^n \operatorname{Tr}\left(\begin{pmatrix} \varphi(e_{i_0,i_0}) & 0 \\ 0 & 0 \end{pmatrix}\right) = (-1)^n \sum_{k=1}^{q} \operatorname{Tr}(\varphi(e_{k,k})).$$

Therefore

$$\langle ch_*(\mathbf{z}_0), [e]\rangle = \sum_{k=1}^{q} \operatorname{Tr}(\varphi(e_{k,k}))$$

as claimed. If A is unital, and we take $e = 1$, then the right hand side is $\operatorname{Tr}(\varphi(1))$. ■

Example 2 We also describe a canonical odd Fredholm module

$$\mathbf{z}_1 \in KK^1(A \times_\alpha \mathbf{Z}, \mathbf{C}).$$

For convenience we suppose that A is unital, so that elements $a \in A$ and the canonical unitary V implementing the automorphism α (via $VaV^* = \alpha(a)$) can be exhibited as elements of the crossed product algebra, rather than just as multipliers.

We have

$$\mathbf{z}_1 = \left(H_1 = l^2(\mathbf{Z}, H), \pi_1, F_1\right).$$

Take $\pi_1 \colon A \times_\alpha \mathbf{Z} \to \mathbf{B}(l^2(\mathbf{Z}, H))$ to be defined by

$$(\pi_1(a)\xi)(n) = \varphi(\alpha^{-n}(a))\,\xi(n), \quad (\pi_1(V)\xi)(n) = \xi(n-1),$$

for $\xi \in l^2(\mathbf{Z}, H)$, $a \in A$, and V the canonical unitary implementing the action α of $\mathbf{Z}$ on A.

In other words, π_1 is the usual representation of $A \times_\alpha \mathbf{Z}$ induced from the representation φ of A. Take

$$F_1\xi(n) = \operatorname{sign}(n)\xi(n) = \begin{cases} \xi(n) & n \geq 0, \\ -\xi(n) & n < 0. \end{cases}$$

It is immediate that $[F_1, \pi_1(a)] = 0$ for all $a \in A$. Further, $[F_1, \pi_1(V)]$ is a finite-rank operator and hence compact, provided H is finite-dimensional. Nontriviality of $\mathbf{z}_1$ (as long as H is nonzero) follows from:

Lemma 2 $\langle ch_*(\mathbf{z}_1), [V] \rangle = \dim(H)$.

Proof Again, $ch_*\colon KK^1(A, \mathbf{C}) \to HC^{odd}(A)$, is the odd Chern character as defined in [3, p. 296], mapping the odd K-homology of A into odd periodic cyclic cohomology. It is straightforward to calculate this pairing directly; however it is quicker to appeal to Connes' index theorem [3, p. 296], which states that

$$\langle ch_*(\mathbf{z}_1), [V] \rangle = \mathrm{Index}(EVE),$$

where $E = \frac{1}{2}(1 + F)$ is the natural orthogonal projection $l^2(\mathbf{Z}, H) \to l^2(\mathbf{N}, H)$. We have

$$\mathrm{Index}(EVE) = \dim\ker(EVE) - \dim\ker(EV^*E) = \dim H - 0 = \dim H,$$

which proves the result. This shows that, provided A is unital, $\mathbf{z}_1$ is a nontrivial Fredholm module, and also that $[V] \neq 0 \in K_1(A \times_\alpha \mathbf{Z})$. ■

The Fredholm module $\mathbf{z}_0$ can be defined more generally, by taking $\varphi\colon A \to \mathbf{K}(H)$, with $\mathbf{K}(H)$ the algebra of compact operators on a (not necessarily finite-dimensional) Hilbert space H. This will not work for $\mathbf{z}_1$, since in this situation the commutator $[F, \pi_1(V)]$ fails to be compact, unless H is finite dimensional. A very useful special case is when we just have $\varphi\colon A \to \mathbf{C}$.

3 Six Term Cyclic Sequence for K-Homology

We now consider the six term cyclic exact sequence for K-homology of crossed products by $\mathbf{Z}$, dual to the Pimsner-Voiculescu sequence for K-theory, as described in [2, p. 199].

Let A be a unital C^*-algebra. Then associated to each crossed product algebra $A \times_\alpha \mathbf{Z}$ is the following semisplit short exact sequence of C^*-algebras, the Pimsner-Voiculescu "Toeplitz extension" [5]:

$$0 \to A \otimes \mathbf{K} \to T_\alpha \to A \times_\alpha \mathbf{Z} \to 0.$$

Here T_α is the C^*-subalgebra of $(A \times_\alpha \mathbf{Z}) \otimes T$ generated by $a \otimes 1$, $a \in A$ and $V \otimes f$, where V is the unitary implementing the action of α on A, and f is the non-unitary isometry generating the ordinary Toeplitz algebra T, that is $f \in \mathbf{B}(l^2(\mathbf{N}))$, $fe_n = e_{n+1}$. This extension defines the Toeplitz element $\mathbf{x} \in KK^1(A \times_\alpha \mathbf{Z}, A)$.

Applying the K-functor gives the Pimsner-Voiculescu six term cyclic sequence for K-theory. The corresponding six term cyclic sequence for K-homology is:

$$\begin{array}{ccccc} KK^0(A, \mathbf{C}) & \xleftarrow{id-\alpha^*} & KK^0(A, \mathbf{C}) & \xleftarrow{i^*} & KK^0(A \times_\alpha \mathbf{Z}, \mathbf{C}) \\ {\scriptstyle\partial_0}\downarrow & & & & \uparrow{\scriptstyle\partial_1} \\ KK^1(A \times_\alpha \mathbf{Z}, \mathbf{C}) & \xrightarrow{i^*} & KK^1(A, \mathbf{C}) & \xrightarrow{id-\alpha^*} & KK^1(A, \mathbf{C}) \end{array}$$

Here i denotes the canonical inclusion map $i\colon A \hookrightarrow A \times_\alpha \mathbf{Z}$. The vertical maps ∂_0 and ∂_1 are given by taking the Kasparov product with the Toeplitz element:

$$\partial_i\colon KK^i(A,\mathbf{C}) \to KK^{i+1}(A \times_\alpha \mathbf{Z}, \mathbf{C}), \quad \mathbf{z} \mapsto \mathbf{x}\widehat{\otimes}_A \mathbf{z}.$$

This sequence formulated in terms of Ext appears in the original paper of Pimsner and Voiculescu [5]. However, the relationship between Ext and the Fredholm module picture of K-homology is not transparent.

Let A be a unital C^*-algebra, with a finite-dimensional representation $\varphi\colon A \to \mathbf{B}(H)$. We assume for convenience that $\varphi(A)H = H$. Then the Fredholm modules $\mathbf{z}_0$ and $\mathbf{z}_1$ described above (Examples 1 and 2) are related via the morphism ∂_0 as follows.

Proposition 3 *Under the map ∂_0 we have $\partial_0(\mathbf{z}_0) = \mathbf{z}_1$.*

Proof We describe the Pimsner-Voiculescu Toeplitz element $\mathbf{x} \in KK^1(A\times_\alpha\mathbf{Z}, A)$ as the Kasparov triple

$$(E_1, \phi_1, F_1) \in \mathbf{E}(A\times_\alpha\mathbf{Z}, A\widehat{\otimes}\mathbf{C}_1). \tag{2}$$

Here, $\mathbf{E}(B, D)$ denotes the set of Kasparov triples over a pair of C^*-algebras B, D; [2, p. 143]. We take $E_1 = l^2(\mathbf{Z}, A)\widehat{\otimes}\mathbf{C}_1$, with the obvious $A\widehat{\otimes}\mathbf{C}_1$-valued inner product. Here $\widehat{\otimes}$ is the graded tensor product of Hilbert modules, while $\mathbf{C}_1$ is the Clifford algebra of the one dimensional complex vector space $\mathbf{C}$, and is generated by elements 1 and ϵ, with $\epsilon^2 = 1$. We have $\phi_1\colon A\times_\alpha\mathbf{Z} \to \mathbf{B}(E_1)$ given by

$$\phi_1(x)(\xi\widehat{\otimes}\omega) = {\phi_1}'(x)\xi\widehat{\otimes}\omega$$

where

$$({\phi_1}'(a)\xi)(n) = \alpha^n(a)\xi(n), \quad ({\phi_1}'(V)\xi)(n) = \xi(n+1)$$

for $a \in A$, $\xi \in l^2(\mathbf{Z}, A)$ and V is the canonical unitary multiplier of $A \times_\alpha \mathbf{Z}$. The operator F_1 is given by $F_1 = F\,\widehat{\otimes}\,\epsilon$, with $F\xi(n) = \operatorname{sign}(n)\xi(n)$.

The canonical even Fredholm module $\mathbf{z}_0$ corresponds to the Kasparov triple

$$\left(H \oplus H, \phi_0 = \varphi \oplus 0, F_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) \in \mathbf{E}(A, \mathbf{C}).$$

Recall that, given C^*-algebras A, B and D, and Kasparov triples $(E_1, \phi_1, F_1) \in \mathbf{E}(A, D)$ and $(E_2, \phi_2, F_2) \in \mathbf{E}(D, B)$, then the product is given [2, p. 166], by the triple

$$(E = E_1\widehat{\otimes}_{\phi_2}E_2, \phi = \phi_1\,\widehat{\otimes}\,1, F) \in \mathbf{E}(A, B) \tag{3}$$

where F is a "suitable" combination of F_1 and F_2. Almost all the difficulties involved in calculating the product lie in finding the correct F.

https://doi.org/10.4153/CJM-2004-042-x Published online by Cambridge University Press

Our calculation of the product proceeds in three steps. First, the triples representing $\mathbf{x}$ and $\mathbf{z}_0$ need to be compatible, in that $\mathbf{z}_0$ should be represented by an element of $\mathbf{E}(A \mathbin{\widehat{\otimes}} \mathbf{C}_1, \,\cdot\,)$ rather than $\mathbf{E}(A, \,\cdot\,)$. This is achieved via the morphism [2, p. 160]:

(4) $$\tau_{\mathbf{C}_1} \colon \mathbf{E}(A, \mathbf{C}) \to \mathbf{E}(A \widehat{\otimes} \mathbf{C}_1, \mathbf{C}_1), \quad (E, \phi, F) \mapsto (E \mathbin{\widehat{\otimes}} \mathbf{C}_1, \phi \mathbin{\widehat{\otimes}} \mathrm{id}, F \mathbin{\widehat{\otimes}} 1).$$

where
$$\big((\phi \mathbin{\widehat{\otimes}} \mathrm{id})(a \mathbin{\widehat{\otimes}} \omega)\big)\,(\xi \mathbin{\widehat{\otimes}} \omega') = \phi(a)\xi \mathbin{\widehat{\otimes}} \omega\omega'.$$

The second step is to calculate the product $\mathbf{x} \widehat{\otimes}_{A \mathbin{\widehat{\otimes}} \mathbf{C}_1} \tau_{\mathbf{C}_1}(\mathbf{z}_0)$, following the procedure outlined above (3).

This gives us a triple in $\mathbf{E}(A \times_\alpha \mathbf{Z}, \mathbf{C}_1)$.

The final step is to show that this triple represents the same element of $KK^1(A \times_\alpha \mathbf{Z}, \mathbf{C})$ as $\mathbf{z}_1$.

Step One We apply the map $\tau_{\mathbf{C}_1}$ to get

$$\tau_{\mathbf{C}_1}(\mathbf{z}_0) = \big((H \oplus H)\widehat{\otimes}\mathbf{C}_1, \phi_0 \widehat{\otimes}\, \mathrm{id}, F_0 \widehat{\otimes} 1\big) = (E_2, \phi_2, F_2) \in \mathbf{E}(A \mathbin{\widehat{\otimes}} \mathbf{C}_1, \mathbf{C}_1).$$

Step Two Now we can take the product. We have

$$\partial_0(\mathbf{z}_0) = \mathbf{x} \widehat{\otimes}_{A \mathbin{\widehat{\otimes}} \mathbf{C}_1} \tau_{\mathbf{C}_1}(\mathbf{z}_0) = (E, \phi, F) \in \mathbf{E}(A \times_\alpha \mathbf{Z}, \mathbf{C}_1).$$

Here $E = E_1 \widehat{\otimes}_{\phi_0 \widehat{\otimes} 1} E_2$, $\phi = \phi_1 \mathbin{\widehat{\otimes}} 1$, and F is yet to be found. As elements of

$$E \cong (l^2(\mathbf{Z}, A) \mathbin{\widehat{\otimes}} \mathbf{C}_1) \mathbin{\widehat{\otimes}} ((H \oplus H) \mathbin{\widehat{\otimes}} \mathbf{C}_1)$$

we can identify

$$(\delta_k a_k \mathbin{\widehat{\otimes}} \omega_1) \widehat{\otimes} (\mathbf{v} \widehat{\otimes} \omega_2) \sim (\delta_k \widehat{\otimes} 1) \widehat{\otimes} \big(\phi_0(a_k)\mathbf{v} \widehat{\otimes} \omega_1\omega_2\big)\,.$$

So we can identify E as a submodule of $l^2(\mathbf{Z}, H \oplus H)\widehat{\otimes}\mathbf{C}_1$ via the morphism

$$(\delta_k a_k \widehat{\otimes} \omega_1)\widehat{\otimes}(\mathbf{v}\widehat{\otimes}\omega_2) \mapsto \delta_k \phi_0(a_k)\mathbf{v} \mathbin{\widehat{\otimes}} \omega_1\omega_2.$$

Since by assumption $\varphi(A)H = H$ and $\phi_0(a) = \left(\begin{smallmatrix} \varphi(a) & 0 \\ 0 & 0 \end{smallmatrix}\right)$, the image of this morphism can be naturally identified with $l^2(\mathbf{Z}, H) \mathbin{\widehat{\otimes}} \mathbf{C}_1$. After this identification, $\phi = \phi_1 \widehat{\otimes} 1$ acts via
$$\phi(x)(\xi \mathbin{\widehat{\otimes}} \omega) = \phi'(x)\xi \mathbin{\widehat{\otimes}} \omega$$

with
$$(\phi'(a)\xi)(n) = \varphi(\alpha^n(a))\xi(n), \quad (\phi'(V)\xi)(n) = \xi(n+1)$$

for $a \in A$ and $\xi \in l^2(\mathbf{Z}, H)$.

We use the Connes-Skandalis formalism of connections [2, p. 170], to find a suitable F. By [2, Prop. 18.10.1], such an F will be given by

(5) $$F = F_1 \mathbin{\widehat{\otimes}} 1 + ((1 - {F_1}^2)^{1/2} \mathbin{\widehat{\otimes}} 1)G$$

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where G is an $(F_2 \widehat{\otimes} 1)$-connection. By [2, Prop. 18.3.3], abstractly such a connection G must exist. In this situation, since ${F_1}^2 = 1$, we can just take $F = F_1 \widehat{\otimes} 1$, and there is no need to find a concrete G.

This defines F as an operator on E. Under our identification of E with $l^2(\mathbf{Z}, H) \widehat{\otimes} \mathbf{C}_1$, we have

$$F(\xi \widehat{\otimes} \omega) = F'\xi \widehat{\otimes} \epsilon\omega$$

with $F'\xi(n) = \operatorname{sign}(n)\xi(n)$, and ϵ is the generator of the Clifford algebra $\mathbf{C}_1$ of $\mathbf{C}$ defined previously. We note that E was originally defined as a submodule of $l^2(\mathbf{Z}, H \oplus H) \widehat{\otimes} \mathbf{C}_1$, and this submodule is invariant under the actions of $A \times_\alpha \mathbf{Z}$ and F defined above. Hence identifying E with $l^2(\mathbf{Z}, H) \widehat{\otimes} \mathbf{C}_1$ is well-defined.

Therefore, we have calculated the product and obtained a triple

$$(E, \phi, F) \in \mathbf{E}(A \times_\alpha \mathbf{Z}, \mathbf{C}_1) \tag{6}$$

representing $\partial_0(\mathbf{z}_0)$.

Step Three An odd Fredholm module $(H_1, \pi_1, F_1) \in KK^1(A \times_\alpha \mathbf{Z}, \mathbf{C})$ is represented by the Kasparov triple $(H_1 \widehat{\otimes} \mathbf{C}_1, \pi_1 \widehat{\otimes} 1, F_1 \widehat{\otimes} \epsilon) \in \mathbf{E}(A \times_\alpha \mathbf{Z}, \mathbf{C}_1)$. It is immediate to see that the Kasparov triple described in (6) corresponding to the product $\partial_0(\mathbf{z}_0)$ represents the Fredholm module $\mathbf{z}_1$ exactly as in Example 2. Hence $\partial_0(\mathbf{z}_0) = \mathbf{z}_1$ as Fredholm modules. This completes the proof. ∎

4 Application to the Rotation Algebras A_θ

We now illustrate this work with the example of the rotation algebras A_θ. Since the A_θ are deformations of the commutative algebra $A_0 = C(\mathbf{T}^2)$ (the case $\theta = 0$) we consider this case first.

Proposition 4 *We have* $KK^i(A_0, \mathbf{C}) \cong \mathbf{Z}^2$, $i = 0, 1$.

Proof It is well known that the K-groups $K_i(A_0)$ $(i = 0, 1)$ are both isomorphic to $\mathbf{Z}^2$. The generators of $K_0(A_0)$ are $[1]$ and the Bott projector $[Bott]$. The generators of $K_1(A_0)$ are $[U]$ and $[V]$.

Now, it follows from Rosenberg and Schochet's universal coefficient theorem [10], [2, p. 234], that for a C^*-algebra A whose K-groups are free abelian, then $KK^i(A, \mathbf{C}) \cong K_i(A)$ (as abelian groups). Hence the result. ∎

We describe the generators of the K-homology. First of all, we have a canonical "zero-dimensional" even Fredholm module $\mathbf{z}_0$ (Example 1) corresponding to the $*$-homomorphism $\varphi\colon A_0 \to \mathbf{C}$ given by $U, V \mapsto 1$. As a special case of Lemma 1, we have

$$\langle ch_*(\mathbf{z}_0), [1]\rangle = 1 = \langle ch_*(\mathbf{z}_0), [Bott]\rangle.$$

Since the pairings with the generators of K-theory are both 1, it follows from Connes' index theorem [3, p. 296] that this Fredholm module is a generator of K-homology,

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in the sense that if $\mathbf{z}$ is another Fredholm module, with $\mathbf{z}_0 = n\mathbf{z}$ for some $n \in \mathbf{Z}$, then $n = \pm 1$.

For the odd K-homology, we first describe A_0 as a crossed product by (a trivial action of) $\mathbf{Z}$ in two obvious ways, first as $C^*(V) \times_{id} \mathbf{Z}$, second as $C^*(U) \times_{id} \mathbf{Z}$, where the trivial action of $\mathbf{Z}$ is implemented by U and then by V respectively. We denote the corresponding odd Fredholm modules of Example 2 by $\mathbf{z}_1$ and $\mathbf{z}_1{}'$. We have $\mathbf{z}_1 = (l^2(\mathbf{Z}), \pi_1, F)$, where

$$\pi_1(U)e_k = e_{k+1}, \quad \pi_1(V) = I,$$

and $\mathbf{z}_1{}' = (l^2(\mathbf{Z}), \pi_1{}', F)$ with

$$\pi_1{}'(U) = I, \quad \pi_1{}'(V)e_k = e_{k+1},$$

and in each case $Fe_k = \operatorname{sign}(k)e_k$. These Fredholm modules generate the odd K-homology $KK^1(A_0, \mathbf{C})$. By Lemma 2 we have

$$\begin{aligned} &\langle ch_*(\mathbf{z}_1), [U]\rangle = 1, \qquad &&\langle ch_*(\mathbf{z}_1), [V]\rangle = 0, \\ &\langle ch_*(\mathbf{z}_1{}'), [U]\rangle = 0, \qquad &&\langle ch_*(\mathbf{z}_1{}'), [V]\rangle = 1. \end{aligned} \tag{7}$$

The second generator of the even K-homology is the Fredholm module **Dirac**, which is the bounded formulation of the Dirac operator on $\mathbf{T}^2$:

$$\mathbf{Dirac} = (H, \pi, F)$$

where $H = l^2(\mathbf{Z}^2) \oplus l^2(\mathbf{Z}^2)$, with A_0 acting on the orthonormal basis $\{e_{m,n}\}_{(m,n)\in\mathbf{Z}^2}$ for $l^2(\mathbf{Z}^2)$ via

$$Ue_{m,n} = e_{m+1,n}, \quad Ve_{m,n} = e_{m,n+1},$$

and we take

$$\pi(a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \quad F = \begin{pmatrix} 0 & F_0 \\ F_0{}^* & 0 \end{pmatrix}$$

where F_0 is the diagonal operator defined by

$$F_0 e_{m,n} = \begin{cases} \dfrac{m + in}{(m^2 + n^2)^{1/2}} e_{m,n} & (m, n) \neq (0, 0), \\ e_{0,0} & (m, n) = (0, 0). \end{cases}$$

We can use the Baum-Connes assembly map [1] to identify the K-homology and K-theory of A_0. For a general discrete group Γ, the assembly map is a group homomorphism

$$\mu : KK^i(C_0(B\Gamma), \mathbf{C}) \to K_i(C_r^*(\Gamma)),$$

where $B\Gamma$ is the classifying space of the group Γ. Now, $A_0 = C(\mathbf{T}^2) = C^*(\mathbf{Z}^2)$, and it is immediate that $B\mathbf{Z}^2 = \mathbf{T}^2$. For torsion-free finitely-generated abelian groups, it is

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well known that the assembly map is an isomorphism, and basically acts as a Fourier transform. We have

$$\mu : KK^i(A_0, \mathbf{C}) \cong K_i(A_0)$$

$$\mathbf{z}_0 \mapsto \pm[1], \quad \mathbf{z}_1 \mapsto \pm[U], \quad \mathbf{z}_1{}' \mapsto \pm[V], \quad \mathbf{Dirac} \mapsto \pm[Bott].$$

Hence for the commutative situation everything is transparent.

We apply this knowledge to the case $\theta \neq 0$.

Proposition 5 *For $0 \leq \theta \leq 1$, the K-homology groups of the A_θ are $KK^i(A_\theta, \mathbf{C}) \cong \mathbf{Z}^2$.*

Proof This again follows as a corollary of Rosenberg and Schochet's universal coefficient theorem, since the K-groups of the A_θ are both $\mathbf{Z}^2$ for all values of θ [8]. ■

Note that the three generators [1], [U] and [V] of $K_*(A_0)$ are still generators of $K_*(A_\theta)$ for θ irrational, but the generator $[Bott]$ of $K_0(A_0)$ is replaced by $[p]$, where $p \in A_\theta$ is the Powers-Rieffel projection [4] with trace θ.

Analogously, three of the four generators of the K-homology of A_0 generalize immediately to the case $\theta \neq 0$. The odd K-homology is still generated by Fredholm modules $\mathbf{z}_1 = (l^2(\mathbf{Z}), \pi_1, F)$, and $\mathbf{z}_1{}' = (l^2(\mathbf{Z}), \pi_1{}', F)$ where

$$\begin{aligned} \pi_1(U)e_k &= e_{k+1}, & \pi_1(V)e_k &= \lambda^k e_k, \\ \pi_1{}'(U)e_k &= \lambda^{-k} e_k, & \pi_1{}'(V)e_k &= e_{k+1} \end{aligned} \tag{8}$$

and in each case $Fe_k = \operatorname{sign}(k)e_k$. The pairings of the Chern characters of these Fredholm modules with the generators of $K_1(A_\theta)$ are unchanged from (7), and the proofs carry over immediately to the noncommutative situation.

The Fredholm module **Dirac** must be slightly modified.

$$\mathbf{Dirac} = (H, \pi, F) \tag{9}$$

where $H = L^2(A_\theta, \tau) \oplus L^2(A_\theta, \tau)$, with τ being the canonical trace on A_θ, given by

$$\tau(\Sigma a_{m,n} U^m V^n) = a_{0,0}.$$

We identify $L^2(A_\theta, \tau)$ with $l^2(\mathbf{Z}^2)$ with orthonormal basis $\{e_{m,n}\}_{(m,n)\in\mathbf{Z}^2}$, with A_θ acting via

$$Ue_{m,n} = e_{m+1,n}, \quad Ve_{m,n} = \lambda^m e_{m,n+1}$$

and we take

$$\pi(a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \quad F = \begin{pmatrix} 0 & F_0 \\ F_0{}^* & 0 \end{pmatrix}$$

where F_0 is the diagonal operator defined by

$$F_0 e_{m,n} = \begin{cases} \dfrac{m+in}{(m^2+n^2)^{1/2}} e_{m,n} & (m,n) \neq (0,0), \\ e_{0,0} & (m,n) = (0,0). \end{cases}$$

https://doi.org/10.4153/CJM-2004-042-x Published online by Cambridge University Press

We consider the six term exact sequence on K-homology, in the case where our algebra A is $C(\mathbf{T})$, thought of as $C^*(U)$ for some generating unitary U, with $\mathbf{Z}$-action given by the automorphism $\alpha(U) = \lambda U$, where $\lambda = e^{2\pi i\theta}$. So $A \times_\alpha \mathbf{Z} \cong A_\theta$, the rotation algebra.

$$\begin{array}{ccccc} KK^0(A,\mathbf{C}) & \xleftarrow{id-\alpha^*} & KK^0(A,\mathbf{C}) & \xleftarrow{i^*} & KK^0(A_\theta,\mathbf{C}) \\ \partial_0 \downarrow & & & & \uparrow \partial_1 \\ KK^1(A_\theta,\mathbf{C}) & \xrightarrow{i^*} & KK^1(A,\mathbf{C}) & \xrightarrow{id-\alpha^*} & KK^1(A,\mathbf{C}) \end{array}$$

Since $K_0(A)$ and $K_1(A)$ are both isomorphic to $\mathbf{Z}$, generated by $[1]$ and $[U]$ respectively, the universal coefficient theorem tells us that the K-homology groups $KK^0(A,\mathbf{C})$ and $KK^1(A,\mathbf{C})$ are also both $\mathbf{Z}$. The generator of $KK^0(A,\mathbf{C})$ is the canonical Fredholm module $\mathbf{w}_0$ (Example 1), corresponding to the unital *-homomorphism $\varphi\colon A \to \mathbf{C}$ given by $U \mapsto 1$. The generator $\mathbf{w}_1$ of $KK^1(A,\mathbf{C})$ is the canonical odd Fredholm module (Example 2) $(l^2(\mathbf{Z}),\pi,F)$, with U acting on $l^2(\mathbf{Z})$ as the bilateral shift, $\pi(U)e_k = e_{k+1}$, and $Fe_k = \operatorname{sign}(k)e_k$. Lemmas 1 and 2 show that these Fredholm modules are in fact the generators of the K-homology groups.

We saw previously that $KK^1(A_\theta,\mathbf{C}) \cong \mathbf{Z}^2$, with generators $\mathbf{z}_1$, $\mathbf{z}_1{}'$ defined in (8). We know from Theorem 3 that $\partial_0(\mathbf{w}_0) = \mathbf{z}_1{}'$. The inclusion $i\colon A \hookrightarrow A_\theta$ induces maps

$$i^*\colon KK^j(A_\theta,\mathbf{C}) \to KK^j(A,\mathbf{C}) \quad (j = 0,1).$$

Lemma 6 *$i^*(\mathbf{z}_1{}') = \mathbf{0}$, $i^*(\mathbf{z}_1) = \mathbf{w}_1$ and $(id - \alpha^*)(\mathbf{w}_1) = \mathbf{0}$.*

Proof We have $i^*(\mathbf{z}_1{}') = (l^2(\mathbf{Z}),\pi_1 \circ i, F)$, a trivial Fredholm module, since $[F,\pi_1 \circ i(U)] = 0$. We can also see that $i^*(\mathbf{z}_1) = \mathbf{w}_1$. We have $\alpha^*(\mathbf{w}_1)$ is the Fredholm module $(l^2(\mathbf{Z}),\pi\circ\alpha, F)$, with $\pi\circ\alpha(U)e_k = \lambda e_{k+1}$. Hence the Fredholm modules $\mathbf{w}_1$ and $\alpha^*(\mathbf{w}_1)$ are unitarily equivalent, via the unitary $Qe_k = \lambda^k e_k$, and therefore represent the same element of K-homology.

So $(id - \alpha^*)(\mathbf{w}_1) = \mathbf{0} \in KK^0(A,\mathbf{C})$. ■

We also saw that the even K-homology of A_θ is $KK^0(A_\theta,\mathbf{C}) \cong \mathbf{Z}^2$. One generator, **Dirac**, was described previously (9).

Proposition 7 *For the map $\partial_1\colon KK^1(A,\mathbf{C}) \to KK^0(A_\theta,\mathbf{C})$ we have $\partial_1(\mathbf{w}_1) = \mathbf{Dirac}$ as elements of K-homology.*

Proof We recall that $\partial_1(\mathbf{w}_1) = \mathbf{x}\widehat{\otimes}_{A\,\widehat{\otimes}\,\mathbf{C}_1}\mathbf{w}_1$, where $\mathbf{x} \in KK^1(A_\theta, A)$ is the Pimsner-Voiculescu Toeplitz element, described in (2).

The Toeplitz element $\mathbf{x}$ is represented by the Kasparov triple

$$(10) \qquad (E_1,\phi_1,F_1) = \big(l^2(\mathbf{Z},A)\,\widehat{\otimes}\,\mathbf{C}_1, \phi_1{}'\,\widehat{\otimes}\,1, F\,\widehat{\otimes}\,\epsilon\big) \in \mathbf{E}(A_\theta, A\,\widehat{\otimes}\,\mathbf{C}_1)$$

https://doi.org/10.4153/CJM-2004-042-x Published online by Cambridge University Press

with

$$\phi_1(a)(\xi \mathbin{\widehat{\otimes}} \omega) = {\phi_1}'(a)\xi \mathbin{\widehat{\otimes}} \omega$$

where ${\phi_1}' \colon A_\theta \to \mathbf{B}(l^2(\mathbf{Z}, A))$ is defined by

$$({\phi_1}'(U)\xi)(n) = \lambda^{-n} U\xi(n), \quad ({\phi_1}'(V)\xi)(n) = \xi(n+1),$$

(remember that $A = C^*(U)$) and $(F\xi)(n) = \operatorname{sign}(n)\xi(n)$.

The Fredholm module $\mathbf{w}_1 = (H = l^2(\mathbf{Z}), \pi, F)$ is represented by the Kasparov triple

$$\mathbf{w}_1 = \big(H \mathbin{\widehat{\otimes}} \mathbf{C}_1, \pi \mathbin{\widehat{\otimes}} 1, F \mathbin{\widehat{\otimes}} \epsilon\big) \in \mathbf{E}(A, \mathbf{C}_1)$$

where $(\pi \mathbin{\widehat{\otimes}} 1)(a)(\xi \mathbin{\widehat{\otimes}} \omega) = \pi(a)\xi \mathbin{\widehat{\otimes}} \omega$, and $(F\xi)(n) = \operatorname{sign}(n)\xi(n)$, for $\xi \in H$, $\omega \in \mathbf{C}_1$. To take the Kasparov product we need the Kasparov triple

$$\tau_{\mathbf{C}_1}(\mathbf{w}_1) \in \mathbf{E}(A \mathbin{\widehat{\otimes}} \mathbf{C}_1, \mathbf{C}_1 \mathbin{\widehat{\otimes}} \mathbf{C}_1).$$

We have from (4) that:

$$\tau_{\mathbf{C}_1}(\mathbf{w}_1) = \big(H \mathbin{\widehat{\otimes}} (\mathbf{C}_1 \mathbin{\widehat{\otimes}} \mathbf{C}_1), \pi \mathbin{\widehat{\otimes}} (1 \mathbin{\widehat{\otimes}} \mathrm{id}), F \mathbin{\widehat{\otimes}} (\epsilon \mathbin{\widehat{\otimes}} 1)\big) \in \mathbf{E}(A \mathbin{\widehat{\otimes}} \mathbf{C}_1, \mathbf{C}_1 \mathbin{\widehat{\otimes}} \mathbf{C}_1).$$

We can identify $\mathbf{C}_1 \mathbin{\widehat{\otimes}} \mathbf{C}_1$ with $M_2(\mathbf{C})$, via the map

$$\epsilon \mathbin{\widehat{\otimes}} 1 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad 1 \mathbin{\widehat{\otimes}} \epsilon \mapsto \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

and realise $\tau_{\mathbf{C}_1}(\mathbf{w}_1)$ as an element of $\mathbf{E}(A \mathbin{\widehat{\otimes}} \mathbf{C}_1, M_2(\mathbf{C}))$. This identification gives

$$\text{(11)} \qquad \begin{aligned} \tau_{\mathbf{C}_1}(\mathbf{w}_1) &\cong \left(H \mathbin{\widehat{\otimes}} M_2(\mathbf{C}), \pi \mathbin{\widehat{\otimes}} \rho, F \mathbin{\widehat{\otimes}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) \\ &= (E_2, \phi_2, F_2) \in \mathbf{E}(A \mathbin{\widehat{\otimes}} \mathbf{C}_1, M_2(\mathbf{C})) \end{aligned}$$

where

$$(\pi \mathbin{\widehat{\otimes}} \rho)(a \mathbin{\widehat{\otimes}} 1)(\xi \mathbin{\widehat{\otimes}} T) = \pi(a)\xi \mathbin{\widehat{\otimes}} T,$$

$$(\pi \mathbin{\widehat{\otimes}} \rho)(a \mathbin{\widehat{\otimes}} \epsilon)(\xi \mathbin{\widehat{\otimes}} T) = \pi(a)\xi \mathbin{\widehat{\otimes}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} T.$$

The Kasparov product $\mathbf{x} \widehat{\otimes}_{A \mathbin{\widehat{\otimes}} \mathbf{C}_1} \tau_{\mathbf{C}_1}(\mathbf{w}_1)$ of the two triples (10), (11) is given by [2, p. 166] the triple

$$\text{(12)} \qquad (E = E_1 \widehat{\otimes}_{\phi_2} E_2, \phi = \phi_1 \mathbin{\widehat{\otimes}} 1, F) \in \mathbf{E}(A_\theta, M_2(\mathbf{C}))$$

where the difficulty lies in finding a suitable F. We have

$$E_1 \widehat{\otimes}_{\phi_2} E_2 \cong (l^2(\mathbf{Z}, A) \mathbin{\widehat{\otimes}} \mathbf{C}_1) \widehat{\otimes}_{\phi_2} (H \mathbin{\widehat{\otimes}} M_2(\mathbf{C})),$$

with

$$(\delta_k a_k \mathbin{\widehat{\otimes}} 1) \mathbin{\widehat{\otimes}} (\xi \mathbin{\widehat{\otimes}} T) \sim (\delta_k \mathbin{\widehat{\otimes}} 1) \mathbin{\widehat{\otimes}} (\pi(a_k)\xi \mathbin{\widehat{\otimes}} T),$$
$$(\delta_k a_k \mathbin{\widehat{\otimes}} \epsilon) \mathbin{\widehat{\otimes}} (\xi \mathbin{\widehat{\otimes}} T) \sim (\delta_k \mathbin{\widehat{\otimes}} 1) \mathbin{\widehat{\otimes}} (\pi(a_k)\xi \mathbin{\widehat{\otimes}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} T).$$

Hence we can identify $(l^2(\mathbf{Z}, A) \mathbin{\widehat{\otimes}} \mathbf{C}_1) \widehat{\otimes}_{\phi_2} (H \mathbin{\widehat{\otimes}} M_2(\mathbf{C}))$ as a submodule of

$$l^2(\mathbf{Z}^2) \mathbin{\widehat{\otimes}} M_2(\mathbf{C})$$

via the map

$$(\delta_k a_k \mathbin{\widehat{\otimes}} 1) \mathbin{\widehat{\otimes}} (e_l \mathbin{\widehat{\otimes}} T) \mapsto (\delta_k \otimes \pi(a_k) e_l) \mathbin{\widehat{\otimes}} T.$$

Under these identifications, we have that $\phi = \phi_1 \mathbin{\widehat{\otimes}} 1$ acts by

$$\phi(U)((\delta_k \otimes e_l) \mathbin{\widehat{\otimes}} T) = (\lambda^{-k} \delta_k \otimes e_{l+1}) \mathbin{\widehat{\otimes}} T,$$
$$\phi(V)((\delta_k \otimes e_l) \mathbin{\widehat{\otimes}} T) = (\delta_{k+1} \otimes e_l) \mathbin{\widehat{\otimes}} T.$$

We calculate the operator F for the product via the Connes-Skandalis formalism of connections [2, p. 170]. We know that there exists an F_2-connection G for E_1, and from (5), having found such a G, an appropriate F for the product is given by

$$F = F_1 \mathbin{\widehat{\otimes}} 1 + ((1 - {F_1}^2)^{1/2} \mathbin{\widehat{\otimes}} 1)G.$$

Since ${F_1}^2 = 1$ in our situation, we can take $F = F_1 \mathbin{\widehat{\otimes}} 1$. There is no need to explicitly find G; knowledge of its existence is enough.

We also have $F_1 \mathbin{\widehat{\otimes}} 1$ acting on E (as a submodule of $l^2(\mathbf{Z}^2) \mathbin{\widehat{\otimes}} M_2(\mathbf{C})$) by

$$(F_1 \mathbin{\widehat{\otimes}} 1)((\delta_k \otimes e_l) \mathbin{\widehat{\otimes}} T) = (\operatorname{sign}(k) \delta_k \otimes e_l) \mathbin{\widehat{\otimes}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} T.$$

Note that the submodule we have identified with E is invariant under the action of A_θ and of F.

We have calculated the product as a Kasparov triple in $\mathbf{E}(A_\theta, M_2(\mathbf{C}))$. We need to show that this represents the same element of K-homology as the Fredholm module **Dirac**. We will exhibit a homotopy of Kasparov triples from (12) to a new element $\mathbf{y}_1$ of $\mathbf{E}(A_\theta, M_2(\mathbf{C}))$. Then we use the KK-equivalence of $M_2(\mathbf{C})$ and $\mathbf{C}$ to obtain a Kasparov triple in $\mathbf{E}(A_\theta, \mathbf{C})$, still representing the product, which also represents **Dirac**.

The homotopy of Kasparov triples $\{\mathbf{y}_t\}_{0 \le t \le 1}$ is given by

$$\mathbf{y}_t = (E, \phi, {F_t}') \in \mathbf{E}(A_\theta, M_2(\mathbf{C}))$$

with

$${F_t}'((\delta_k \otimes e_l) \mathbin{\widehat{\otimes}} T) = (k^2 + t^2 l^2)^{-1/2} (\delta_k \otimes e_l) \mathbin{\widehat{\otimes}} \begin{pmatrix} 0 & ik + tl \\ -ik + tl & 0 \end{pmatrix} T.$$

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Now, $\mathbf{y}_0$ is the triple representing the product $\partial_1(\mathbf{w}_1)$ calculated in (12), while $\mathbf{y}_1$ is the triple

$$(E, \phi, F_1{}') \in \mathbf{E}(A_\theta, M_2(\mathbf{C}))$$

which therefore also represents the product.

We now use the KK-equivalence of $M_2(\mathbf{C})$ and $\mathbf{C}$ to realise the product as an element of $\mathbf{E}(A_\theta, \mathbf{C})$. The KK-equivalence is implemented (on the right) by the Kasparov triple

$$\mathbf{z} = (\mathbf{C}^2, \mathrm{id}, 0) \in \mathbf{E}(M_2(\mathbf{C}), \mathbf{C}).$$

Taking the product with $\mathbf{z}$ gives us the triple

$$(E' = E \widehat{\otimes}_{\mathrm{id}} \mathbf{C}^2, \phi' = \phi \,\widehat{\otimes}\, 1, F' = F_1{}' \,\widehat{\otimes}\, 1) \in \mathbf{E}(A_\theta, \mathbf{C}).$$

The same argument as above tells us that this is the appropriate F' for the product. We can identify $E' = (l^2(\mathbf{Z}^2) \,\widehat{\otimes}\, M_2(\mathbf{C}))\widehat{\otimes}_{\mathrm{id}} \mathbf{C}^2$ with $l^2(\mathbf{Z}^2) \oplus l^2(\mathbf{Z}^2)$ via the map

$$(\xi \,\widehat{\otimes}\, I_2) \,\widehat{\otimes} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \mapsto \begin{bmatrix} \alpha\xi \\ \beta\xi \end{bmatrix}.$$

Under this identification, we have A_θ acting via $\phi' = \pi' \oplus \pi'$, with

$$\pi'(U)(\delta_k \otimes e_l) = \lambda^{-k} \delta_k \otimes e_{l+1}, \quad \pi'(V)(\delta_k \otimes e_l) = \delta_{k+1} \otimes e_l,$$

and $F' = \left(\begin{smallmatrix} 0 & F'' \\ F''^* & 0 \end{smallmatrix}\right)$, where

$$F''(\delta_k \otimes e_l) = \begin{cases} \dfrac{ik + l}{(k^2 + l^2)^{1/2}} \delta_k \otimes e_l & (k, l) \neq (0, 0), \\ \delta_0 \otimes e_0 & (k, l) = (0, 0). \end{cases}$$

Now, let $\{e_{p,q}\}$ be an arbitrary new orthonormal basis for $l^2(\mathbf{Z}^2)$, and define a unitary operator $Q\colon l^2(\mathbf{Z}^2) \to l^2(\mathbf{Z}^2)$ by

$$Q(\delta_k \otimes e_l) = \lambda^{kl} e_{l,k}.$$

Then the triple (E', ϕ', F') is unitarily equivalent to the triple $(E, \phi, F) \in \mathbf{E}(A_\theta, \mathbf{C})$, where

$$E = E' = l^2(\mathbf{Z}^2) \oplus l^2(\mathbf{Z}^2),$$

with $\phi = \pi \oplus \pi$ acting via

$$\pi(U) e_{p,q} = e_{p+1,q}, \quad \pi(V) e_{p,q} = \lambda^p e_{p,q+1},$$

and

$$F = \begin{pmatrix} 0 & F_0{}^* \\ F_0 & 0 \end{pmatrix}, \quad F_0 e_{p,q} = \begin{cases} \dfrac{p + iq}{(p^2 + q^2)^{1/2}} e_{p,q} & (p, q) \neq (0, 0), \\ e_{0,0} & (p, q) = (0, 0). \end{cases}$$

We recognise this as a triple representing the Fredholm module **Dirac**, as defined in (9). So we have shown by direct computation that $\partial_1(\mathbf{w}_1) = \textbf{Dirac}$ as elements of K-homology. ∎

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Lemma 8 $i^*(\mathbf{Dirac}) = \mathbf{0} \in KK^0(A, \mathbf{C})$.

Proof Since $\mathbf{Dirac} = \partial_1(\mathbf{w}_1)$, and $i^* \circ \partial_1 = 0$, this follows from Proposition 7. However, it is instructive to prove this directly.

We have $i^*(\mathbf{Dirac}) \in KK^0(A, \mathbf{C})$ is the Fredholm module

$$\left(l^2(\mathbf{Z}^2) \oplus l^2(\mathbf{Z}^2), \pi_0 \oplus \pi_0, F = \begin{pmatrix} 0 & F_0 \\ F_0 & 0 \end{pmatrix}\right) \tag{13}$$

where $\pi_0(U)e_{p,q} = e_{p+1,q}$, and

$$F_0 e_{p,q} = \begin{cases} \dfrac{p+iq}{(p^2+q^2)^{1/2}} e_{p,q} & (p,q) \neq (0,0), \\ e_{0,0} & (p,q) = (0,0). \end{cases}$$

Now, $K_0(A) \cong \mathbf{Z}$, generated by $[1]$, $KK^0(A, \mathbf{C}) \cong \mathbf{Z}$, generated by $\mathbf{w}_0$, and $\langle ch_*(\mathbf{w}_0), [1]\rangle = 1$. If $i^*(\mathbf{Dirac})$ is a nontrivial element of K-homology, then we will have $i^*(\mathbf{Dirac}) = n\mathbf{w}_0$, for some $n \neq 0$, and so $\langle ch_*(i^*(\mathbf{Dirac})), [1]\rangle = n$. But we see from (13) that $[F, (\pi_0 \oplus \pi_0)(1)] = 0$. It follows that $\langle ch_*(i^*(\mathbf{Dirac})), [1]\rangle = 0$, and so $i^*(\mathbf{Dirac})$ represents a trivial element of K-homology. ■

We want to describe a second generator of $KK^0(A_\theta, \mathbf{C}) \cong \mathbf{Z}^2$. We will again denote this generator by $\mathbf{z}_0$. In the case $\theta = 0$, we take $\mathbf{z}_0$ to be the canonical even Fredholm module (Lemma 1) over $C(\mathbf{T}^2)$, and for other values of θ we want the corresponding $\mathbf{z}_0$ to be a "continuous deformation" of this. However, it is difficult to describe such a Fredholm module explicitly.

It is easy to see that the map $(id - \alpha^*)\colon KK^0(A, \mathbf{C}) \to KK^0(A, \mathbf{C})$ is the zero map. Hence the map i^* is surjective. Since $i^*(\mathbf{Dirac}) = \mathbf{0}$, we may impose for all values of θ that $i^*(\mathbf{z}_0) = \mathbf{w}_0 \in KK^0(A, \mathbf{C})$.

Under the map $i_*\colon K_0(A) \to K_0(A_\theta)$, we have $i_*[1] = [1]$, and we calculate that

$$1 = \langle ch_*(\mathbf{w}_0), [1]\rangle = [1]\widehat{\otimes}_A \mathbf{w}_0 = [1]\widehat{\otimes}_A i^*(\mathbf{z}_0) = i_*[1]\widehat{\otimes}_{A\times_\alpha \mathbf{Z}} \mathbf{z}_0 = \langle ch_*(\mathbf{z}_0), [1]\rangle.$$

We also need to know $\langle ch_*(\mathbf{z}_0), [p]\rangle$. Since in the case $\theta = 0$ the Powers-Rieffel projection p of [4, p .170], is just $p = 0$, and we want our $\mathbf{z}_0$ to be a deformation of the $\theta = 0$ case, we will require that

$$\langle ch_*(\mathbf{z}_0), [p]\rangle = 0.$$

In the case θ is rational, $\theta = m/n$, with m, n relatively prime integers, $n > 0$, then A_θ is the algebra of continuous sections of a vector bundle over $\mathbf{T}^2$, whose fibres are full matrix algebras $M_n(\mathbf{C})$.

We construct a Fredholm module $\mathbf{z}_0{}'$ over A_θ as follows:

$$\mathbf{z}_0{}' = \left(\mathbf{C}^n \oplus \mathbf{C}^n, \pi_0{}' = \varphi \oplus 0, F_0{}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$$

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with

$$\varphi(U) = \begin{pmatrix} 0 & 0 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}, \quad \varphi(V) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda^{n-1} \end{pmatrix}.$$

Then an easy calculation (see Lemma 1) shows that

$$\langle ch_*(\mathbf{z}_0{}'), [1]\rangle = n,$$

and further, under $i^*\colon KK^0(A_\theta, \mathbf{C}) \to KK^0(A, \mathbf{C})$, we have $i^*(\mathbf{z}_0{}') = n\mathbf{w}_0$. Since i^* is surjective, $\mathbf{z}_0{}'$ cannot be a generator of $KK^0(A_\theta, \mathbf{C})$. In particular, $\mathbf{z}_0{}' \neq \mathbf{z}_0$.

It would be good to have an explicit description of the Fredholm module $\mathbf{z}_0$. In the final section of this paper, we describe an approach to this via Pimsner and Voiculescu's work on embedding the irrational rotation algebras in AF-algebras [6]. The general question of concrete realizations (as Fredholm modules) of the K-homology of AF-algebras is not well-studied and is an interesting topic for future research.

5 K-Homology of the A_θ via AF-Embeddings

We conclude our study of the K-homology of the rotation algebras by exploiting the AF-embedding technique of Pimsner and Voiculescu [6] to try to find the missing generator of the even K-homology.

Given an irrational $\theta \in (0, 1)$, Pimsner and Voiculescu constructed an embedding of the irrational rotation algebra A_θ in an AF-algebra C_θ as follows.

We begin by considering the continued fraction expansion

$$\theta = \lim_{n\to\infty} [a_0, a_1, \dots, a_n] = \lim_{n\to\infty} \left(a_0 + \cfrac{1}{a_1 + \frac{1}{\dots + \frac{1}{a_n}}} \right)$$

where $a_0 \in \mathbf{Z}$, and $a_1, \dots, a_n \in \mathbf{N}$. The rational approximations $\frac{p_n}{q_n} = [a_0, \dots, a_n]$ are given recursively by

$$(14) \qquad \begin{gathered} p_0 = a_0,\ q_0 = 1,\ p_1 = a_0a_1 + 1,\ q_1 = a_1, \\ p_n = a_np_{n-1} + p_{n-2},\ q_n = a_nq_{n-1} + q_{n-2}. \end{gathered}$$

We define a sequence $\{C_n\}_{n\geq 1}$ of finite-dimensional C^*-algebras by

$$C_n = M_{q_n}(\mathbf{C}) \oplus M_{q_{n-1}}(\mathbf{C}).$$

The maps $\phi_{n+1,n}\colon C_n \to C_{n+1}$ are given by

$$(15) \quad \phi_{n+1,n}\colon \begin{pmatrix} A_n & 0 \\ 0 & B_n \end{pmatrix} \mapsto \begin{pmatrix} W_{n+1} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A_n & & & & 0 \\ & \ddots & & & \\ & & A_n & & \\ & & & B_n & \\ 0 & & & & A_n \end{pmatrix} \begin{pmatrix} W_{n+1}^* & 0 \\ 0 & I \end{pmatrix}$$

where the A_n's in the top left corner occur with multiplicity a_n, and $W_{n+1} \in M_{q_{n+1}}(\mathbf{C})$ is a unitary.

The AF-algebra C_θ is defined to be $C_\theta = \lim_{n\to\infty} C_n$. For each n we have an inclusion map $\phi_n \colon C_n \to C_\theta$. Pimsner and Voiculescu [6] proved the following result:

Theorem 9 *There is an injective $*$-homomorphism $\rho\colon A_\theta \to C_\theta$, such that*

$$\rho_*\colon K_0(A_\theta) \to K_0(C_\theta)$$

is an isomorphism of abelian groups, and such that, furthermore, if τ and σ are the canonical normalized traces on A_θ and C_θ respectively, then $\tau_* = \sigma_*\rho_*$ is an order isomorphism of $K_0(A_\theta)$ onto $\mathbf{Z} + \mathbf{Z}\theta$.

So $K_0(C_\theta) \cong \mathbf{Z}^2$, generated by $[\rho(1)]$ and $[\rho(p)]$, where $[1]$ and $[p]$ generate $K_0(A_\theta)$, and further $K_1(C_\theta) \cong 0$ (since C_θ is AF). Hence by the universal coefficient theorem we have $KK^0(C_\theta, \mathbf{C}) \cong \mathbf{Z}^2$. (We will also calculate this directly, in a way that will be more useful for our purposes.) Since C_θ is AF, for each $[x] \in K_0(C_\theta)$ there exists n, and $[x_n] \in K_0(C_n)$, such that $[\phi_n(x_n)] = [x]$. In fact, under $\phi_1\colon C_1 \to C_\theta$, we have:

Lemma 10 *$[\phi_1(1)] = [\rho(1)]$, and $[\phi_1(p_1)] = [\rho(p)]$, where 1 is the unit of C_1, and p_1 is the rank one projection in $C_1 \cong M_{a_1}(\mathbf{C}) \oplus \mathbf{C}$ given by*

$$p_1 = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

We have

$$C_1 \to C_2 \to \cdots \to C_n \to \cdots \to C_\theta \xleftarrow{\rho} A_\theta$$

If we could exhibit the Fredholm modules generating the even K-homology of C_θ, then we could pull these back via ρ to get Fredholm modules over A_θ.

For each n, we have $KK^0(C_n, \mathbf{C}) \cong \mathbf{Z}^2$, generated by Fredholm modules $\mathbf{z}_1^{(n)}$ and $\mathbf{z}_2^{(n)}$.

These are defined as follows (see Example 1):

$$\mathbf{z}_1^{(n)} = \left(\mathbf{C}^{q_n} \oplus \mathbf{C}^{q_n}, \begin{pmatrix} A_n & 0 \\ 0 & B_n \end{pmatrix} \mapsto \begin{pmatrix} A_n & 0 \\ 0 & 0 \end{pmatrix}, F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right),$$

$$\mathbf{z}_2^{(n)} = \left(\mathbf{C}^{q_{n-1}} \oplus \mathbf{C}^{q_{n-1}}, \begin{pmatrix} A_n & 0 \\ 0 & B_n \end{pmatrix} \mapsto \begin{pmatrix} B_n & 0 \\ 0 & 0 \end{pmatrix}, F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right).$$

Under the maps $\phi_{n+1,n}\colon C_n \to C_{n+1}$ (15) we have

$$\phi^*_{n+1,n}(\mathbf{z}_1^{(n+1)}) = a_n\mathbf{z}_1^{(n)} + \mathbf{z}_2^{(n)}, \tag{16}$$

$$\phi^*_{n+1,n}(\mathbf{z}_2^{(n+1)}) = \mathbf{z}_1^{(n)}. \tag{17}$$

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Hence all the maps

$$\phi^*_{n+1,n} \colon KK^0(C_{n+1}, \mathbf{C}) \cong \mathbf{Z}^2 \to KK^0(C_n, \mathbf{C}) \cong \mathbf{Z}^2,$$

are surjective, because the matrix $\left(\begin{smallmatrix} a_n & 1 \\ 1 & 0 \end{smallmatrix}\right)$ is always invertible.

We now appeal to the following special case of a much more general result of Rosenberg and Schochet [10].

Proposition 11 *Suppose that $A = \lim_{\rightarrow} A_n$ is an AF-algebra. Then the following sequences in K-homology are exact:*

$$0 \to \lim_{\leftarrow}{}^1 KK^1(A_n, \mathbf{C}) \to KK^0(A, \mathbf{C}) \to \lim_{\leftarrow} KK^0(A_n, \mathbf{C}) \to 0,$$

$$0 \to \lim_{\leftarrow}{}^1 KK^0(A_n, \mathbf{C}) \to KK^1(A, \mathbf{C}) \to \lim_{\leftarrow} KK^1(A_n, \mathbf{C}) \to 0.$$

The left hand term is Milnor's $\lim^1_{\leftarrow}$, and the right hand term is the inverse limit of the K-homology groups.

It follows from [11, p. 80] that if the maps $KK^i(A_{n+1}, \mathbf{C}) \to KK^i(A_n, \mathbf{C})$ are all surjective, then both $\lim^1$ terms vanish, and hence $KK^i(A, \mathbf{C}) \cong \lim_{\leftarrow} KK^i(A_n, \mathbf{C})$, $(i = 0, 1)$. This is true in our situation, and furthermore we have $\lim_{\leftarrow} KK^0(C_n, \mathbf{C}) \cong \mathbf{Z}^2$, $\lim_{\leftarrow} KK^1(C_n, \mathbf{C}) \cong 0$. Hence $KK^0(C_\theta, \mathbf{C}) \cong \mathbf{Z}^2$, and $KK^1(C_\theta, \mathbf{C}) \cong 0$, which we knew already, but via the definition of the inverse limit we can now visualize the elements of $KK^0(C_\theta, \mathbf{C})$.

Lemma 12 *An element $\mathbf{z} \in KK^0(C_\theta, \mathbf{C})$ is represented by a sequence of Fredholm modules $\{\mathbf{z}_n\}_{n \geq 1}$, with $\mathbf{z}_n \in KK^0(C_n, \mathbf{C})$, such that under each of the inclusion maps $\phi_n \colon C_n \to C_\theta$, we have $\phi^*_n(\mathbf{z}) = \mathbf{z}_n$. It follows immediately that for each of the maps $\phi_{n+k,n} \colon C_n \to C_{n+k}$, we have $\phi^*_{n+k,n}(\mathbf{z}_{n+k}) = \mathbf{z}_n$.*

Proof This follows immediately from the definition of the inverse limit of a sequence of abelian groups. Recall that, for a tower of abelian groups

$$\cdots \longrightarrow G_n \xrightarrow{f_n} G_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_0} G_0,$$

the inverse limit $\lim_{\leftarrow} G_n$ is isomorphic to the abelian group consisting of all sequences $\{g_n\}_{n \geq 0}$, with $g_n \in G_n$ for each n, such that $g_{n-1} = f_n(g_n)$. ■

Furthermore, given any $[x] \in K_0(C_\theta)$, there exists n, and $[x_n] \in K_0(C_n)$, such that $[\phi_n(x_n)] = [x]$. So

$$\langle ch_*(\mathbf{z}), [x] \rangle = \langle ch_*(\mathbf{z}), [\phi_n(x_n)] \rangle = \langle ch_*(\phi^*_n(\mathbf{z})), [x_n] \rangle = \langle ch_*(\mathbf{z}_n), [x_n] \rangle.$$

We want to find a Fredholm module $\mathbf{z}_0 \in KK^0(C_\theta, \mathbf{C})$ such that $\rho^*(\mathbf{z}_0) = \mathbf{w}_0 \in KK^0(A_\theta, \mathbf{C})$. We need

$$\langle ch_*(\mathbf{z}_0), [\rho(1)]\rangle = \langle ch_*(\mathbf{w}_0), [1]\rangle = 1,$$
$$\langle ch_*(\mathbf{z}_0), [\rho(p)]\rangle = \langle ch_*(\mathbf{w}_0), [p]\rangle = 0,$$

We can take $\mathbf{z}_1 = \phi_1^*(\mathbf{z}_0) = \mathbf{z}_2^{(1)}$, since

$$\langle ch_*(\mathbf{z}_1^{(1)}), [1]\rangle = a_1,\ \langle ch_*(\mathbf{z}_1^{(1)}), [p_1]\rangle = 1,$$
$$\langle ch_*(\mathbf{z}_2^{(1)}), [1]\rangle = 1,\ \langle ch_*(\mathbf{z}_2^{(1)}), [p_1]\rangle = 0.$$

We can calculate the corresponding $\mathbf{z}_n = \phi_n^*(\mathbf{z}_0) \in KK^0(C_n, \mathbf{C})$ in the same way, provided we know all the a_n's. We have $\mathbf{z}_n = x\mathbf{z}_1^{(n)} + y\mathbf{z}_2^{(n)}$ where

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{n-1} & 1 \\ 1 & 0 \end{pmatrix}^{-1} \cdots \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} q_n & q_{n-1} \\ p_n & p_{n-1} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$= (-1)^n \begin{pmatrix} p_{n-1} & -q_{n-1} \\ -p_n & q_n \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = (-1)^n \begin{pmatrix} -q_{n-1} \\ q_n \end{pmatrix}$$

(this follows from the relations (14)). In this way we obtain an element of $\lim_{n\leftarrow\infty} KK^0(C_n, \mathbf{C})$ representing $\mathbf{z}_0 \in KK^0(C_\theta, \mathbf{C})$. It is not clear how to pull this back via ρ to $KK^0(A_\theta, \mathbf{C})$.

Acknowledgements

I would like to thank my advisor, Professor Marc Rieffel, for his advice and support throughout my time at Berkeley. I am very grateful for his help. I would also like to thank Erik Guentner, Nathaniel Brown and Frederic Latremoliere for many useful discussions.

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https://doi.org/10.4153/CJM-2004-042-x Published online by Cambridge University Press