Proceedings of the Edinburgh Mathematical Society (1990) 33, 443–460 ©

MEASURE VALUED SOLUTIONS OF ASYMPTOTICALLY HOMOGENEOUS SEMILINEAR HYPERBOLIC SYSTEMS IN ONE SPACE VÁRIABLE

by F. DEMENGEL and J. RAUCH

(Received 7th March 1989)

We study systems which in characteristic coordinates have the form

$$\partial u/\partial t + A(t,x)\,\partial u/\partial x + F(t,x,u) = g(t,x)$$

where A is a $k\times k$ diagonal matrix with distinct real eigenvalues. The nonlinearity F is assumed to be asymptotically homogeneous in the sense, that it is a sum of two terms, one positively homogeneous of degree one in u and a second which is sublinear in u and vanishes when $u=0$. In this case, $F(t,x,u(t))$ is meaningful provided that $u(t)$ is a Radon measure, and, for Radon measure initial data there is a unique solution (Theorem 2.1).

The main result asserts that if μ_n is a sequence of initial data such that, in characteristic coordinates, the positive and negative parts of each component, $(\mu_k^n)^{\pm}$, converge weakly to $\mu^{\pm}$, then the solutions coverge weakly and the limit has an interesting description given by a **nonlinear superposition principle.**

Simple weak converge of the initial data does not imply weak convergence of the solutions.

1980 *Mathematics subject classification* (1985 Revision): 35B25, 35L45, 35D99.

Introduction

This paper is devoted to the study of solutions to semilinear hyperbolic systems in one space variable whose initial data are or approach measures. The results continue the line of investigation initiated in [5], [6]. The systems have the form

$$\partial u/\partial t + A(t,x)\,\partial u/\partial x + F(t,x,u) = g(t,x)$$

where $t,x\in\mathbb{R}^2$, $u(t,x)=(u_1,\dots u_k)$ is $\mathbb{R}^k$ valued, and A is a $k\times k$ matrix valued function with k distinct real eigenvalues for each t,x. The nonlinearity is assumed to be asymptotically homogeneous in the sense that it is a sum of two terms, one positively homogeneous of degree one in u and a second which is sublinear and vanishes when $u=0$. In this case, $F(t,x,u(t))$ is meaningful provided that $u(t)$ is a Radon measure, in fact if F_1 is the homogeneous part and F_2 the sublinear part then $F(u)=F_1(u)+F_2(u_{ac})$ where u_{ac} is the absolutely continuous part and $F_1(u)$ is defined as in [1], [3]. This definition is recalled in the next section.

https://doi.org/10.1017/S0013091500004855 Published online by Cambridge University Press

Rauch and Reed observed that if F_1 is linear and L^1 initial data converged to a measure μ in a suitably restricted way then the classical solutions converged to a limit determined by μ. The limit had a relatively simple description given by a nonlinear superposition principle. The singular part of the limiting measure is determined as a solution of a linear initial value problem and the absolutely continuous part as the solution of a semilinear equation in which the singular part enters as a source term. The results of that paper are here extended in three ways.

1. The term homogeneous of degree one is allowed to be nonlinear.

2. The limiting function is interpreted as a measure valued solution and suitable existence, uniqueness and continuous dependence theorems (for norm convergence of the data) are proved (Theorem 2.1). A nonlinear superposition principle is proved (Section 3).

3. For weakly convergent data we have sharp results telling when the resulting solutions converge.

On the other hand, some of the sharpest results in [**6**] asserting that weakly convergent solutions converge strongly away from characteristics issued from "bad points" are not generalized. It is our feeling that such results are correct in the present context.

It is easy to see that weak convergence alone is insufficient to guarantee convergence of the resulting solutions. This is typical of nonlinear equations and is illustrated by the following example.

Example. The scalar initial value problem

$$u_t = |u|, \qquad u(0,\cdot) = \mu$$

has solution $u = \mu^+ e^t + \mu^- e^{-t}$ where $\mu^\pm$ are the positive and negative parts of μ. Consider a sequence of Cauchy data μ^n converging weakly to a limit μ. In order for the solutions u^n to converge weakly to u, one must have weak convergence of the positive and negative parts, $(\mu^n)^\pm$, to the corresponding parts $\mu^\pm$. Weak convergence of μ^n to μ is not enough.

In [**2, 4**] we studied the convergence of functions $F(\mu^n)$ for weakly convergent arguments and F as above. We found necessary and sufficient conditions on μ^n in order that $F(\mu^n)$ converge weakly to $F(\mu)$ for all such F. As such terms appear in our differential equations it is natural to suspect that we must assume that the μ^n at least satisfy these conditions. We were surprised to discover that in fact one can get by with less. The reason is that the conditions are placed on the initial data and the differential equation forces a great deal of structure on the solution in space-time so that even though the trace at $t=0$ may violate our condition the solution in $\mathbb{R}^2$ may satisfy it. What we require of the initial data is easiest to describe in characteristic coordinates for $\mathbb{R}^k$. In these coordinates, the matrix A is diagonal and the condition on the initial data is that the positive and negative parts of each component, $(\mu_k^n)^\pm$, converge weakly to $\mu_k^\pm$. A part of our results were announced in [**3**].

1. Basic definitions

This paper is devoted to the study of measure valued solutions of $k \times k$ semilinear strictly hyperbolic systems in one space dimension:

$$\partial_t u + A(t,x)\partial_x u + F(t,x,u) = g(t,x). \tag{1.1}$$

Here $A \in C^1([0,T] \times \mathbb{R}: \mathrm{Hom}(\mathbb{R}^k))$ has k distinct real eigenvalues $\lambda_1(t,x) < \cdots < \lambda_k(t,x)$ with the same smoothness as A. The nonlinear function $F \in C([0,T] \times \mathbb{R} \times \mathbb{R}^k)$ will be supposed asymptotically homogeneous in the sense that

$$F(t,x,0) = 0, \text{ and, } \lim_{\lambda \to \infty} F(t,x,\lambda\xi)/|\lambda\xi| \doteq F_\infty(t,x,\xi)$$

uniformly on compact subsets of $[0,t] \times \mathbb{R} \times \mathbb{R}^k$. Then, F_∞ is uniformly continuous and homogeneous in u, while $G \equiv F - F_\infty$ is continuous and sublinear in u. Both functions vanish when $u = 0$. We further assume that F is uniformly Lipschitzian in u.

We denote by $(t, X_i(t,x))$ the flow of the ith characteristic vector field $\partial_t + \lambda_i(t,x)\partial_x$. Thus

$$\partial_t X_i(t,x) = \lambda_i(t, X_i(t,x)), \quad X_i(0,x) = x.$$

Then $t \mapsto (t, X_i(t,x))$ is the ith characteristic through $(0,x)$. We let $\eta(t,x)$ be the foot of the ith characteristic through t,x so

$$X_i(t, \eta_i(t,x)) = x.$$

Given a compact space interval I, $|I| \neq 0$, and $T > 0$ we denote by R the domain of determinacy of $\{T\} \times I$. R is bounded on the left by the λ_k characteristic through (T, $\inf I$), and on the right by the λ_1 characteristic through ($T, \sup I$). We denote by R_t the x-cross-section of R at time t, that is the interval of x such that $(t,x) \in R$.

The solutions, u, are viewed as functions of $t \in [0,T]$ whose value at time t is a Borel measure on R_t.

Convention. $M(R_t: \mathbb{R}^k)$ denotes the space of finite $\mathbb{R}^k$ valued Borel measures on R_t. With the total variation norm, it is the set of all continuous maps from $C(R_t: \mathbb{R})$ to $\mathbb{R}^k$. $M(R_t: \mathbb{R}^k)_*$ denotes the same space endowed with the associated weak star topology.

Example. $u(t) = \delta(x - t)$ solves $(\partial_t + \partial_x)u = 0$ and is discontinuous with values in M and continuous with values in M_*.

If J is a compact interval and u is a continuous function on $[0,T]$ with values in $M(J)_*$, then, the uniform boundedness principle implies that

$$\sup_{t \in [0,T]} \|u(t)\|_{M(J)} < \infty.$$

https://doi.org/10.1017/S0013091500004855 Published online by Cambridge University Press

$C([0,T]:M(J)_*)$ is a Banach space in that norm, but, the functions $u(t)$ and $u_\varepsilon(t) \doteq j_\varepsilon(x) *_x u(t)$ are not close in that topology. For us a weak topology on $C([0,T]:M_*)$ will be more important.

There are (at least) two natural weak topologies on $C([0,T]:M(J)_*)$. The smaller of the two is defined by the seminorms

$$\sup_{t\in[0,T]} |\langle u(t), \phi\rangle|, \quad \phi\in C(J).$$

The larger is defined by seminorms for which the test function may depend on time,

$$\sup_{t\in[0,T]} |\langle u(t), \psi(t,\cdot)\rangle|, \quad \psi\in C([0,T]\times J). \tag{1.2}$$

Note that $\langle u(t), \psi(t,\cdot)\rangle$ is a continuous function of time.

On subsets of $C([0,T]:M_*)$ which are bounded in $L^\infty([0,T]:M)$, the two topologies agree. Our applications will involve such bounded sets.

Convention. The topology on $C([0,T]:M(J)_*)$ will be that defined by the seminorms (1.2).

Convergence in this topology is a stronger conclusion than convergence in the smaller topology.

Mollification in x shows that $C([0,T]:C^\infty(J))$ is dense in $C([0,T]:M(J)_*)$.

The solutions, u, are functions of $t\in[0,T]$ whose value at time t is a measure on R_t. As $u(t)$ is defined on a time dependent domain, R_t, the definition of the basic spaces of distributions is slightly awkward. All of our solutions will at least lie in $C^{-1}(R:\mathbb{R}^k)$ the dual of $C^1(R:\mathbb{R}^k)$. Choose

$$\chi: R\to[0,T]\times I$$

a C^1 diffeomorphism such that χ preserves the lines $t=$constant. Then χ is given by $t'=t$, $x'=x'(t,x)$. Note that since the characteristics need only be C^1, we cannot expect to find a more regular χ.

Definition. The spaces $L^1([0,T]:M(R_t))$, $C([0,T]:M(R_t)_*),\dots$, etc. are defined by transforming to $[0,T]\times I$. For example, $u\in C([0,T]: M(R_t:R^k)_*)$ if and only if $u\in C^{-1}(R:\mathbb{R}^k)$ and $u\circ\chi^{-1}\in C([0,T]: M(I:\mathbb{R}^k)_*)$.

Remarks. 1. $u\circ\chi^{-1}$ is automatically in $C^{-1}([0,T]\times I:\mathbb{R}^k)$.

2. Having chosen the weak topology on $C([0,T]:M(I)_*)$ as defined by test functions $\psi(t,x)$, it is easy to show that the spaces and the topologies induced by different maps χ are the same.

3. The norms $\|u(t)\|_{M(R_t:\mathbb{R}^k)}$ and $\|u\circ\chi^{-1}(t)\|_{M(I:\mathbb{R}^k)}$ are uniformly equivalent for $t\in[0,T]$.

Following [1] and [2,4], the map $u \mapsto F(t,x,u)$ yields a well-defined globally Lipschitzian map of $L^1([0,T]:M(R_t:\mathbb{R}^k))$ to itself. In the definition, t is a parameter. First F is split into homogeneous and sublinear parts,

$$F(t,x,u(t)) \doteq F_\infty(t,x,u(t)) + G(t,x,u(t)).$$

$G(t,x,u(t))$ is defined to be equal to $G(t,x,g(t,x))\,dx$ where $u(t)=g(t,x)\,dx+u(t)_{\text{sing}}$ is the Lebesgue decomposition. $F_\infty(t,x,u(t))$ is defined by choosing $v\in M(R_t:\mathbb{R})$ with $|u(t)|\ll v$ then

$$F_\infty(t,x,u(t)) \doteq F(t,x,du(t)/dv)v.$$

The map is globally Lipschitzian in the sense that if

$$|F(t,x,\xi)-F(t,x,\eta)| \leqq \Lambda|\xi-\eta| \quad \forall t,x,\xi,\eta$$

then

$$\|(F(t,x,u)-F(t,x,v))(t)\|_{M(R_t:\mathbb{R}^k)}$$
$$\leqq \|\nabla_x\chi^{-1}\|_{L^\infty}\|\nabla_x\chi\|_{L^\infty}\Lambda\|u(t)-v(t)\|_{M(R_t:\mathbb{R}^k)} \quad \text{a.e. } t.$$

That this definition is natural is best seen by showing that $F(t,x,u)=\lim F(t,x,u_\varepsilon)$ where u_ε are suitable regularizations of u. Mollifying in x is the most obvious choice but there are problems at ∂R_t. To overcome this we write u first as a sum of two terms each vanishing near one of the boundaries of R_t. Toward this end choose $\phi_-\in C^\infty(R)$ with $0\leqq\phi_-\leqq 1$, ϕ_- identically one near the left hand boundary of R, and, identically zero near the right hand boundary. Let $\phi_+\doteq 1-\phi_-$ and $u_\pm \doteq \phi_\pm u$. Choose $j^+\in C_0^\infty(\mathbb{R})$, $j^+\geqq 0$, $\int j^+=1$ and $\operatorname{supp} j^+\subset[0,1]$. Let $j^-(s)\doteq j^+(-s)$, $j_\varepsilon^\pm(s)\doteq\varepsilon^{-1}j^\pm(s)$. Then for $\varepsilon<\operatorname{dist}(\operatorname{supp}\phi_-, r.h.\,\partial R)$,

$$u_\varepsilon \doteq j_\varepsilon^+ *_x u_- + j_\varepsilon^- *_x u_+. \tag{1.3}$$

Then (see [3]), in $M(R_t:\mathbb{R}^k)_*$,

$$F(t,x,u_\varepsilon(t)) \to F(t,x,u(t)) \quad \text{a.e. } t.$$

2. Existence, uniqueness, and continuity for the norm topology

Theorem 2.1. (a) *For $g\in L^1([0,T]:M(R_t:\mathbb{R}^k))$ and μ_0 in $M(R_0:\mathbb{R}^k)$ there is one and only one solution $u\in L^1([0,T]:M(R_t:\mathbb{R}^k))$ to* (1.1) *with $u(0)=\mu_0$.*

(b) *This solution lies in $C([0,T]:M(R_t:\mathbb{R}^k)_*)$.*

(c) *The map $g,\mu_0\mapsto u$ is uniformly Lipschitzian and even more, there is a constant c such that for any two such solutions u^1,u^2 and any $t\in[0,T]$,*

https://doi.org/10.1017/S0013091500004855 Published online by Cambridge University Press

$$\|u^1(t)-u^2(t)\|_{M(R_t)} \leqq c\left\{\|u^1(0)-u^2(0)\|_{M(R_t)}+\int_0^t \|g^1(t)-g^2(t)\|_{M(R_t)}\,dt\right\}.$$

The proof of this result uses two lemmas concerning the initial value problem for the linear operator $\partial_t + A\partial_x \doteq L$.

Lemma 2.2. *If* $u \in L^1([0,T]:M(R_t)]$ *and* $Lu \in L^1([0,T]:M(R_t))$ *then* $u \in C([0,T]:C^{-1}(R_t))$.

Proof. In the coordinates so that $R=[0,T]\times I$, we have

$$u \in L^1([0,T]:M(I)).$$

$$u_t = -\tilde{A}\partial_x u - \tilde{B}u + Lu \in L^1([0,T]:C^{-1}(I))$$

so $u \in W^{1,1}([0,T]:C^{-1}(I)) \subset C([0,T]:C^{-1}(I))$. □

Remark. This shows that if $u \in L^1([0,T]:M(R_t))$ satisfies (1.1) then $u \in C([0,T]:C^{-1}(R_t))$ and it makes sense to speak of the Cauchy data, $u(0)$.

Lemma 2.3. *For any* $1 \leqq j \leqq k$, $v_0 \in M(R_0)$ *and* $g \in L^1([0,T]:M(R_t))$ *there is one and only one* $v \in L^1([0,T]:\ M(R_t))$ *satisfying*

$$(\partial_t + \lambda_j \partial_x)v = g \text{ in } C^{-1}(\operatorname{Int} R), \quad u(\underline{t},\cdot)=u_0.$$

This v *lies in* $C([0,T]:M(R_t)_*)$ *and there is a* $c>0$ *independent of* $t \geqq \underline{t} \geqq 0$, g, u_0 *so that*

$$\|v(t)\|_{M(R_t)} \leqq c\left(\|v(t)\|_{M(R_{t'})}+\int_{\underline{t}}^t \|g(s)\|_{M(R_s)}\,ds\right). \tag{2.1}$$

Proof. A C^1 change of coordinates $t'=t$, $x'=x'(t,x)$ converts $\partial_t+\lambda\partial_x$ to $\partial_{t'}$. The partial differential equation becomes the trivial equation $\partial_{t'} v = g$. □

Proof of Theorem 2.1. Choose a $U \in C^1(R:GL(\mathbb{R}^k))$ so that $U^{-1}AU = \operatorname{diag}(\lambda_1,\lambda_2,\ldots,\lambda_k)$ then $\tilde{u} \doteq U^{-1}u$ must satisfy

$$\partial_t \tilde{u} + \operatorname{diag}(\lambda_j)\partial_x \tilde{u} + \tilde{F}(t,x,\tilde{u}) = \tilde{g}, \quad \tilde{u}(0,\cdot)=\tilde{u}_0 \tag{2.2}$$

where

$$\tilde{g}=U^{-1}g, \quad \tilde{u}_0 = U^{-1}u_0 \tag{2.3}$$

$$\tilde{F}(t,x,\tilde{u}) = U^{-1}F(t,x,U\tilde{u}) + U^{-1}(U_t + AU_x)\tilde{u}. \tag{2.4}$$

In this way we may replace the original problem with one where $A=\operatorname{diag}(\lambda_i)$. We drop the tildes and suppose that $A=\operatorname{diag}(\lambda_i)$.

The solution is the limit as $n\to\infty$ of Picard iterates u^n defined by $u^n(0,\cdot)=u_0$ and

$$(\partial_t+\lambda_i\partial_x)u_i^0=g_i \qquad \text{for } n=0$$

$$(\partial_t+\lambda_i\partial_x)u_i^n+F_i(t,x,u^{n-1})=g_i \quad \text{for } n\geqq 1.$$

Lemma 2.3 shows that $u^n\in C([0,T]:M(R_t)_*)$ and that for $n\geqq 1$

$$\|(u^{n+1}-u^n)(t)\|_{M(R_t)}\leqq c\int_0^t \|F(t,x,u^n)-F(t,x,u^{n-1})\|_{M(R_s)}\,ds$$

$$\leqq c\int_0^t \|u^n(s)-u^{n-1}(s)\|_{M(R_s)}\,ds.$$

If

$$K\equiv\int_0^T \|u^1(s)-u^0(s)\|_{M(R_s)}\,ds$$

we prove inductively, using (2.1), that

$$\|u^{n+1}(t)-u^n(t)\|_{M(R_t)}\leqq\frac{(ct)^n}{n!}K.$$

This suffices to show the convergence of u^n in $C([0,T]:M(R_t)_*)$ to a solution.

For the continuous dependence, suppose that u^1 and u^2 are two solutions. Then

$$(\partial_t+\lambda_i\partial_x)(u_i^1-u_i^2)=F_i(t,x,u^1)-F_i(t,x,u^2)+g^1-g^2$$

$$(u_i^1-u_i^2)(0,\cdot)=0.$$

The estimate from Lemma 2.3 yields

$$\|u^1(t)-u^2(t)\|_{M(R_t)}\leqq c\int_0^t \|u^1(s)-u^2(s)\|_{M(R_s)}+\|g^1(s)-g^2(s)\|_{M(R_s)}\,ds.$$

Gronwall's inequality yields the inequality of part (c). □

3. Formulas for the singular and absolutely continuous parts of *u*

We work in coordinates for $\mathbb{R}^k$ which diagonalize A. As before we drop the tildes so the system takes the form

$$(\partial_t + \lambda_i \partial_x) u_i + F_i(t, x, u) = g_i.$$

One of the striking results in [6] is that if $F = B(t, x)u +$ sublinear with B a $k \times k$ matrix there is a sort of superposition principle. The part of u singular with respect to Lebesgue measure satisfies a linear equation even though the overall dynamics is nonlinear. Precisely, if $D(t, x)$ is the diagonal part of B, the singular part satisfies

$$(\partial_t + (\text{diag}\,\lambda_i)\partial_x + D) u_{\text{sing}} = 0.$$

We begin by defining analogues of the diagonal and off diagonal parts of B.

Definition. The *diagonal part*, D, *of* F is defined by $D = (D_1, \ldots, D_k)$ where

$$D_i(t, x, u_1, \ldots, u_k) = (F_\infty)_i(t, x, 0, \ldots, 0, u_i, 0, \ldots, 0).$$

The *off diagonal part* E is defined to be $F - D$.

Let $L \equiv \partial_t + (\text{diag}\,\lambda_i)\partial_x$.

Theorem 3.1. *Suppose that u is the solution u from part* (a) *of Theorem* 2.1. *and that* $u(t)_s + u(t)_{a \cdot c}$ *is the Lebesgue decomposition of* $u(t)$ *with respect to dx. Then,*

(a) $u_s = \Gamma$, *the unique* $C([0, T]: M(R_t)_*)$ *solution to:*

$$L\Gamma + D(t, x, \Gamma) = 0, \quad \Gamma(s) = \mu_s \tag{3.1}$$

where $\mu_s + \mu_{ac}$ *is the Lebesgue decomposition of* μ_0.

(b) $u_{ac} = v$ *is the unique* $C([0, T]: M(R_t)_*)$ *solution to*

$$Lv + F(t, x, v) = g + E(t, x, \Gamma), \quad v(0) = \mu_{ac}.$$

u_{ac} *lies in* $C([0, T]: L^1(R_t))$.

This result asserts that the components of the singular part satisfy simple uncoupled equations

$$(\partial_t + \lambda_i \partial_x) u_s^i + (F_\infty)_i(t, x, 0, \ldots, 0, u_i, 0, \ldots, 0) = 0.$$

In [6] the singular part satisfied a *linear* equation. A similar result can also be proved here.

For $1 \leqq i \leqq k$ we define

$$d_i^{\pm}(t, x) \doteq \pm (F_\infty)_i(t, x, 0, \ldots, 0, \pm 1, 0,, \ldots, 0)$$

where the ± 1 goes in the ith slot. Then since F_∞ is positively homogeneous of degree one,

$$(F_\infty)_i(t,x,0,\ldots,s,\ldots,0)=\begin{cases} sd_i^+ & \text{if} \quad s\geqq 0 \\ sd_i^- & \text{if} \quad s\leqq 0. \end{cases}$$

Corollary 3.2. *Let $\Gamma^\pm \in C([0,T]:M(R_t)_*)$ be the unique solutions of linear initial value problems*

$$L\Gamma^\pm + \operatorname{diag}(d_i^\pm)\Gamma^\pm = 0, \quad \Gamma^\pm(0)=(\mu_0)_s^\pm.$$

Then $\Gamma=\Gamma^+ + \Gamma^-$ is the Jordan decomposition of $\Gamma(t)$ in part (a) *of Theorem* 3.1.

Proof of Corollary. Since $\Gamma_i^+(s)\perp\Gamma_i^-(s)$ and $(\partial_t+\lambda_i\partial_x)\Gamma_i^\pm + d_i^\pm\Gamma_i^\pm=0$ we see that for all t, $\Gamma_i^+\perp\Gamma_i^-$. It follows from the definition of homogeneous functions of measures that (see [**2**, ex. 1.2])

$$\begin{aligned} D_i(\Gamma_i^+ + \Gamma_i^-) &= (F_\infty)_i(0,\ldots,\Gamma_i,\ldots,0) \\ &= (F_\infty)_i(0,\ldots,\Gamma_i^+,\ldots,0) + (F_\infty)_i(0,\ldots,\Gamma_i^-,\ldots,0) \\ &= d_i^+\Gamma^+ + d_i^-\Gamma^-. \end{aligned}$$

Thus

$$L(\Gamma^+ + \Gamma^-) + D(\Gamma^+ + \Gamma^-) = 0, \quad (\Gamma^+ + \Gamma^-)(0)=\mu_s$$

proving the corollary. □

Proof of Theorem. Define Γ as in the corollary, and define $v\in C([0,T]:M(R_t)_*)$ to be the unique solution of

$$Lv + F(t,x,v) = g - E(t,x,\Gamma), \quad v(0)=(\mu_0)_{ac}. \tag{3.2}$$

We know that $\Gamma(t)\perp dx$ for all t so if we can show that $v(t)\ll dx$ for all t we would have $\Gamma\perp v$ and therefore from the definition of the measure $F(\mu)$ (see [**1**], [**3**]), $F(v+\Gamma)= F(v)+F_\infty(\Gamma)$. Thus,

$$\begin{aligned} L(v+\Gamma)+F(v+\Gamma) &= Lv + L\Gamma + F(v+\Gamma) \\ &= (g - F(v) - E(\Gamma)) - D(\Gamma) + (F(v)+F_\infty(\Gamma)) \\ &= g. \end{aligned}$$

so $v+\Gamma$ solves the initial value problem defining u and $v\perp\Gamma$ so v and Γ give the

Lebesgue decomposition of u. Thus, to complete the proof of both parts of Theorem 3.1 it is sufficient to show that $v \in C([0,T]:L^1(R_t))$. Define $\zeta \in C([0,T]:M(R_t)_*)$ by

$$L\zeta = E(\Gamma), \quad \zeta(s) = 0.$$

The crucial step in the proof is to show that $\zeta \in C([0,T]:L^1(R_t))$.

Claim 3.3. $\zeta \in C([0,T]:L^1(R_t))$.

Assuming the claim, we complete the proof of Theorem 3.1. Define w to be the unique $C([0,T]:L^1(R_t))$ solution to

$$Lw + F(t,x,w+\zeta) = g, \quad w(0) = (\mu_0)_{ac}.$$

Then, $w+\zeta$ lies in $C([0,T]:L^1(R_t))$ and satisfies (3.2). By unicity $v = w + \zeta \in C([0,T]:L^1(R_t))$.

It remains to prove Claim 3.3. The key step is the following claim.

Claim 3.4. *For almost all $t \in [0,T]$ the measures $\Gamma_1(t), \Gamma_2(t), \ldots, \Gamma_k(t)$ are mutually singular.*

Assuming Claim 3.4, we have

$$E_i(\Gamma) = \sum_{\pm} \sum_j E_i(0, \ldots, \Gamma_j^{\pm}, 0, \ldots, 0).$$

Let $e_{ij}^{\pm} \equiv E_i(0, \ldots, 0, \pm 1, \ldots, 0)$ so $e_{ii}^{\pm} \equiv 0$. Then

$$E_i(\Gamma) = \sum \sum e_{ij}^{\pm}(t,x)\Gamma_j^{\pm},$$

so,

$$\zeta_i = (\partial_t + \lambda_i \partial_x)^{-1} \left(\sum_{\pm} \sum_{j \neq i} e_{ij}^{\pm} \Gamma_j^{\pm} \right).$$

That each summand lies in $C([0,T]:L^1(R_t))$ is not difficult to prove (see [**6**, p. 163]). This is the conclusion of Claim 3.3.

To complete the proof of the theorem, it remains to prove Claim 3.4.

Proof of Claim 3.4. From the formula in Corollary 3.2, we see that for all j, $\Gamma_j^{\pm} \ll (\Gamma_j')^{\pm}$ where Γ' is the solution to $L\Gamma' = 0$, $\Gamma'(0) = (\mu_0)_s$. Thus, replacing Γ by Γ' it is sufficient to prove the lemma when $D \equiv 0$.

Fix $i \neq j$. Define $x'(t,x)$ by

$$(\partial_t + \lambda_i(t,x)\,\partial_x)x' = 0, \quad x'(0,x) = x.$$

Then the change of variables $t' = t$, $x' = x'(t,x)$ transforms $\partial_t + \lambda_i \partial_x$ to $\partial_{t'}$.

We suppose that the reductions of the previous paragraphs have been performed, and, we drop the primes. Then $D \equiv 0$, and $\Gamma_i(t) \equiv \Gamma_i(0)$. Choose a Lebesgue null set N such that $|\Gamma_i(0)|(\mathbb{R}\backslash N) = 0$. Recall that $(t, X_j(t,x))$ is the jth characteristic through $(0,x)$, hence, transverse to the x-axis. To show that $\Gamma_j \perp \Gamma_i$ a.e. t, it suffices to show for Lebesgue almost all $t \in [0,T]$, that $|\Gamma_i(0)|(X_j(t,N)) = 0$. By Fubini's Theorem it suffices to show that $(|\Gamma_i(0)| \times dt)(\tilde{N}) = 0$ where

$$\tilde{N} = \{(t, x_j(t,x)) : x \in N\}$$

has $X_j(t,N)$ as cross-section at height t.

Extend $\lambda_j(t,x)$ to be a C^1 function on $\mathbb{R}^2$ with

$$0 < \inf|\lambda_j| \leqq \sup|\lambda_j| < \infty.$$

For $\bar{x} \in \mathbb{R}$ define a diffeomorphism from $\{0\} \times \mathbb{R}$ to $\mathbb{R} \times \{\bar{x}\}$ sending $(0,x)$ to the unique intersection point of the jth characteristic through $(0,x)$ with the vertical line $x = \bar{x}$. The image of N under this map is a Lebesgue null set in $\{x = \bar{x}\}$. However, the image of N is exactly the cross-section $\tilde{N}_{\bar{x}}$, so $dt(\tilde{N}_{\bar{x}}) = 0$ for all $\bar{x} \in \mathbb{R}$. Fubini's Theorem implies that $(|\Gamma_i(0)| \times dt)(\tilde{N}) = 0$. This completes the proof of Claim 3.4 and therefore the proof of Theorem 3.1. □

4. Continuous dependence for some weakly convergent data

A singular measure cannot be approximated in norm by absolutely continuous measures. In particular, if $\mu_0 \in M(R_0 : \mathbb{R}^k)$, $\operatorname{supp}\mu_0 \subset \operatorname{Int} R_0$, is not absolutely continuous then the standard regularizations $j_\varepsilon * \mu_0$ converge weakly to μ_0 but not in norm. To show that the notion of solution studied in Section 2 is natural, it is desirable that the solution u^ε with initial data $j_\varepsilon * \mu_0$ converges to the solution u with data μ_0. The next theorem implies that result.

Theorem 4.1. *Suppose that $g \in L^1([0,T] : M(R_t))$, $L = \partial_t + \operatorname{diag}(\lambda_i(t,x))\partial_x$, and $\mu^n \to \mu$ in $M(R_0)_*$ have Lebesgue decompositions $h^n dx + \mu_s^n$ and $h dx + \mu_s$ respectively. Let u^n, u be the solutions of* 1.1 *with initial data μ^n, μ. Then, u^n converges to u in* $C([0,T] : M(R_t)_*)$ *provided that*

(i) *$h^n \to h$ in measure, and*

(ii) *for each $j = 1, 2, \ldots, k$, $(\mu_j^n)^\pm \to (\mu_j)^\pm$ in $M(R_0)_*$.*

Remarks. (1) The conditions (i), (ii) of this theorem are necessary and sufficient in order that $\Phi(\mu_i^n) \to \Phi(\mu_i)$ in M_* for every asymptotically homogeneous function $\Phi \in C(\mathbb{R})$ of a *single variable*. They do not imply that $F(\mu^n) \to F(\mu)$ for asymptotically homogeneous functions $F \in C(\mathbb{R}^k)$ (see [3]). The fact that $F(\mu^n)$ need not converge to $F(\mu)$ renders somewhat surprising the fact that $u^n \to u$.

(2) The hypothesis $(\mu_j^n)^\pm \to (\mu_j)^\pm$ is made in coordinates for $\mathbb{R}^k$ in which the coefficient matrix A is diagonal. In general coordinates the condition is expressed as follows. Let $b(t,x)$ be a C^1 eigenbasis of $\mathbb{R}^k$ associated with $A(t,x)$. Define scalar valued μ_j^n by $\mu^n = \sum \mu_j^n b_j(t,x)$. The condition is then $(\mu_j^n)^\pm \to (\mu_j)^\pm$ in $M(R_0)_*$.

(3) If $\mu \subset M(R_0)$ and μ^n are constructed by mollifying as in (1.3), then it is not hard to show that the hypotheses of Theorems 4.1, 4.2 are satisfied (see [**4**, ex. 1.27 and ex. 2.6]). Thus, the desired continuity described in the first paragraph of this section is valid.

(4) Two examples showing that hypotheses (i), (ii) cannot be much relaxed are presented at the end of this section.

Theorem 4.1 is an immediate consequence of the next result which provides more detailed information.

Theorem 4.2. *With the notation of Theorem* 4.1 *define* $v^n, \Gamma^n \in C([0,T]:M(R_t)_*)$ *by*

$$L\Gamma^n + D(\Gamma^n) = 0, \qquad \Gamma^n(0,\cdot) = \mu_s^n + (h^n - h)\,dx, \tag{4.1}$$

$$Lv^n + F(v^n) = g - E(\Gamma^n), \quad v^n(0,\cdot) = h^n. \tag{4.2}$$

Let $\Gamma(t) + v(t)$ *be the Lebesgue decomposition of* $u(t)$ *described in Theorem* 3.1. *Then* $v^n \in C([0,T]:L^1(R_t))$ *and*

$$(\Gamma_j^n)^\pm \to \Gamma_j^\pm \text{ in } C([0,T]:M(R_t)_*),$$

$$v^n \to v \text{ in } C([0,T]:L^1(R_t)), \text{ and}$$

$$u^n - v^n - \Gamma^n \to 0 \text{ in } C([0,T]:L^1(R_t)).$$

Proof. The equation for Γ^n yields for $1 \leqq j \leqq k$

$$(\partial_t + \lambda_j \partial_x + d_j^\pm)(\Gamma_j^n)^\pm = 0, \quad (\Gamma_j^n)^\pm(0,\cdot) = (h^n - h)_j^\pm\,dx + (\mu_s^n)_j^\pm.$$

Subtracting the equation for $\Gamma_j^\pm$ in Corollary 3.2 yields the linear initial problem,

$$(\partial_t + \lambda_j \partial_x + d_j^\pm)((\Gamma_j^n)^\pm - \Gamma_j^\pm) = 0,$$

$$(\Gamma_j^n)^\pm(0,\cdot) - \Gamma_j^\pm(0,\cdot) = (\mu^n - h\,dx)^\pm - \mu_s^\pm.$$

The hypotheses (i) and (ii) imply that $\Gamma_j^n(0)^\pm - \Gamma_j(0)^\pm$ converges to zero in $M(R_0)_*$ (direct proof or see Proposition 3.3 of [3]). It follows that $(\Gamma_j^n)^\pm - \Gamma_j^\pm$ tends to zero in. $C([0,T]:M(R_t)_*)$.

Since μ_s^n is singular and $h^n \to h$ in measure, the explicitly solvable initial value problem for $(\Gamma_j^n)^{\pm}$ shows that $\Gamma_a^n(t)\to 0$ in measure uniformly for $0\leqq t\leqq T$, where $\Gamma_a^n+\Gamma_s^n$ is the Lebesgue decomposition of $\Gamma^n(t)$.

To prove that $v^n\in C([0,T]:L^1(R_t))$, decompose E into homogeneous and sublinear parts, $E=E_\infty+e$. Then,

$$E(\Gamma^n)=E_\infty(\Gamma_s^n)+E_\infty(\Gamma_a^n)+e(\Gamma_a^n)=E_\infty(\Gamma_s^n)+C([0,T]:L^1(R_t)).$$

Then

$$Lv^n+F(v^n)+E(\Gamma_s^n)\in C([0,T]:L^1(R_t)),\quad v^n(0,\cdot)\in L^1(R_0).$$

The proof that $v^n\in C([0,T]:L^1(R_t))$ is then exactly like the proof of Theorem 3.2(b).

The next, and crucial, step is to show that the family $\{v^n\}$ is precompact in $C([0,T]:L^1(R_t))$. To do this, the crux is to analyze the effect of the source term $E(\Gamma^n)$. Let ζ^n, $\zeta\in C([0,T]:M(R_t)_*)$ be defined as solutions of the linear initial value problems

$$L\zeta^n=-E(\Gamma^n),\quad \zeta^n(0,\cdot)=h$$
$$L\zeta=-E(\Gamma),\quad \zeta(0,\cdot)=h.$$

As in the last paragraph, $\zeta^n\in C([0,T]:L^1(R_t))$. We turn next to the proof that

$$\zeta^n\to\zeta\quad\text{in}\quad C([0,T]:L^1(R_t)). \tag{4.3}$$

The $\Gamma(t)$ are singular so $E(\Gamma(t))=E_\infty(\Gamma(t))$. For t outside a Lebesgue null set, the measures $\Gamma_1(t),\dots,\Gamma_k(t)$ are mutually singular so

$$E_\infty(\Gamma(t))=\sum_j E_\infty(0,,\dots,0,\Gamma_j,0,\dots,0)\doteq\sum_j E_\infty(\Gamma_j)$$
$$=\sum_j e_j^{\pm}(t,x)\,\Gamma_j^{\pm}\quad\text{a.e.}\quad t\in[0,T],$$

the $\Gamma_j(t)$ appearing in the jth slot. The key to estimating v_n-v is that $E(\Gamma^n)$ can be replaced by $\sum E_\infty(\Gamma_j^n)$ with small error.

Claim 4.4. For Lebesgue almost all $t\in[0,T]$

$$\left\|E((\Gamma^n(t))-\sum_{j=1}^{k}E(\Gamma_j^n)\right\|_{M(R_t)}\to 0. \tag{4.5}$$

To prove the claim we use the fact that for almost all $t\in[0,T]$, Γ_1^n, $\Gamma_2^n,\dots,\Gamma_k^n$ are *separated* according to the following definition.

Definition. Suppose X is a compact Hausdorff space. The sequences $u^n, v^n\in M(X:\mathbb{R}^k)$

are *separated* if and only if for any $\delta>0$ there is Borel function $0\leqq\phi\leqq 1$ and N so that for all $n>N$, $(1-\phi)\mu^n$ a ϕv^n have norm less than δ.

Examples. 1. If in M_* $(\mu_i^n)^{\pm}\to(\mu_i)^{\pm}$, $(v_i^n)^{\pm}\to(v)^{\pm}$, and, $\mu\perp v$ then μ^n, v^n are separated. (Hint: choose $\phi\in C(x)$, $0\leqq\phi\leqq 1$ with $\|(1-\phi)\mu\|+\|\phi v\|<\delta$.)

2. The converse is not true even for non-negative measures μ^n, v^n. For example

$$\mu^n\equiv\sum_{1\leqq i\leqq n} n^{-1}\delta_{i/n}\to\chi_{[0,1]}\equiv v^n.$$

In this case the limits are not mutually singular but the sequences are separated as one sees upon taking $\phi_1\equiv\chi_{\mathbb{Q}\cap[0,1]}$ and $\phi_2\equiv\chi_{[0,1]\setminus\mathbb{Q}}$.

Lemma 4.3. *Suppose that X is compact Hausdorff and that $H\in C(X\times\mathbb{R}^k)$ is positively homogeneous and uniformly Lipschitzian with respect to the second variable. Suppose that $\mu^n, v^n\in M(X:\mathbb{R}^k)$ are separated. Then, as $n\to\infty$*

$$\|H(x,\mu^n+v^n)-H(x,\mu^n)-H(x,v^n)\|_{M(X,\mathbb{R})}\to 0.$$

Proof. Let Λ be a Lipschitz constant for H. For any $\delta>0$ choose ϕ, $N(\delta)$ as in the definition of separated. Let $\alpha^n\doteq H(\mu^n+v^n)-H(\mu^n)-H(v^n)$. Then

$$|\phi\alpha^n|\leqq\phi\{|H(\mu^n+v^n)-H(\mu^n)|+|H(v^n)|\}\leqq 2\Lambda|\phi v^n|.$$

$$|(1-\phi)\alpha^n|\leqq(1-\phi)\{|H(\mu^n+v^n)-H(v^n)|+|H(\mu^n)|\}\leqq 2\Lambda|(1-\phi)\mu^n|.$$

Thus for $n>N(\delta)$,

$$\|\alpha^n\|=\|\phi\alpha^n+(1-\phi)\alpha^n\|\leqq 4\Lambda\delta. \qquad \square$$

From Claim 3.4 we see that there is a Lebesgue null set $\mathcal{N}\subset[0,T]$ so that for $t\notin\mathcal{N}$ and $i\neq j$ the sequence $\Gamma_i(t)\perp\Gamma_j(t)$. Then for $t\notin\mathcal{N}$, the sequences $\Gamma_i^n(t)$ and $\Gamma_j^n(t)$ are separated. A simple induction using Lemma 4.3 yields (4.4) for $t\notin\mathcal{N}$, completing the proof of Claim 4.3.

We return to the proof of (4.3). Since $\|\Gamma^n(t)\|_{M(R_t)}$ is bounded independent of n and $t\in[0,T]$, it follows from Theorem 2.1(b), Lemma 4.3, and Lebesgue's dominated Convergence Theorem that

$$\sup_{t\in[0,T]}\left\|L^{-1}(E(\Gamma^n)-\sum_{j=1}^{k}E(\Gamma_j^n))\right\|_{M(R_t)}\to 0.$$

Thus to prove that $\zeta^n\to\zeta$, it suffices to prove that for $1\leqq j\leqq k$, $L^{-1}(E(\Gamma_j^n))\to L^{-1}(E(\Gamma_j))$. Since the $\Gamma^n\to 0$ in measure uniformly and e is sublinear we have $e(\Gamma_j^n)\to 0$ in

$C([0,T]:L^1(R_t))$ (see [6 pp. 159–160]). Thus it remains to study $L^{-1}E_\infty(\Gamma_j^n)$. Considering the ith component, we find

$$(L^{-1}E_\infty(\Gamma_j^n))_i=(\partial_t+\lambda_i\partial_x)^{-1}\left(\sum_{\pm} e_{ij}^{\pm}(t,x)(\Gamma_j^n)^{\pm}\right).$$

Note that $e_{ii}\equiv 0$ so these expressions integrate $e_{ij}^{\pm}((\Gamma_j^n)^{\pm})$ along characteristics transverse to those along which Γ_j^n propagates. It follows as on p. 163 of [6] that the boundedness of $(\Gamma_j^n)^{\pm}(0,\cdot)$ in $M(R_0)$ implies the precompactness of $(\zeta_j^n)^{\pm}$ in $C([0,T]:L^1(R_t))$. The convergence $(\zeta_j^n)^{\pm}\to\zeta^{\pm}$ in $C([0,T]:M(R_t)_*)$, is an immediate consequence of the convergence $(\Gamma_j^n)^{\pm}\to\Gamma^{\pm}$ in $C([0,T]:M(R_t)_*)$ and the continuity of $(\partial_t+\lambda\partial_x)^{-1}$ from $C([0,T]:M(R_t)_*)$ to itself. Together with precompactness, this implies that $(\zeta_j^n)^{\pm}\to\zeta^{\pm}$ in $C([0,T]:L^1(R_t))$, proving (4.3).

We next prove that $v^n\to v$ in $C([0,T]:L^1(R_t))$. Towards that end, note that

$$L(v^n-\zeta^n)+F(v^n)=0,\quad (v^n-\zeta^n)(0,\cdot)=0.$$

Define $\eta^n\doteq v^n-\zeta^n$, and, $\eta\in C([0,T]:L^1(R_t))$ the solution of

$$L\eta+F(\eta+\zeta)=0,\ \eta(0,\cdot)=0.$$

Then $\eta+\zeta$ satisfies the same initial value problem as v so we must have $v=\eta+\zeta$. In addition,

$$\begin{aligned}|L(\eta^n-\eta)|&=|F(\eta+\zeta)-F(\eta^n+\zeta^n)|\\&\leqq C(|\eta^n-\eta|+|\zeta^n-\zeta|).\end{aligned}$$

Since $\zeta^n-\zeta\to 0$ in $C([0,T]:L^1(R_t))$ and $(\eta^n-\eta)(0,\cdot)=0$, Theorem 2.1(b) together with Gronwall's inequality imply that $\eta^n-\eta\to 0$ in $C([0,T]:L^1(R_t))$. Thus

$$v^n=\eta^n+\zeta^n\to\eta+\zeta=v \text{ in } C([0,T]:L^1(R_t)).$$

To complete the proof of Theorem 4.2 it remains to show that $u^n-v^n-\Gamma^n\to 0$ in $C([0,T]:L^1(R_t))$. Subtracting the sum of 4.1 and 4.2 from $Lu^n+F(u^n)=g$ yields for $v\doteq u^n-v^n-\Gamma^n$,

$$Lv^n=F(v^n)+F(\Gamma^n)-F(v^n+v^n+\Gamma^n). \tag{4.6}$$

Let $\Gamma_a^n(t)+\Gamma_s^n(t)$ be the Lebesgue decomposition of $\Gamma^n(t)$. Write

$$F(v^n+v^n+\Gamma^n)=F(v^n+v^n+\Gamma^n)-F(v^n+\Gamma^n)+F(v^n+\Gamma_a^n)+F_\infty(\Gamma_s^n)$$

and

$$F(\Gamma^n)=F(\Gamma_a^n)+F_\infty(\Gamma_s^n).$$

Thus, Lv^n is equal to

$$F(v^n)+F(\Gamma_a^n)-(F(v^n+v^n+\Gamma^n)-F(v^n+\Gamma^n))-F(v^n+\Gamma_a^n).$$

Estimate the term in parentheses by $\Lambda|v^n|$ and then apply the estimate from Lemma 2.3 to find

$$\|v^n(t)\|_{M(R_t)}\leqq c\int_0^t \|v^n(s)\|_{M(R_s)}\,ds$$

$$+c\int_0^t \|F(v^n+\Gamma_a^n)-F(v^n)-F(\Gamma_a^n)\|_{M(R_s)}\,ds. \tag{4.7}$$

Gronwall's inequality implies that to complete the proof it suffices to show the second integral in 4.7 converges to zero uniformly for $0\leqq t\leqq T$. Now, replacing v^n by v in two occurrences in the second integral changes the value of the integrand by at most $2\Lambda\|v^n-v\|_{L^1(R_t)}$ where Λ is a Lipschitz constant for F. Since this norm tends to zero uniformly for $t\in[0,T]$ it suffices to show that

$$\int_0^T \|F(v+\Gamma_a^n)-F(v)-F(\Gamma_a^n)\|_{M(R_s)}\,ds\to 0 \tag{4.8}$$

as $n\to\infty$.

At the very beginning to the proof we observed that $\Gamma^n(t)$, and therefore $\Gamma_a^n(t)$, converge to zero in measure uniformly for $t\in[0,T]$. Write $F=F_\infty+G$ with G sublinear. Then as on pp. 159–160 of [6] we have

$$G(v+\Gamma_a^n)\to G(v) \text{ in } C([0,T]:L^1(R_t))$$

$$G(\Gamma_a^n)\to 0 \qquad \text{in } C([0,T]:L^1(R_t))$$

so the contribution of the sublinear part of 4.8 tends to zero.

A key ingredient in the proof for the F_∞ part is the following lemma.

Lemma 4.4. *Suppose that X is a compact Hausdorff space, dx is a Radon measure on*

X and that $H \in C(X \times \mathbb{R}^k)$ is positively homogeneous and uniformly Lipschitzian with respect to the second variable. If $\alpha \in L^1(X, dx)$ and β_n is a sequence of measurable functions tending to zero in dx measure then

$$H(x, \alpha+\beta^n)-H(x,\alpha)-H(x,\beta^n)$$

tends to zero strongly in $L^1(X, dx)$.

Proof of Lemma 4.4. For $\delta > 0$ define

$$G_\delta^n = \{x \in X : |\beta(x)| > \delta/\text{meas}(X)\}.$$

For any δ choose $\delta_1(\delta)$ so that if A is a Borel set with measure less than δ_1 then $|\int_A \alpha\, dx| < \delta$. Choose $N = N(\delta)$ so that for $n > N$, $\text{meas}(G_n^\delta) < \delta_1$.

On G_n^δ write

$$|H(\alpha+\beta_n)-H(\alpha)-H(\beta_n)| \leqq |H(\alpha+\beta_n)-H(\beta_n)|+|H(\alpha)| \leqq 2\Lambda|\alpha|$$

and the integral over G_n^δ is then dominated by $2\Lambda\delta$.

On $X \backslash G_n^\delta$ write

$$|H(\alpha+\beta_n)-H(\alpha)-H(\beta_n)| \leqq |H(\alpha+\beta_n)-H(\alpha)|+|H(\beta_n)| \leqq 2\Lambda|\beta_n|$$

whose integral is at most $2\Lambda\delta$. Thus for $n > N(\delta)$,

$$\int |H(\alpha+\beta_n)-H(\alpha)-H(\beta_n)|\,dx \leqq 4\Lambda\delta. \qquad \square$$

As a corollary of the proof we have:

Lemma 4.5. *Suppose that X, dx are as above and $H \in C([0,T] \times X \times \mathbb{R}^k)$ is positively homogeneous and uniformly Lipschitzian with respect to the third variable. If $\alpha \in C([0,T] : L^1(X : \mathbb{R}^k, dx))$ and β^n is a sequence of measurable functions with $\beta^n(t)$ tending to zero in dx measure uniformly for $t \in [0,T]$, then*

$$H(\alpha+\beta^n)-H(\alpha)-H(\beta^n)$$

tends to zero in $C([0,T] : L^1(X, dx))$.

Applying this lemma with $\alpha = v$, $\beta^n = \Gamma_a^n$ shows that the F_∞ contribution to the integrand of (4.8) tends uniformly to zero. It follows that (4.8) is true and the proof of Theorem 4.2 is complete. $\square$

The hypotheses (i), (ii) of Theorems 4.1, 4.2 are stronger than $\mu^n \to \mu$ in M_*. The next examples show that $\mu^n \to \mu$ in M_* is insufficient and that neither (i) nor (ii) can be appreciably weakened.

Example 4.9. Consider the initial value problem

https://doi.org/10.1017/S0013091500004855 Published online by Cambridge University Press

$$(\partial_t + \partial_x)u^n = |u^n|, \; u^n(0,\cdot) = \delta_{1/n} - \delta_{-1/n}.$$

Then $u^n(0,\cdot)\to 0$ in M_* but $u^n(0,\cdot)^{\pm}$ does not converge to 0 so hypothesis (ii) is violated. The solution is

$$u^n(0,\cdot) = e^t\delta_{x+1/n} - e^{-t}\delta_{x-1/n}$$

$$u^n(t)\to(e^t - e^{-t})\delta_x$$

which is different from zero, the solution with data $0 = M_* - \lim u^n(0,\cdot)$.

Example 4.10. Let $F(s) = \min(s^2, 1)$ so F is a sublinear and consider the initial value problems

$$\partial_t u^n = F(u^n), \quad u^n(0,\cdot) = g^n\,dx$$

$$g^n = \sum_0^{n-1} (1 + (-1)^j)\chi_{[j/n,\, j+1/n[}.$$

Then in M_*, $u^n(0,\cdot)\to\chi_{[0,1]}dx \doteq g$ but g^n does not converge to $\chi_{[0,1]}$ in measure so (i) is violated. Note that $F(g^n)$ does not converge to $F(g)$ in M_*. The solutions are given by

$$u^n(t,\cdot) = \sum_0^{n-1}\left(1 + \frac{t}{2}\right)(1 + (-1)^j)\chi_{[j/n,(j+1)/n[}\,dx$$

so $u^n(t,\cdot)\rightharpoonup(1 + t/2)\chi_{[0,1]}dx$ which is different from $u = (1+t)\chi_{[0,1]}dx$, the solution of the limit problem $\partial_t u = F(u)$, $u(0,\cdot) = g\,dx$.

REFERENCES

1. N. Bourbaki, *Integration*, chapter 5 (second edition) (Hermann, Paris), 65–67.

2. F. Demengel and J. Rauch, Continuité faible des fonctions de mesures (non convexes), *C.R. Acad. Sci. Paris, Sér. I. Math.* **305** (1987), 63–66.

3. F. Demengel and J. Rauch, Superposition d'ondes singulières, *C.R. Acad. Sci. Paris, Sér. I. Math.* **304** (1987), 447–449.

4. F. Demengel and J. Rauch, Weak convergence of asymptotically homogeneous functions of measures, *Nonlinear Anal.* (1990), to appear.

5. M. Oberguggenberger, Weak limits of solutions of semilinear hyperbolic systems, *Math. Ann.* **274** (1986), 599–607.

6. J. Rauch and M. Reed, Nonlinear superposition and absorption of delta waves in one space dimension, *J. Funct. Anal.* **73** (1987), 152–178.

Département de Mathematiques
Université de Paris Sud
Orsay, 91405 Cedex
France

Department of Mathematics
University of Michigan
Ann Arbor, MI 48109
U.S.A.

https://doi.org/10.1017/S0013091500004855 Published online by Cambridge University Press