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Finite Semisimple Loop Algebras of Indecomposable RA Loops

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Abstract. There are seven classes of finite indecomposable RA loops upto isomorphism. In this paper, we completely characterize the structure of the unit loop of loop algebras of these seven classes of loops over finite fields of characteristic greater than 2.

1 Introduction

A loop is a binary system $(L, *)$ with an identity element and the property that for each a and b in L, there exist unique elements x and y in L such that $a * x = b$ and $y * a = b$. A group is nothing but an associative loop. Let R be a commutative, associative ring with identity. A loop ring $R[L]$ can be constructed from a loop L precisely in the same manner as the group ring $R[G]$ is constructed from a group G. If $R = F$, a field, then we call $F[L]$ a loop algebra. An alternative ring is a ring in which $x(xy) = x^2y$ and $(yx)x = yx^2$ are identities. A loop whose loop ring $R[L]$ over some commutative, associative ring R with unity and of characteristic different from 2 is alternative, but not associative is called an RA loop. We refer the reader to [3] for more details.

Goodaire [2] has determined the loop of units in the integral alternative loop rings of the six smallest order loops when the loop rings have non-trivial units. The unit loop $\mathcal{U}(\mathbb{Z}[M(Q_8, 2)])$ has been studied by Jespers and Leal [4], where $M(Q_8, 2) = M(Q_8, -1, 1)$ denotes the Moufang Loop obtained from Q_8, the quaternion group of order 8, and the inverse involution on Q_8. Recently the semisimple loop algebras of RA loops have been studied by Ferraz, Goodaire, and Milies [1]. The structure of the unit loops of the loop algebras of RA loops of order 32 and 64 have been determined in [8] and [9]. Jespers, Leal, and Milies [5] have proved that, up to isomorphism, there are seven classes of indecomposable RA loops, given in Table 1. The purpose of this paper is to determine the structure of the unit loops of finite semisimple loop algebras of these seven classes of loops.

Throughout F denotes the finite field containing $q = p^n$ elements with $p > 2$. Let $M(G, *, g_0)$ denote the Moufang loop obtained from a non-abelian group G, $g_0 \in \mathcal{Z}(G)$, and the involution $*$ on G. Also, we use the following notations:

- C_n: cyclic group of order n

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Table 1: The seven classes of finite indecomposable RA loops.

Loop	Group	$\mathcal{Z}(\mathcal{L}_i)$	x^2	y^2	$u^2 = g_0$
$\mathcal{L}_1$	G_1	$\langle t_1 \rangle$	1	1	1
$\mathcal{L}_2$	G_2	$\langle t_1 \rangle$	t_1	t_1	t_1
$\mathcal{L}_3$	G_3	$\langle t_1 \rangle \times \langle t_2 \rangle$	1	t_2	1
$\mathcal{L}_4$	G_4	$\langle t_1 \rangle \times \langle t_2 \rangle$	t_1	t_2	t_1
$\mathcal{L}_5$	G_5	$\langle t_1 \rangle \times \langle t_2 \rangle \times \langle t_3 \rangle$	t_2	t_3	1
$\mathcal{L}_6$	G_5	$\langle t_1 \rangle \times \langle t_2 \rangle \times \langle t_3 \rangle$	t_2	t_3	t_1
$\mathcal{L}_7$	$G_5 \times \langle w \rangle$	$\langle t_1 \rangle \times \langle t_2 \rangle \times \langle t_3 \rangle \times \langle w \rangle$	t_2	t_3	w

- ξ_k: primitive k^{th} root of unity
- F_n: extension field of F of degree n over F
- $\mathfrak{Z}(R)$: Zorn's vector matrix algebra over a commutative and associative ring R (with unity)
- $\mathrm{GLL}(2, R)$: General Linear Loop of degree 2 over R
- $\phi(n)$: Euler's phi function
- $\mathrm{ord}_n(q)$: order of q modulo n
- (x, y): commutator of elements x and y in a loop L.

2 Preliminaries

In this section, we discuss some useful results. An indecomposable RA loop is a 2-loop. From [3, Chapter IV, Theorem 3.1], a loop L is an RA loop if and only if $L = M(G, *, g_0)$ for a non-abelian group G. A non-abelian group G forms an RA loop $M(G, *, g_0)$ if and only if $G/\mathcal{Z}(G) \cong C_2 \times C_2$. In [3, Chapter V, Theorem 1.2], it has been proved that $G/\mathcal{Z}(G) \cong C_2 \times C_2$ if and only if $G = D \times C$, where D is an indecomposable 2-group and C is a cyclic group that, if nontrivial, is a 2-group. Such a group D is given by $D = \langle x, y, \mathcal{Z}(D) \rangle$, where

- $\mathcal{Z}(D) = \langle t_1 \rangle \times \langle t_2 \rangle \times \langle t_3 \rangle$,
- $o(t_i) = 2^{m_i}$ for $i = 1, 2, 3$, $m_1 \geq 1$, $m_2, m_3 \geq 0$,
- $s = (x, y) = t_1^{2^{m_1 - 1}}$,
- $x^2 = t_1^{\alpha_1} t_2^{\alpha_2}$, $\alpha_i \geq 0$,
- $y^2 = t_1^{\beta_1} t_2^{\beta_2} t_3^{\beta_3}$, $\beta_i \geq 0$.

Up to isomorphism, there are at the most five classes of these indecomposable 2-groups. Table 2 contains these groups along with their presentations.
Further, these groups form the seven non-isomorphic classes of indecomposable RA loops given in Table 1.

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Table 2: Finite indecomposable groups G with $\dfrac{G}{\mathcal{Z}(G)} \cong C_2 \times C_2$.

Group	Presentation
G_1	$\langle x, y, t_1 \mid x^2, y^2, t_1^{2^{m_1}}, t_1 \text{ central}, (x,y) = t_1^{2^{m_1-1}} \rangle$
G_2	$\langle x, y, t_1 \mid x^2 = y^2 = t_1, t_1^{2^{m_1}}, t_1 \text{ central}, (x,y) = t_1^{2^{m_1-1}} \rangle$
G_3	$\langle x, y, t_1, t_2 \mid x^2, y^2 = t_2, t_1^{2^{m_1}}, t_2^{2^{m_2}},$ $t_1, t_2 \text{ central}, (x,y) = t_1^{2^{m_1-1}} \rangle$
G_4	$\langle x, y, t_1, t_2 \mid x^2 = t_1, y^2 = t_2, t_1^{2^{m_1}}, t_2^{2^{m_2}},$ $t_1, t_2 \text{ central}, (x,y) = t_1^{2^{m_1-1}} \rangle$
G_5	$\langle x, y, t_1, t_2, t_3 \mid x^2 = t_2, y^2 = t_3, t_1^{2^{m_1}}, t_2^{2^{m_2}}, t_3^{2^{m_3}},$ $t_1, t_2, t_3 \text{ central}, (x,y) = t_1^{2^{m_1-1}} \rangle$

The following lemma gives the decomposition of semisimple loop algebras of RA loops.

Lemma 2.1 ([3, Chapter VI, Corollary 4.8]) Let $L = M(G, *, g_0)$ be a finite RA loop with commutator-associator subloop $L' = \{1, s\} = G'$ and $H = L$ or G. If char $K \nmid |H|$, then

$$K[H] = K[H]\Big(\frac{1+s}{2}\Big) \oplus K[H]\Big(\frac{1-s}{2}\Big),$$

where $K[H]\big(\frac{1+s}{2}\big) \cong K[H/H']$ is a direct sum of fields and $K[H]\big(\frac{1-s}{2}\big) = \Delta_K(H, H')$ is a direct sum of Cayley–Dickson algebras if $H = L$ and a direct sum of quaternion algebras otherwise.

Using this theorem we can determine L/L' for an indecomposable RA loop, once we know the center of the loop L.

Theorem 2.2 ([3, Chapter XI, Proposition 2.3]) Let L be a finite indecomposable RA loop. Write

$$\mathcal{Z}(L) = \langle t_1 \rangle \times \langle t_2 \rangle \times \langle t_3 \rangle \times \langle w \rangle \cong C_{2^a} \times C_{2^b} \times C_{2^c} \times C_{2^d},$$

with $a \geq 1$, $b, c, d \geq 0$, and $L' \subseteq \langle t_1 \rangle$. Then

$$L/L' \cong \begin{cases} C_{2^{a-1}} \times C_{2^{b+1}} \times C_{2^{c+1}} \times C_{2^{d+1}} & \text{if } L \in \mathcal{L}_1 \cup \mathcal{L}_3 \cup \mathcal{L}_5 \cup \mathcal{L}_7, \\ C_{2^a} \times C_{2^{b+1}} \times C_{2^{c+1}} \times C_{2^d} & \text{otherwise.} \end{cases}$$

The next proposition tells us how to find the center of $\Delta_K(L, L')$ for an indecomposable RA 2-loop L using $\mathcal{Z}(L)$.

Proposition 2.3 ([3, Chapter XI, Proposition 1.3]) Let L be a finite RA 2-loop and K be a field such that char $K \neq 2$. Write $\mathcal{Z}(L) = \langle t_1 \rangle \times \cdots \times \langle t_r \rangle$, where $\langle t_i \rangle \cong C_{2^{n_i}}$, for

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$i = 1, 2, \ldots, r$. Assume that $L' \subseteq \langle t_1 \rangle$. Then

$$\mathcal{Z}\big(\Delta_K(L, L')\big) \cong \frac{2^{n_1-1}}{[K(\xi_{2^{n_1}}) : K]} K(\xi_{2^{n_1}})[C_{2^{n_2}} \times \cdots \times C_{2^{n_r}}].$$

Corollary [10, Corollary, p. 152] says that every Cayley–Dickson algebra over a finite field is split. From [3, Chapter I, Corollary 4.17], any split Cayley–Dickson algebra over a field K of characteristic different from 2 is isomorphic to Zorn's vector matrix algebra $\mathfrak{Z}(K)$. Also, we have the center of Zorn's vector matrix algebra $\mathfrak{Z}(K)$ is isomorphic to K.

Remark 2.4 It follows that every non-commutative, non-associative component in the decomposition of finite semisimple loop algebra $F[L]$ is isomorphic to Zorn's vector matrix algebra $\mathfrak{Z}(F_i)$, where F_i denotes the extension field of F.

The following theorem gives the Wedderburn decomposition of finite semisimple group algebras of abelian groups.

Theorem 2.5 (Perlis–Walker; [7, Theorem 3.5.4]) Let G be a finite abelian group of order n and $K[G]$ be a semi-simple group algebra. Then

$$K[G] \cong \bigoplus_{j|n} a_j K(\xi_j),$$

where $a_j = \frac{n_j}{[K(\xi_j):K]}$, n_j being the number of elements of order j in G.

To calculate $[K(\xi_j) : K]$, we will use the next theorem.

Definition 2.6 Let n be a positive integer. The splitting field K of $x^n - 1$ over a field F is called the n-th cyclotomic field over F.

Theorem 2.7 ([6, Theorem 2.47]) Let K be a finite field containing $q = p^n$ elements with $\gcd(q, n) = 1$ and let

$$\mathcal{Q}_n(x) = \prod_{\substack{s=1 \\ \gcd(s,n)=1}}^{n} (x - \xi^s)$$

denote the n-th cyclotomic polynomial. Then $\mathcal{Q}_n$ factors into $\phi(n)/d$ distinct monic irreducible polynomials in $F[x]$ of the same degree d, K is the splitting field of any such irreducible factor over F, and $[K : F] = d$, where $d = \operatorname{ord}_n(q)$.

3 The Unit Loops of $F[\mathcal{L}_1]$ and $F[\mathcal{L}_2]$

In this section we determine the structure of the unit loop of finite semisimple loop algebras of those indecomposable RA loops whose center is isomorphic to $C_{2^{m_1}}$.

Lemma 3.1 $F[C_{2^a}] \cong F \oplus \bigoplus_{i=1}^{a} \frac{\phi(2^i)}{d_i} F_{d_i}$, where $d_i = \operatorname{ord}_{2^i}(q)$.

Proof Follows from Theorem 2.5 and 2.7. ∎

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Theorem 3.2 The unit loop $\mathcal{U}(F[\mathcal{L}_1])$ satisfies

$$\mathcal{U}(F[\mathcal{L}_1]) \cong 8F^* \times \bigtimes_{i=1}^{m_1-1} \frac{8\phi(2^i)}{d_i} F_{d_i}^* \times \frac{2^{m_1-1}}{d_{m_1}} \mathrm{GLL}(2, F_{d_{m_1}})$$

where $d_i = \mathrm{ord}_{2^i}(q)$.

Proof We have $\mathcal{Z}(\mathcal{L}_1) = \mathcal{Z}(G_1) = \langle t_1 \rangle \cong C_{2^{m_1}}$. Write

$$\mathcal{Z}(\mathcal{L}_1) \cong C_{2^{m_1}} \times C_{2^0} \times C_{2^0} \times C_{2^0}.$$

Using Theorem 2.2, we get

$$\begin{aligned} \mathcal{L}_1/\mathcal{L}_1' &\cong C_{2^{m_1-1}} \times C_2 \times C_2 \times C_2 \\ F[\mathcal{L}_1/\mathcal{L}_1'] &\cong 8F[C_{2^{m_1-1}}] \\ &\cong 8F \oplus \bigoplus_{i=1}^{m_1-1} \frac{8\phi(2^i)}{d_i} F_{d_i} \end{aligned}$$

Note that $\mathcal{Z}\big(\Delta_F(\mathcal{L}_1, \mathcal{L}_1')\big) \cong \frac{2^{m_1-1}}{[F(\xi_{2^{m_1}}):F]} F(\xi_{2^{m_1}}) \cong \frac{2^{m_1-1}}{d_{m_1}} F_{d_{m_1}}$. Thus, using Remark 2.4,

$$\Delta_F(\mathcal{L}_1, \mathcal{L}_1') \cong \frac{2^{m_1-1}}{d_{m_1}} \mathfrak{Z}(F_{d_{m_1}}).$$

Hence $F[\mathcal{L}_1] \cong 8F \bigoplus_{i=1}^{m_1-1} \frac{8\phi(2^i)}{d_i} F_{d_i} \oplus \frac{2^{m_1-1}}{d_{m_1}} \mathfrak{Z}(F_{d_{m_1}})$. ∎

Theorem 3.3 The unit loop $\mathcal{U}(F[\mathcal{L}_2])$ satisfies

$$\mathcal{U}(F[\mathcal{L}_2]) \cong 4F^* \times \bigtimes_{i=1}^{m_1} \frac{4\phi(2^i)}{d_i} F_{d_i}^* \times \frac{2^{m_1-1}}{d_{m_1}} \mathrm{GLL}(2, F_{d_{m_1}})$$

where $d_i = \mathrm{ord}_{2^i}(q)$.

Proof Note that $\mathcal{Z}(\mathcal{L}_2) = \mathcal{Z}(G_2) = \langle t_1 \rangle \cong C_{2^{m_1}}$. Write

$$\mathcal{Z}(\mathcal{L}_2) \cong C_{2^{m_1}} \times C_{2^0} \times C_{2^0} \times C_{2^0}.$$

Thus,

$$\begin{aligned} \mathcal{L}_2/\mathcal{L}_2' &\cong C_{2^{m_1}} \times C_2 \times C_2 \\ F[\mathcal{L}_2/\mathcal{L}_2'] &\cong 4F[C_{2^{m_1}}] \\ &\cong 4F \oplus \bigoplus_{i=1}^{m_1} \frac{4\phi(2^i)}{d_i} F_{d_i}. \end{aligned}$$

Now, $\mathcal{Z}\big(\Delta_F(\mathcal{L}_2, \mathcal{L}_2')\big) \cong \frac{2^{m_1-1}}{[F(\xi_{2^{m_1}}):F]} F(\xi_{2^{m_1}}) \cong \frac{2^{m_1-1}}{d_{m_1}} F_{d_{m_1}}$. Therefore, $\Delta_F(\mathcal{L}_2, \mathcal{L}_2') \cong \frac{2^{m_1-1}}{d_{m_1}} \mathfrak{Z}(F_{d_{m_1}})$. Hence, $F[\mathcal{L}_2] \cong 4F \oplus \bigoplus_{i=1}^{m_1} \frac{4\phi(2^i)}{d_i} F_{d_i} \oplus \frac{2^{m_1-1}}{d_{m_1}} \mathfrak{Z}(F_{d_{m_1}})$. ∎

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4 The Unit Loops of $F[\mathcal{L}_3]$ and $F[\mathcal{L}_4]$

In this section, we describe the structure of the unit loop of finite semisimple loop algebras of those indecomposable RA loops whose center is isomorphic to $C_{2^{m_1}} \times C_{2^{m_2}}$.

Lemma 4.1 Let $G = C_{2^a} \times C_{2^b}$. Then

$$F[C_{2^a} \times C_{2^b}] \cong F \bigoplus_{i=0}^{a} \bigoplus_{\substack{j=0 \\ i+j>0}}^{b} \frac{\phi(2^i)\phi(2^j)}{d(i,j)} F_{d(i,j)}$$

where $d(i,j) = \mathrm{ord}_{2^{\max\{i,j\}}}(q)$.

Proof Using Theorem 2.5 and 2.7, we have

$$\begin{aligned}
F[C_{2^a} \times C_{2^b}] &\cong (F[C_{2^a}])[C_{2^b}] \\
&\cong \Big(F \oplus \bigoplus_{i=1}^{a} \frac{\phi(2^i)}{[F(\xi_{2^i}) : F]} F(\xi_{2^i}) \Big) [C_{2^b}] \\
&\cong F[C_{2^b}] \oplus \bigoplus_{i=1}^{a} \frac{\phi(2^i)}{[F(\xi_{2^i}) : F]} F(\xi_{2^i})[C_{2^b}] \\
&\cong \Big(F \oplus \bigoplus_{k=1}^{b} \frac{\phi(2^k)}{[F(\xi_{2^k}) : F]} F(\xi_{2^k}) \Big) \\
&\quad \oplus \bigoplus_{i=1}^{a} \frac{\phi(2^i)}{[F(\xi_{2^i}) : F]} \Big(F(\xi_{2^i}) \oplus \bigoplus_{j=1}^{b} \frac{\phi(2^j)}{[F(\xi_{2^i}, \xi_{2^j}) : F]} F(\xi_{2^i}, \xi_{2^j}) \Big) \\
&\cong F \oplus \bigoplus_{i=1}^{a} \frac{\phi(2^i)}{[F(\xi_{2^i}) : F]} F(\xi_{2^i}) \oplus \bigoplus_{k=1}^{b} \frac{\phi(2^k)}{[F(\xi_{2^k}) : F]} F(\xi_{2^k}) \\
&\quad \oplus \bigoplus_{i=1}^{a} \bigoplus_{j=1}^{b} \frac{\phi(2^i)\phi(2^j)}{[F(\xi_{2^{\max\{i,j\}}}) : F]} F(\xi_{2^{\max\{i,j\}}}) \\
&\cong F \oplus \bigoplus_{i=0}^{a} \bigoplus_{\substack{j=0 \\ i+j>0}}^{b} \frac{\phi(2^i)\phi(2^j)}{d(i,j)} F_{d(i,j)}. \qquad \blacksquare
\end{aligned}$$

Theorem 4.2 The unit loop $\mathcal{U}(F[\mathcal{L}_3])$ satisfies

$$\mathcal{U}(F[\mathcal{L}_3]) \cong 4F \times \bigtimes_{i=0}^{m_1-1} \bigtimes_{\substack{j=0 \\ i+j>0}}^{m_2+1} \frac{4\phi(2^i)\phi(2^j)}{d(i,j)} F^*_{d(i,j)} \times \bigtimes_{k=0}^{m_2} \frac{2^{m_1-1}\phi(2^k)}{d(m_1,k)} \mathrm{GLL}(2, F_{d(m_1,k)}),$$

where $d(i,j) = \mathrm{ord}_{2^{\max\{i,j\}}}(q)$.

Proof Note that $\mathcal{Z}(\mathcal{L}_3) = \mathcal{Z}(G_3) = \langle t_1 \rangle \times \langle t_2 \rangle \cong C_{2^{m_1}} \times C_{2^{m_2}}$. Write

$$\mathcal{Z}(\mathcal{L}_3) \cong C_{2^{m_1}} \times C_{2^{m_2}} \times C_{2^0} \times C_{2^0}.$$

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Thus,

$$\begin{aligned}\mathcal{L}_3/\mathcal{L}_3' &\cong C_{2^{m_1-1}} \times C_{2^{m_2+1}} \times C_2 \times C_2,\\ F[\mathcal{L}_3/\mathcal{L}_3'] &\cong 4F[C_{2^{m_1-1}} \times C_{2^{m_2+1}}]\\ &\cong 4\Big(F \oplus \bigoplus_{i=0}^{m_1-1} \bigoplus_{\substack{j=0\\ i+j>0}}^{m_2+1} \frac{\phi(2^i)\phi(2^j)}{d(i,j)} F_{d(i,j)}\Big).\end{aligned}$$

Now

$$\begin{aligned}\mathcal{Z}\big(\Delta_F(\mathcal{L}_3,\mathcal{L}_3')\big) &\cong \frac{2^{m_1-1}}{[F(\xi_{2^{m_1}}):F]} F(\xi_{2^{m_1}})[C_{2^{m_2}}]\\ &\cong \frac{2^{m_1-1}}{[F(\xi_{2^{m_1}}):F]} F(\xi_{2^{m_1}}) \oplus \bigoplus_{k=1}^{m_2} \frac{2^{m_1-1}\phi(2^k)}{[F(\xi_{2^{\max\{m_1,k\}}}):F]} F(\xi_{2^{\max\{m_1,k\}}})\\ &\cong \bigoplus_{k=0}^{m_2} \frac{2^{m_1-1}\phi(2^k)}{d(m_1,k)} F_{d(m_1,k)}.\end{aligned}$$

Thus, $\Delta_F(\mathcal{L}_3,\mathcal{L}_3') \cong \bigoplus_{k=0}^{m_2} \frac{2^{m_1-1}\phi(2^k)}{d(m_1,k)} \mathfrak{Z}(F_{d(m_1,k)})$. Hence

$$F[\mathcal{L}_3] \cong \bigoplus_{i=0}^{m_1-1} \bigoplus_{j=0}^{m_2+1} \frac{4\phi(2^i)\phi(2^j)}{d(i,j)} F_{d(i,j)} \oplus \bigoplus_{k=0}^{m_2} \frac{2^{m_1-1}\phi(2^k)}{d(m_1,k)} \mathfrak{Z}(F_{d(m_1,k)}). \qquad \blacksquare$$

Theorem 4.3 The unit loop $\mathcal{U}(F[\mathcal{L}_4])$ satisfies

$$\mathcal{U}(F[\mathcal{L}_4]) \cong 2F^* \times \bigtimes_{i=0}^{m_1} \bigtimes_{\substack{j=0\\ i+j>0}}^{m_2+1} \frac{2\phi(2^i)\phi(2^j)}{d(i,j)} F^*_{d(i,j)} \times \bigtimes_{k=0}^{m_2} \frac{2^{m_1-1}\phi(2^k)}{d(m_1,k)} \operatorname{GLL}(2, F_{d(m_1,k)})$$

where $d(i,j) = \operatorname{ord}_{2^{\max\{i,j\}}}(q)$.

Proof As $\mathcal{Z}(\mathcal{L}_4) = \mathcal{Z}(G_4) = \langle t_1\rangle \times \langle t_2\rangle \cong C_{2^{m_1}} \times C_{2^{m_2}}$. Write

$$\mathcal{Z}(\mathcal{L}_4) \cong C_{2^{m_1}} \times C_{2^{m_2}} \times C_{2^0} \times C_{2^0}.$$

Thus

$$\begin{aligned}\mathcal{L}_4/\mathcal{L}_4' &\cong C_{2^{m_1}} \times C_{2^{m_2+1}} \times C_2,\\ F[\mathcal{L}_4/\mathcal{L}_4'] &\cong 2F[C_{2^{m_1}} \times C_{2^{m_2+1}}]\\ &\cong 2\Big(F \oplus \bigoplus_{i=0}^{m_1} \bigoplus_{\substack{j=0\\ i+j>0}}^{m_2+1} \frac{\phi(2^i)\phi(2^j)}{d(i,j)} F_{d(i,j)}\Big).\end{aligned}$$

Observe that $\mathcal{Z}(\mathcal{L}_4) = \mathcal{Z}(\mathcal{L}_3)$. So, by the previous theorem,

$$\mathcal{Z}\big(\Delta_F(\mathcal{L}_4,\mathcal{L}_4')\big) = \mathcal{Z}\big(\Delta_F(\mathcal{L}_3,\mathcal{L}_3')\big) \cong \bigoplus_{k=0}^{m_2} \frac{2^{m_1-1}\phi(2^k)}{d(m_1,k)} F_{d(m_1,k)}.$$

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Thus, $\Delta_F(\mathcal{L}_4, \mathcal{L}_4') \cong \bigoplus_{k=0}^{m_2} \frac{2^{m_1-1}\phi(2^k)}{d(m_1,k)} \mathfrak{Z}(F_{d(m_1,k)})$. Hence

$$F[\mathcal{L}_4] \cong 2F \oplus \bigoplus_{i=0}^{m_1} \bigoplus_{\substack{j=0\\ i+j>0}}^{m_2+1} \frac{2\phi(2^i)\phi(2^j)}{d(i,j)} F_{d(i,j)} \oplus \bigoplus_{k=0}^{m_2} \frac{2^{m_1-1}\phi(2^k)}{d(m_1,k)} \mathfrak{Z}(F_{d(m_1,k)}). \qquad \blacksquare$$

5 The Unit Loops of $F[\mathcal{L}_5]$ and $F[\mathcal{L}_6]$

In this section we determine the structure of the unit loop of finite semisimple loop algebras of those indecomposable RA loops whose center is isomorphic to $C_{2^{m_1}} \times C_{2^{m_2}} \times C_{2^{m_3}}$.

Working similar to Lemma 4.1, the following lemma is straightforward.

Lemma 5.1 Let $G = C_{2^a} \times C_{2^b} \times C_{2^c}$. Then

$$F[C_{2^a} \times C_{2^b} \times C_{2^c}] \cong F \oplus \bigoplus_{i=0}^{a} \bigoplus_{\substack{j=0\\ i+j+k>0}}^{b} \bigoplus_{k=0}^{c} \frac{\phi(2^i)\phi(2^j)\phi(2^k)}{d(i,j,k)} F_{d(i,j,k)}$$

where $d(i,j,k) = \mathrm{ord}_{2^{\max\{i,j,k\}}}(q)$.

Theorem 5.2 The unit loop $\mathcal{U}(F[\mathcal{L}_5])$ satisfies

$$\mathcal{U}(F[\mathcal{L}_5]) \cong 2F^* \times \bigtimes_{i=0}^{m_1-1} \bigtimes_{\substack{j=0\\ i+j+k>0}}^{m_2+1} \bigtimes_{k=0}^{m_3+1} \frac{2\phi(2^i)\phi(2^j)\phi(2^k)}{d(i,j,k)} F^*_{d(i,j,k)}$$
$$\times \bigtimes_{i_1=0}^{m_2} \bigtimes_{j_1=0}^{m_3} \frac{2^{m_1-1}\phi(2^{i_1})\phi(2^{j_1})}{d(m_1,i_1,j_1)} \mathrm{GLL}(2, F_{d(m_1,i_1,j_1)})$$

where $d(i,j,k) = \mathrm{ord}_{2^{\max\{i,j,k\}}}(q)$.

Proof We have $\mathcal{Z}(\mathcal{L}_5) = \mathcal{Z}(G_5) = \langle t_1\rangle \times \langle t_2\rangle \times \langle t_3\rangle \cong C_{2^{m_1}} \times C_{2^{m_2}} \times C_{2^{m_3}}$. Write

$$\mathcal{Z}(\mathcal{L}_5) \cong C_{2^{m_1}} \times C_{2^{m_2}} \times C_{2^{m_3}} \times C_{2^0}.$$

Thus

$$\begin{aligned}
\mathcal{L}_5/\mathcal{L}_5' &\cong C_{2^{m_1-1}} \times C_{2^{m_2+1}} \times C_{2^{m_3+1}} \times C_2,\\
F[\mathcal{L}_5/\mathcal{L}_5'] &\cong 2F[C_{2^{m_1-1}} \times C_{2^{m_2+1}} \times C_{2^{m_3+1}}]\\
&\cong 2\Big(F \oplus \bigoplus_{i=0}^{m_1-1} \bigoplus_{\substack{j=0\\ i+j+k>0}}^{m_2+1} \bigoplus_{k=0}^{m_3+1} \frac{\phi(2^i)\phi(2^j)\phi(2^k)}{d(i,j,k)} F_{d(i,j,k)}\Big),\\
\mathcal{Z}\big(\Delta_F(\mathcal{L}_5, \mathcal{L}_5')\big) &\cong \frac{2^{m_1-1}}{[F(\xi_{2^{m_1}}):F]} F(\xi_{2^{m_1}})[C_{2^{m_2}} \times C_{2^{m_3}}]\\
&\cong \Big(\frac{2^{m_1-1}}{[F(\xi_{2^{m_1}}):F]} F(\xi_{2^{m_1}})\\
&\qquad \oplus \bigoplus_{i_1=1}^{m_2} \frac{2^{m_1-1}\phi(2^{i_1})}{[F(\xi_{2^{m_1}}, \xi_{2^{i_1}}):F]} F(\xi_{2^{m_1}}, \xi_{2^{i_1}})\Big)[C_{2^{m_3}}]
\end{aligned}$$

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$$\cong \frac{2^{m_1-1}}{[F(\xi_{2^{m_1}}):F]}F(\xi_{2^{m_1}}) \oplus \bigoplus_{j_1=1}^{m_3} \frac{2^{m_1-1}\phi(2^{j_1})}{[F(\xi_{2^{m_1}},\xi_{2^{j_1}}):F]}F(\xi_{2^{m_1}},\xi_{2^{j_1}})$$
$$\oplus \bigoplus_{i_1=1}^{m_2} \frac{2^{m_1-1}\phi(2^{i_1})}{[F(\xi_{2^{m_1}},\xi_{2^{i_1}}):F]}F(\xi_{2^{m_1}},\xi_{2^{i_1}})$$
$$\oplus \bigoplus_{i_1=1}^{m_2}\bigoplus_{k_1=1}^{m_3} \frac{2^{m_1-1}\phi(2^{i_1})\phi(2^{k_1})}{[F(\xi_{2^{m_1}},\xi_{2^{i_1}},\xi_{2^{k_1}}):F]}F(\xi_{2^{m_1}},\xi_{2^{i_1}},\xi_{2^{k_1}})$$
$$\cong \bigoplus_{i_1=0}^{m_2}\bigoplus_{j_1=0}^{m_3} \frac{2^{m_1-1}\phi(2^{i_1})\phi(2^{j_1})}{d(m_1,i_1,j_1)}F_{d(m_1,i_1,j_1)}.$$

Thus, $\Delta_F(\mathcal{L}_5,\mathcal{L}_5') \cong \bigoplus_{i_1=0}^{m_2}\bigoplus_{j_1=0}^{m_3} \frac{2^{m_1-1}\phi(2^{i_1})\phi(2^{j_1})}{d(m_1,i_1,j_1)}\mathfrak{Z}(F_{d(m_1,i_1,j_1)})$. Hence

$$F[\mathcal{L}_5] \cong 2F \oplus \bigoplus_{i=0}^{m_1-1}\bigoplus_{\substack{j=0\\ i+j+k>0}}^{m_2+1}\bigoplus_{k=0}^{m_3+1} \frac{2\phi(2^i)\phi(2^j)\phi(2^k)}{d(i,j,k)}F_{d(i,j,k)}$$
$$\oplus \bigoplus_{i_1=0}^{m_2}\bigoplus_{j_1=0}^{m_3} \frac{2^{m_1-1}\phi(2^{i_1})\phi(2^{j_1})}{d(m_1,i_1,j_1)}\mathfrak{Z}(F_{d(m_1,i_1,j_1)}). \qquad \blacksquare$$

Theorem 5.3 The unit loop $\mathcal{U}(F[\mathcal{L}_6])$ satisfies

$$\mathcal{U}(F[\mathcal{L}_6]) \cong F^* \times \bigtimes_{i=0}^{m_1}\bigtimes_{\substack{j=0\\ i+j+k>0}}^{m_2+1}\bigtimes_{k=0}^{m_3+1} \frac{\phi(2^i)\phi(2^j)\phi(2^k)}{d(i,j,k)}F^*_{d(i,j,k)}$$
$$\times \bigtimes_{i_1=0}^{m_2}\bigtimes_{j_1=0}^{m_3} \frac{2^{m_1-1}\phi(2^{i_1})\phi(2^{j_1})}{d(m_1,i_1,j_1)}\,\mathrm{GLL}(2,F_{d(m_1,i_1,j_1)})$$

where $d(i,j,k) = \mathrm{ord}_{2^{\max\{i,j,k\}}}(q)$.

Proof Note that $\mathcal{Z}(\mathcal{L}_6) = \mathcal{Z}(G_5) = \langle t_1\rangle \times \langle t_2\rangle \times \langle t_3\rangle \cong C_{2^{m_1}} \times C_{2^{m_2}} \times C_{2^{m_3}}$. Write

$$\mathcal{Z}(\mathcal{L}_6) \cong C_{2^{m_1}} \times C_{2^{m_2}} \times C_{2^{m_3}} \times C_{2^0}.$$

Thus,

$$\mathcal{L}_6/\mathcal{L}_6' \cong C_{2^{m_1}} \times C_{2^{m_2+1}} \times C_{2^{m_3+1}}.$$

$$F[\mathcal{L}_6/\mathcal{L}_6'] \cong F[C_{2^{m_1}} \times C_{2^{m_2+1}} \times C_{2^{m_3+1}}]$$
$$\cong F \oplus \bigoplus_{i=0}^{m_1}\bigoplus_{\substack{j=0\\ i+j+k>0}}^{m_2+1}\bigoplus_{k=0}^{m_3+1} \frac{\phi(2^i)\phi(2^j)\phi(2^k)}{d(i,j,k)}F_{d(i,j,k)}.$$

As $\mathcal{Z}(\mathcal{L}_6) = \mathcal{Z}(\mathcal{L}_5)$, by the previous theorem, we have

$$\mathcal{Z}\big(\Delta_F(\mathcal{L}_6,\mathcal{L}_6')\big) = \mathcal{Z}\big(\Delta_F(\mathcal{L}_5,\mathcal{L}_5')\big) \cong \bigoplus_{i_1=0}^{m_2}\bigoplus_{j_1=0}^{m_3} \frac{2^{m_1-1}\phi(2^{i_1})\phi(2^{j_1})}{d(m_1,i_1,j_1)}F_{d(m_1,i_1,j_1)}.$$

So

$$\Delta_F(\mathcal{L}_6,\mathcal{L}_6') \cong \bigoplus_{i_1=0}^{m_2}\bigoplus_{j_1=0}^{m_3} \frac{2^{m_1-1}\phi(2^{i_1})\phi(2^{j_1})}{d(m_1,i_1,j_1)}\mathfrak{Z}(F_{d(m_1,i_1,j_1)}).$$

Hence

$$F[\mathcal{L}_6] \cong F \oplus \bigoplus_{\substack{i=0 \\ i+j+k>0}}^{m_1} \bigoplus_{j=0}^{m_2+1} \bigoplus_{k=0}^{m_3+1} \frac{\phi(2^i)\phi(2^j)\phi(2^k)}{d(i,j,k)} F_{d(i,j,k)}$$

$$\oplus \bigoplus_{i_1=0}^{m_2} \bigoplus_{j_1=0}^{m_3} \frac{2^{m_1-1}\phi(2^{i_1})\phi(2^{j_1})}{d(m_1,i_1,j_1)} \mathfrak{Z}(F_{d(m_1,i_1,j_1)}). \qquad \blacksquare$$

6 The Unit Loop of $F[\mathcal{L}_7]$

In this section, we determine the structure of the unit loop of finite semisimple loop algebra of indecomposable RA loop whose center is isomorphic to

$$C_{2^{m_1}} \times C_{2^{m_2}} \times C_{2^{m_3}} \times C_{2^d}.$$

The following lemma is useful.

Lemma 6.1 Let $G = C_{2^a} \times C_{2^b} \times C_{2^c} \times C_{2^d}$. Then

$$F[C_{2^a} \times C_{2^b} \times C_{2^c} \times C_{2^d}] \cong F \bigoplus_{i=0}^{a} \bigoplus_{j=0}^{b} \bigoplus_{\substack{k=0 \\ i+j+k+l>0}}^{c} \bigoplus_{l=0}^{d} \frac{\phi(2^i)\phi(2^j)\phi(2^k)\phi(2^l)}{d(i,j,k,l)} F_{d(i,j,k,l)}$$

where $d(i,j,k,l) = \operatorname{ord}_{2^{\max\{i,j,k,l\}}}(q)$.

Theorem 6.2 The unit loop $\mathcal{U}(F[\mathcal{L}_7])$ satisfies

$$\mathcal{U}(F[\mathcal{L}_7]) \cong F^* \times \bigtimes_{i=0}^{m_1-1} \bigtimes_{j=0}^{m_2+1} \bigtimes_{\substack{k=0 \\ i+j+k+l>0}}^{m_3+1} \bigtimes_{l=0}^{d+1} \frac{\phi(2^i)\phi(2^j)\phi(2^k)\phi(2^l)}{d(i,j,k,l)} F^*_{d(i,j,k,l)}$$

$$\times \bigtimes_{i_1=0}^{m_2} \bigtimes_{j_1=0}^{m_3} \bigtimes_{k_1=0}^{d} \frac{2^{m_1-1}\phi(2^{i_1})\phi(2^{j_1})\phi(2^{k_1})}{d(m_1,i_1,j_1,k_1)} \operatorname{GLL}(2, F_{d(m_1,i_1,j_1,k_1)})$$

where $d(i,j,k,l) = \operatorname{ord}_{2^{\max\{i,j,k,l\}}}(q)$.

Proof We have

$$\begin{aligned} \mathcal{Z}(\mathcal{L}_7) &= \mathcal{Z}(G_5 \times \langle w \rangle) \\ &= \langle t_1 \rangle \times \langle t_2 \rangle \times \langle t_3 \rangle \times \langle w \rangle \\ &\cong C_{2^{m_1}} \times C_{2^{m_2}} \times C_{2^{m_3}} \times C_{2^d}. \end{aligned}$$

Thus,

$$\begin{aligned} \mathcal{L}_7/\mathcal{L}_7' &\cong C_{2^{m_1-1}} \times C_{2^{m_2+1}} \times C_{2^{m_3+1}} \times C_{2^{d+1}} \\ F[\mathcal{L}_7/\mathcal{L}_7'] &\cong F[C_{2^{m_1-1}} \times C_{2^{m_2+1}} \times C_{2^{m_3+1}} \times C_{2^{d+1}}] \\ &\cong F \oplus \bigoplus_{i=0}^{m_1-1} \bigoplus_{j=0}^{m_2+1} \bigoplus_{\substack{k=0 \\ i+j+k+l>0}}^{m_3+1} \bigoplus_{l=0}^{d+1} \frac{\phi(2^i)\phi(2^j)\phi(2^k)\phi(2^l)}{d(i,j,k,l)} F_{d(i,j,k,l)}. \end{aligned}$$

Now

$$\mathcal{Z}\big(\Delta_F(\mathcal{L}_7,\mathcal{L}_7')\big) \cong \frac{2^{m_1-1}}{[F(\xi_{2^{m_1}}):F]}F(\xi_{2^{m_1}})[C_{2^{m_2}}\times C_{2^{m_3}}\times C_{2^d}]$$
$$\cong \bigoplus_{i_1=0}^{m_2}\bigoplus_{j_1=0}^{m_3}\bigoplus_{k_1=0}^{d}\frac{2^{m_1-1}\phi(2^{i_1})\phi(2^{j_1})\phi(2^{k_1})}{d(m_1,i_1,j_1,k_1)}F_{d(m_1,i_1,j_1,k_1)}.$$

So

$$\Delta_F(\mathcal{L}_7,\mathcal{L}_7') \cong \bigoplus_{i_1=0}^{m_2}\bigoplus_{j_1=0}^{m_3}\bigoplus_{k_1=0}^{d}\frac{2^{m_1-1}\phi(2^{i_1})\phi(2^{j_1})\phi(2^{k_1})}{d(m_1,i_1,j_1,k_1)}\mathfrak{Z}(F_{d(m_1,i_1,j_1,k_1)}).$$

Hence

$$F[\mathcal{L}_7] \cong F \oplus \bigoplus_{i=0}^{m_1-1}\bigoplus_{j=0}^{m_2+1}\bigoplus_{\substack{k=0\\ i+j+k+l>0}}^{m_3+1}\bigoplus_{l=0}^{d+1}\frac{\phi(2^i)\phi(2^j)\phi(2^k)\phi(2^l)}{d(i,j,k,l)}F_{d(i,j,k,l)}$$
$$\oplus \bigoplus_{i_1=0}^{m_2}\bigoplus_{j_1=0}^{m_3}\bigoplus_{k_1=0}^{d}\frac{2^{m_1-1}\phi(2^{i_1})\phi(2^{j_1})\phi(2^{k_1})}{d(m_1,i_1,j_1,k_1)}\mathfrak{Z}(F_{d(m_1,i_1,j_1,k_1)}). \qquad \blacksquare$$

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