Can. J. Math. Vol. **49** (4), 1997 pp. 653–674.

ON q-CARLESON MEASURES FOR SPACES OF M-HARMONIC FUNCTIONS

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ABSTRACT. In this paper we study the q-Carleson measures for a space h^p_α of M-harmonic potentials in the unit ball of $\mathbf{C}^n$, when $q < p$. We obtain some computable sufficient conditions, and study the relations among them.

1. **Introduction.** If $H^q(\mathbf{D})$ is the Hardy space on the unit disc, Carleson [Ca] showed that the positive measures on $\mathbf{D}$ so that

$$\int_{\mathbf{D}} |f(z)|^q \, d\mu(z) \leq C\|f\|^q_{H^q(\mathbf{D})},$$

were characterized by the now so-called Carleson condition: there exists $C > 0$ such that for any interval I in the unit circle, if $T(I)$ is the non-tangential tent over I, $\mu\big(T(I)\big) \leq C|I|$.

In general, if Ω is a region, μ is a finite positive Borel measure on Ω and X is a Banach space of continuous functions on Ω, and $1 \leq q < +\infty$, we say that μ is a q-Carleson measure for X if there exists $C > 0$ so that for any function f in X

$$\int_\Omega |f(z)|^q \, d\mu(z) \leq C\|f\|^q_X.$$

When X is the space h^q_α of Poisson transforms of Riesz potentials R_α of L^q-functions in the unit circle, Stegenga [St] characterized the q-Carleson measures for h^q_α. He proved that they coincide with the positive measures on $\mathbf{D}$ for which there exists $C > 0$ such that for any open set A in the unit circle $\mu\big(T(A)\big) \leq CC_{\alpha q}(A)$. Here $C_{\alpha q}$ denotes an appropriate Riesz capacity. The above results can be thoroughly extended to $\mathbf{R}^{n+1}_+$.

When X is the space of Riesz potentials R_α of L^p-functions in $\mathbf{R}^n$, and $p < q$, Adams [Ad] showed that the q-Carleson measures for this space coincide with the space of finite positive measures μ on $\mathbf{R}^n$ so that $\mu\Big(\big(B(\omega, r)\big)\Big) \leq Cr^{(n-\alpha p)\frac{q}{p}}$ for any ω in $\mathbf{R}^n$, $r > 0$. Here $B(\omega, r)$ denotes the ball centered at ω of radius r.

For the same spaces, the case $q < p$ is more difficult. Maz'ya and Netrusov [Ma] and [MaNe] characterized the positive q-Carleson measures in $\mathbf{R}^n$ for Riesz potentials R_α of L^p-functions in $\mathbf{R}^n$. They proved that they coincide with the space of finite positive measures satisfying

$$\int_0^{\mu(\mathbf{R}^n)} \left(\frac{t^{\frac{p}{q}}}{\chi(t)}\right)^{\frac{q}{p-q}} \frac{dt}{t} < +\infty,$$

Both authors partially supported by Grant PB95-0956-C02-01, Ministerio de Educación y Cultura, Spain.
Received by the editors October 19, 1995; revised November 7, 1996.
AMS subject classification: Primary: 32A35, 31C15.

where $\chi(t)$ is the infimum of the Riesz capacity $C_{\alpha p}(A)$, for A any open set in $\mathbf{R}^n$ so that $t \leq \mu(A)$. See also [Ve] for other approach to this problem, and [Lu1] for a related problem.

Let $\mathbf{B}^n$ be the unit ball in $\mathbf{C}^n$, and $\mathbf{S}^n$ the unit sphere. The normalized Lebesgue measure on $\mathbf{B}^n$ will be denoted by dV and the normalized Lebesgue measure on $\mathbf{S}^n$ will be denoted by $d\sigma$. For each $\zeta \in \mathbf{S}^n$, $r > 0$ we will consider the non-isotropic ball $B(\zeta, r) = \{\omega \in \mathbf{S}^n ; |1 - \zeta\bar{\omega}| < r\}$. For each $0 < \alpha < n$, it is introduced in [AhCo] a non-isotropic Bessel kernel given by

$$K_\alpha(z, \zeta) = \frac{1}{|1 - z\zeta|^{n-\alpha}}, \quad z \in \bar{\mathbf{B}}^n,\ \zeta \in \mathbf{S}^n.$$

If $1 \leq p < +\infty$, and if $f \in L^p(d\sigma)$, the non-isotropic convolution is given by

$$K_\alpha * f(z) = \int_{\mathbf{S}^n} K_\alpha(z, \zeta) f(\zeta)\, d\sigma(\zeta), \quad z \in \bar{\mathbf{B}}^n.$$

For $0 < \alpha < n$, $1 < p < +\infty$, and $A \subset \mathbf{S}^n$, the non-isotropic Riesz capacity $C_{\alpha p}(A)$ is defined by $C_{\alpha p}(A) = \inf \|f\|_p^p$, where the infimum is taken with respect to the positive functions $f \in L^p(d\sigma)$ so that $K_\alpha * f \geq 1$ on A.

We will study the q-Carleson measures for the space h^p_α of Poisson-Szegö transforms of convolutions of L^p functions with K_α, that is, the space of finite positive measures μ such that $\|P[K_\alpha * f]\|_{L^q(d\mu)} \leq C\|f\|_{L^p(d\sigma)}$. The Hardy-Sobolev space H^p_α, $\alpha > 0$ is the space of holomorphic functions f in $\mathbf{B}^n$ satisfying that if $f = \sum_k f_k$ is its homogeneous expansion, then $R^\alpha f = \sum_k (k+1)^\alpha f_k$ is in $H^p(\mathbf{B}^n)$. The norm of a function f in H^p_α will be denoted by $\|f\|_{p,\alpha} := \|R^\alpha f\|_{H^p}$.

If $q > p$ the methods in [Ad] give that μ is a q-Carleson measure for h^p_α if and only if $\mu\Big(T\big(B(\omega, r)\big)\Big) \leq Cr^{(n-\alpha p)\frac{q}{p}}$, where $T\big(B(\omega, r)\big)$ is a non-isotropic tent over $B(\omega, r)$ that we will later define. If $q \leq p$ a characterization analogous to the ones obtained by [St] and [MaNe] can be obtained for h^p_α. Such characterization involves the non-isotropic Riesz capacity of an arbitrary open set and then it is difficult to compute. In [CaOr1] we obtained some computable sufficient conditions for a positive measure to be p-Carleson for h^p_α.

The purpose of this work is to obtain in the remaining cases $q < p$ accurate computable sufficient conditions for a measure μ to be q-Carleson for h^p_α.

We have organized the paper in the following way. In Section 2 we give another characterization of the 1-Carleson measures for h^p_α which does not involve capacity, and we show that for $q > 1$ the corresponding condition is always sufficient. We also observe that every q-Carleson measure for h^p_α is also a q-Carleson measure for H^p_α, but in general they do not coincide. In Section 3 we obtain some non trivial necessary conditions which will suggest the computable sufficient conditions. These conditions are of geometric type and in terms of an $L^{\frac{p}{p-q}}$-modulus of continuity of the measure μ. Finally we give some examples which show the independency of the different sufficient conditions, as well as their accuracy.

https://doi.org/10.4153/CJM-1997-031-3 Published online by Cambridge University Press

As a final remark on notation, we adopt the usual convention of writing by the same letter various absolute constants which values may differ in each occurrence. We will write $A \preceq B$ if there exists an absolute constant $C > 0$ so that $A \leq CB$.

2. **Conditions arising from duality.** Let μ be a q-Carleson measure for h^p_α, $1 \leq q \leq p < +\infty$, and take $A \subset \mathbf{S}^n$ an open set.

We define the admissible tent over $A \subset \mathbf{S}^n$ given by

$$T(A) = T^\beta(A) = \mathbf{B}^n \Big\backslash \bigcup_{\zeta \notin A} D(\zeta),$$

where $D(\zeta) = D^\beta(\zeta) = \{z \in \mathbf{B}^n \; ; \; |1 - z\bar{\zeta}| < \beta(1 - |z|)\}$, $\beta > 1$. Let f be any test function for $C_{\alpha p}(A)$. Then $K_\alpha * f \geq 1$ on A, and consequently there exists $b > 0$ and $P[K_\alpha * f] \geq b$ on the admissible tent $T(A)$ over A. Since μ is q-Carleson, this gives that $\mu\big(T(A)\big) \preceq \|f\|_p^q$, and taking infimum with respect to f, we get that $\mu\big(T(A)\big) \preceq C_{\alpha p}(A)^{\frac{q}{p}}$.

This necessary condition turns out to be sufficient when $p = q$ (see [St]), but fails to be in general sufficient when $q < p$. The methods in Maz'ya and Maz'ya and Netrusov (see [Ma] and [MaNe]) for the isotropic case can be adapted to show that a finite positive Borel measure on $\mathbf{S}^n$ is a q-Carleson measure for h^p_α if and only if

$$\int_0^{\mu(\mathbf{B}^n)} \left(\frac{t^{\frac{p}{q}}}{\chi(t)} \right)^{\frac{q}{p-q}} \frac{dt}{t} < +\infty, \tag{2.1}$$

where $\chi(t) = \inf C_{\alpha p}(A)$, and the infimum is taken with respect to the open sets $A \subset \mathbf{S}^n$ satisfying that $t \leq \mu\big(T(A)\big)$.

The necessary condition we have just written is equivalent to $t^{\frac{p}{q}} \leq \chi(t)$.

In [AhCo] it is proved that for any $f \in H^p_\alpha$ there exists $g \in L^p_+(d\sigma)$ so that $|f(z)| \preceq P[K_\alpha * g](z)$ for each $z \in \mathbf{B}^n$ and $\|f\|_{p,\alpha} \preceq \|g\|_p$. This gives that any q-Carleson measure for h^p_α is also a q-Carleson measure for H^p_α. If $n - \alpha p < 1$, both classes of Carleson measures coincide [CoVe1].

Our first result shows that as it happens when $p = q$, (see [CaOr1]), when $n - \alpha p \geq 1$ and $q < p$, the q-Carleson measures for h^p_α and H^p_α are different.

PROPOSITION 2.1. *Assume $1 < p$, $n - \alpha p \geq 1$, $1 \leq q < p$. Then there exists a finite positive Borel measure on $\mathbf{B}^n$, μ, which is a q-Carleson measure for H^p_α, but is not a q-Carleson measure for h^p_α.*

PROOF OF PROPOSITION 2.1. Theorem 3.1 and Corollary 3.4 in [AhCo] for the case $1 < p \leq 2$, and Theorem 5 in [CoVe1], prove that there exists an invariant positive measure ν, supported in a compact set $E \subset \mathbf{S}^n$ with $C_{\alpha p}(E) = 0$, so that for any holomorphic function f in $\mathbf{B}^n$,

$$\int_{\mathbf{S}^n} |Nf(\zeta)|^p \, d\sigma(\zeta) \leq C \|R^\alpha f\|_p^p.$$

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(Here Nf denotes the admissible maximal function). The arguments in [CaOr1], shows then that it can be constructed a positive measure μ on $\mathbf{B}^n$ and a sequence of open sets $(E_k)_k$ in $\mathbf{S}^n$ such that

$$\int_{\mathbf{B}^n} |P[K_\alpha * f](z)|^p \, d\mu(z) \preceq \|f\|_p^p, \quad f \in H^p_\alpha,$$

and $C_{\alpha p}(E_k) \longrightarrow 0$, $\inf_k \mu\big(T(E_k)\big) > 0$. Hence μ does not satisfy the necessary condition $\mu\big(T(A)\big) \preceq C_{\alpha p}(A)^{\frac{q}{p}}$, and since $L^p(d\mu) \subset L^q(d\mu)$, we obtain the desired example. ∎

The previous condition (2.1) is, in general, difficult to verify. When $q = 1$ we can give a simple characterization of the 1-Carleson measures for h^p_α, $1 < p < +\infty$, which does not involve capacity, and which is, via duality, a non-isotropic version of Wolff's inequality [HeWo].

THEOREM 2.2. *Let $1 < p < +\infty$, $n - \alpha p > 0$ and assume μ is a finite positive Borel measure on $\mathbf{B}^n$. Then*

$$\int_{\mathbf{B}^n} |P[K_\alpha * f](z)| \, d\mu(z) \preceq \|f\|_p,$$

for each $f \in L^p(d\sigma)$, if and only if the non-isotropic potential

$$W^{\alpha p}_\mu(\eta) = \int_{1-|\eta|}^1 \left(\frac{\mu\Big(T\big(B(\eta_0, \delta)\big)\Big)}{\delta^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{d\delta}{\delta}, \quad \eta_0 = \frac{\eta}{|\eta|} \tag{2.2}$$

belongs to $L^1(d\mu)$.

PROOF OF THEOREM 2.2. Observe that in order to prove that μ is a q-Carleson measure for h^p_α we just need to deal with positive functions. It is proved in [CaOr1] that for $f \in L^p_+(d\sigma)$ and for each $z \in \mathbf{B}^n$,

$$P[K_\alpha * f](z) \simeq \int_{\mathbf{S}^n} \frac{f(\zeta)}{|1 - z\zeta|^{n-\alpha}} \, d\sigma(\zeta).$$

Thus we must characterize the positive measures on $\mathbf{B}^n$ satisfying that there exists $C > 0$ and for any $f \in L^p_+(d\sigma)$,

$$\int_{\mathbf{B}^n} \int_{\mathbf{S}^n} \frac{f(\zeta)}{|1 - z\zeta|^{n-\alpha}} \, d\sigma(\zeta) \, d\mu(z) \leq C \|f\|_p. \tag{2.3}$$

Fubini's theorem shows that the integral on the left side of (2.3) is equal to $\int_{\mathbf{S}^n} K_\alpha[\mu](\zeta) f(\zeta) \, d\sigma(\zeta)$. Consequently the necessary and sufficient condition for a positive measure μ to satisfy condition (2.3) is that $K_\alpha[\mu] \in L^{p'}(d\sigma)$, with $\frac{1}{p} + \frac{1}{p'} = 1$, that is

$$\int_{\mathbf{S}^n} \left(\int_{\mathbf{B}^n} \frac{d\mu(z)}{|1 - z\zeta|^{n-\alpha}} \right)^{p'} d\sigma(\zeta) < +\infty. \tag{2.4}$$

In the real case, the analogue to this integral is the energy of μ

$$E^\alpha_p(\mu) = \int_{\mathbf{R}^n} \left(\int_{\mathbf{R}^n} \frac{d\mu(x)}{|x - y|^{n-\alpha}} \right)^{p'} dy,$$

which, by Wolff's inequality ([HeWo]), is equivalent to

$$\int_{\mathbf{R}^n}\int_0^1\left(\frac{\mu\big(B(x,\delta)\big)}{\delta^{n-\alpha p}}\right)^{\frac{1}{p-1}}\frac{d\delta}{\delta}\,d\mu(x).$$

We will give a sketch of the proof for the non-isotropic case. If μ be a positive finite Borel measure on $\mathbf{B}^n$ satisfying that $W_\mu^{\alpha p} \in L^1(d\mu)$, there exists $\varepsilon_0 < 1$ so that

$$\int_{|1-z\bar{\zeta}|<\varepsilon_0}\frac{d\mu(z)}{|1-z\zeta|^{n-\alpha}} \preceq \int_0^1\frac{\mu\Big(T\big(B(\zeta,\delta)\big)\Big)}{\delta^{n-\alpha}}\frac{d\delta}{\delta}. \tag{2.5}$$

Now, there exist $M > 0$, $C \geq 1$ so that for any finite positive Borel measure μ on $\mathbf{B}^n$,

$$\int_{\mathbf{S}^n}\left\{\int_0^1\left(\frac{\mu\Big(T\big(B(\zeta,\delta)\big)\Big)}{\delta^{n-\alpha}}\right)\frac{d\delta}{\delta}\right\}^{p'}d\sigma(\zeta) \leq M\int_{\mathbf{B}^n}\int_{1-|\eta|}^{C}\left(\frac{\mu\Big(T\big(B(\eta_0,\delta)\big)\Big)}{\delta^{n-\alpha p}}\right)^{p'-1}\frac{d\delta}{\delta}\,d\mu(\eta). \tag{2.6}$$

The proof of the above estimate is analogous to the following non-isotropic version of Wolff's inequality, [HeWo], which is due to Cohn and Verbistky, [CoVe2]:

THEOREM 2.3 ([COVE2]). *Let ν be a finite positive Borel measure on $\mathbf{S}^n$, $1 < p < +\infty$, $0 < s \leq +\infty$ and $\alpha > 0$. Then*

$$\int_{\mathbf{S}^n}\left\{\int_0^1\left(\frac{\nu\big(B(\zeta,1-r)\big)}{(1-r)^{n-\alpha}}\right)^{s}\frac{dr}{1-r}\right\}^{\frac{p'}{s}}d\sigma(\zeta) \leq C\int_{\mathbf{S}^n}\int_0^1\left(\frac{\nu\big(B(\eta,1-r)\big)}{(1-r)^{n-\alpha p}}\right)^{p'-1}\frac{dr}{1-r}\,d\nu(z),$$

where $C > 0$ is a constant independent of μ.

The estimate (2.5) together with (2.6) give the boundedness of the integral in (2.4), and the sufficiency is proved.

For the necessity, Fubini's theorem gives that

$$\int_{\mathbf{S}^n}\left(\int_{\mathbf{B}^n}\frac{d\mu(z)}{|1-z\zeta|^{n-\alpha}}\right)^{p'}d\sigma(\zeta) = \int_{\mathbf{B}^n}\int_{\mathbf{S}^n}\frac{d\sigma(\zeta)}{|1-\eta\zeta|^{n-\alpha}}\left(\int_{\mathbf{B}^n}\frac{d\mu(z)}{|1-z\zeta|^{n-\alpha}}\right)^{p'-1}d\mu(\eta).$$

Now the next lemma and the previous formula give the end of the proof of the theorem.

LEMMA 2.4. *There exists $C > 0$ so that for any positive measure μ on $\mathbf{B}^n$ and any $\eta \in \mathbf{B}^n$,*

$$\int_{\mathbf{S}^n}\frac{1}{|1-\eta\zeta|^{n-\alpha}}\left(\int_{\mathbf{B}^n}\frac{d\mu(z)}{|1-z\zeta|^{n-\alpha}}\right)^{p'-1}d\sigma(\zeta) \geq C\int_{1-|\eta|}^{1}\left(\frac{\mu\Big(T\big(B(\eta_0,\delta)\big)\Big)}{\delta^{n-\alpha p}}\right)^{p'-1}\frac{d\delta}{\delta}.$$

So we are led to prove the lemma:

PROOF OF LEMMA 2.4.

$$\int_{\mathbf{B}^n} \frac{d\mu(z)}{|1-z\zeta|^{n-\alpha}} = \int_0^{+\infty} \mu\Big(\{z\in\mathbf{B}^n\,;\, \frac{1}{|1-z\zeta|^{n-\alpha}}>\lambda\}\Big)\,d\lambda$$
$$\simeq \int_0^{+\infty} \mu(\{z\in\mathbf{B}^n\,;\,|1-z\bar\zeta|<t\})\frac{dt}{t^{n-\alpha+1}}.$$

Now, we can choose $\varepsilon>0$ small enough so that if $z\in T\big(B(\zeta,\varepsilon t)\big)$ we have that $T\big(B(\zeta,\varepsilon t)\big)\subset\{z\in\mathbf{B}^n\,;\,|1-z\bar\zeta|<t\}$. Consequently

$$\int_{\mathbf{S}^n}\frac{1}{|1-\eta\zeta|^{n-\alpha}}\Big(\int_{\mathbf{B}^n}\frac{d\mu(z)}{|1-z\zeta|^{n-\alpha}}\Big)^{p'-1}d\sigma(\zeta)$$
$$\succeq \int_{\mathbf{S}^n}\frac{d\sigma(\zeta)}{|1-\eta\zeta|^{n-\alpha}}\left(\int_{C|1-\eta_0\bar\zeta|}^{+\infty}\frac{\mu\Big(T\big(B(\zeta,\varepsilon t)\big)\Big)}{t^{n-\alpha+1}}\,dt\right)^{p'-1},$$

where $C>0$ is to be chosen. Since $C|1-\eta_0\bar\zeta|<t$, we have that $T\big(B(\zeta,\varepsilon C|1-\eta_0\bar\zeta|)\big)\subset T\big(B(\zeta,\varepsilon t)\big)$, and the previous integral is bounded from below by

$$\int_{\mathbf{S}^n}\frac{1}{|1-\eta\zeta|^{n-\alpha}}\mu\Big(T\big(B(\zeta,\varepsilon C|1-\eta_0\bar\zeta|)\big)\Big)\Big(\int_{C|1-\eta_0\bar\zeta|}^{+\infty}\frac{dt}{t^{n-\alpha+1}}\Big)^{p'-1}d\sigma(\zeta)$$
$$\simeq \int_{\mathbf{S}^n}\frac{1}{|1-\eta\bar\zeta|^{n-\alpha}|1-\eta_0\bar\zeta|^{\frac{n-\alpha}{p-1}}}\mu\Big(T\big(B(\zeta,\varepsilon C|1-\eta_0\bar\zeta|)\big)\Big)^{p'-1}d\sigma(\zeta)$$
$$\ge \int_{\mathbf{S}^n}\frac{1}{|1-\eta\bar\zeta|^{\frac{(n-\alpha)p}{p-1}}}\mu\Big(T\big(B(\zeta,\varepsilon C|1-\eta_0\bar\zeta|)\big)\Big)^{p'-1}d\sigma(\zeta).$$

Next, if $C>0$ is big enough, we have that $T\big(B(\eta_0,|1-\eta_0\bar\zeta|)\big)\subset T\big(B(\zeta,\varepsilon C|1-\eta_0\bar\zeta|)\big)$, and consequently, the last integral is bounded by

$$\int_{\mathbf{S}^n}\mu\Big(T\big(B(\eta_0,|1-\eta_0\bar\zeta|)\big)\Big)^{p'-1}\frac{d\sigma(\zeta)}{\big((1-|\eta|)+|1-\eta_0\bar\zeta|\big)^{\frac{(n-\alpha)p}{p-1}}}$$
$$\ge \int_{|1-\eta_0\bar\zeta|\le 1-|\eta|}\frac{\mu\Big(T\big(B(\eta_0,|1-\eta_0\bar\zeta|)\big)\Big)^{p'-1}}{\big((1-|\eta|)+|1-\eta_0\bar\zeta|\big)^{\frac{(n-\alpha)p}{p-1}}}\,d\sigma(\zeta)$$
$$+\sum_{k\ge 0,\,2^k(1-|\eta|)\le 1}\int_{2^k(1-|\eta|)\le|1-\eta_0\bar\zeta|<2^{k+1}(1-|\eta|)}\frac{\mu\Big(T\big(B(\eta_0,|1-\eta_0\bar\zeta|)\big)\Big)^{p'-1}}{\big(2^k(1-|\eta|)\big)^{\frac{(n-\alpha)p}{p-1}}}\,d\sigma(\zeta)$$
$$\ge \sum_{k\ge 0,\,2^k(1-|\eta|)\le 1}\int_{2^k(1-|\eta|)\le|1-\eta_0\bar\zeta|<2^{k+1}(1-|\eta|)}\frac{\mu\Big(T\Big(B\big(\eta_0,2^k(1-|\eta|)\big)\Big)\Big)^{p'-1}}{\big(2^k(1-|\eta|)\big)^{\frac{(n-\alpha)p}{p-1}}}\,d\sigma(\zeta)$$

$$\succeq \sum_{k\geq 0,\quad 2^k(1-|\eta|)\leq 1} \frac{\mu\Big(T\Big(B\big(\eta_0, 2^k(1-|\eta|)\big)\Big)\Big)^{p'-1}}{\big(2^k(1-|\eta|)\big)^{\frac{n-\alpha p}{p-1}}} \succeq \int_{1-|\eta|}^{1} \left(\frac{\mu\Big(T\big(B(\eta_0, t)\big)\Big)}{t^{n-\alpha p}}\right)^{p'-1} \frac{dt}{t}.$$

■

When $q > 1$ and $\frac{n-\alpha}{p-1} < n$ (which holds if for example $p > 2$), the following theorem gives that the corresponding condition to (2.2) is always sufficient.

THEOREM 2.5. *Let $1 < q < p$ and assume that $n - \alpha p > 0$ and $\frac{n-\alpha}{p-1} < n$. If there exists $C > 0$ such that if μ is a finite positive Borel measure on $\mathbf{B}^n$ satisfying that*

$$W_\mu^{\alpha p}(\eta) = \int_{1-|\eta|}^{1} \left(\frac{\mu\Big(T\big(B(\eta_0, C\delta)\big)\Big)}{\delta^{n-\alpha p}}\right)^{p'-1} \frac{d\delta}{\delta} \in L^{\frac{q}{p-q}(p-1)}(d\mu), \tag{2.7}$$

then μ is a q-Carleson measure for h_α^p.

PROOF OF THEOREM 2.5. A similar argument to the one we have used in (2.2) shows that we just need to check that the linear operator

$$Tf(z) = \int_{\mathbf{S}^n} \frac{f(\zeta)}{|1 - z\zeta|^{n-\alpha}}\, d\sigma(\zeta),$$

is bounded from $L^p(d\sigma)$ to $L^q(d\mu)$. Equivalently we will show that the adjoint operator T^* given by

$$T^* f(\zeta) = \int_{\mathbf{B}^n} \frac{f(z)}{|1 - z\zeta|^{n-\alpha}}\, d\mu(z),$$

is bounded from $L^{q'}(d\mu)$ to $L^{p'}(d\sigma)$, where $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$. That is, we will check that provided μ satisfies (2.7),

$$\int_{\mathbf{S}^n} \left| \int_{\mathbf{B}^n} \frac{f(z)}{|1 - z\zeta|^{n-\alpha}}\, d\mu(z) \right|^{p'} d\sigma(\zeta) \leq C\Big(\int_{\mathbf{B}^n} |f(\omega)|^{q'}\, d\mu(\omega)\Big)^{\frac{p'}{q'}}, \tag{2.8}$$

for some constant $C > 0$.

Now Hölder's inequality followed by Fubini's theorem give that the integral that appears in the left hand side of (2.8) is bounded by

$$\begin{aligned}
&\int_{\mathbf{S}^n} \int_{\mathbf{B}^n} \frac{|f(\omega)|^{p'}}{|1 - \omega\zeta|^{n-\alpha}}\, d\mu(\omega) \Big(\int_{\mathbf{B}^n} \frac{d\mu(z)}{|1 - z\zeta|^{n-\alpha}}\Big)^{p'-1} d\sigma(\zeta) \\
&\quad = \int_{\mathbf{B}^n} \int_{\mathbf{S}^n} \frac{1}{|1 - \omega\zeta|^{n-\alpha}} \Big(\int_{\mathbf{B}^n} \frac{d\mu(z)}{|1 - z\zeta|^{n-\alpha}}\Big)^{p'-1} d\sigma(\zeta) |f(\omega)|^{p'}\, d\mu(\omega).
\end{aligned}$$

Hölder's inequality gives now (2.8) if the function

$$\begin{aligned}
U_{\alpha p}^{\mu}(\omega) &= \int_{\mathbf{S}^n} \frac{1}{|1 - \omega\zeta|^{n-\alpha}} \Big(\int_{\mathbf{B}^n} \frac{d\mu(z)}{|1 - z\zeta|^{n-\alpha}}\Big)^{p'-1} d\sigma(\zeta) \quad \text{belongs to } L^{\frac{q'}{q'-p'}}(d\mu) \\
&= L^{\frac{q}{p-q}(p-1)}(d\mu).
\end{aligned}$$

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Now the proof of Theorem 2.5 is deduced immediately from the previous estimate and the following claim: if $\frac{n-\alpha}{p-1} < n$, there exists $M > 0$, $K > 0$ and $C > 0$ so that for any finite positive measure μ on $\mathbf{B}^n$, and for any $\eta \in \mathbf{B}^n$,

$$U_\alpha^p(\eta) \leq M \int_{1-|\eta|}^{K} \left(\frac{\mu\Big(T\big(B(\eta_0, C\delta)\big)\Big)}{\delta^{n-\alpha p}} \right)^{p'-1} \frac{d\delta}{\delta}.$$

The proof of that follows closely a similar estimate for positive measures on $\mathbf{R}^n$ due to Maz'ya and Khavin, [MaKh]:

THEOREM 2.6 ([MAKH], THEOREM 2.6). *Assume that* $1 < p < +\infty$, $n - \alpha p > 0$ *and* $\frac{n-\alpha}{p-1} < n$*. Then for any positive measure* μ *in* $\mathbf{R}^n$ *and for any point* $x \in \mathbf{R}^n$,

$$U_{\alpha p}^{\mu}(x) := \int_{\mathbf{R}^n} \left(\int_{\mathbf{R}^n} \frac{d\mu(z)}{|y-z|^{n-\alpha}} \right)^{p'-1} \frac{dy}{|x-y|^{n-\alpha}} \leq C \int_0^{+\infty} \left(\frac{\mu\big(B(x,\delta)\big)}{\delta^{n-\alpha p}} \right)^{p'-1} \frac{d\delta}{\delta},$$

where the constant C *depends only on* n, α, *and* p. ∎

3**. Geometric and moduli of continuity type conditions.** Before stating other sufficient conditions, and in order to give a motivation for them, we will obtain some non-trivial necessary conditions for a positive measure to be q-Carleson for h_α^p. The methods to get such necessary conditions, specially Theorem 3.2, are based in [Lu1], where similar problems for the spaces of derivatives of harmonic functions in $\mathbf{R}^n$ are considered. For a sake of completeness we will briefly sketch the main ideas. We first need some definitions.

We will denote by $E_\varepsilon(z)$, $z \in \mathbf{B}^n$ the balls for the pseudohyperbolic metric given by $E_\varepsilon(z) = \{\omega \in \mathbf{C}^n \,;\, |\phi_z(\omega)| < \varepsilon\}$, where ϕ_z is the automorphism of the unit ball in $\mathbf{C}^n$ taking 0 to z and satisfying that $\phi_z \circ \phi_z = \mathrm{Id}_{\mathbf{B}^n}$. We will say that a sequence $(z_k)_k \subset \mathbf{B}^n$ is separated if there exists $0 < \varepsilon < 1$ so that the balls $E_\varepsilon(z_k)$ for the pseudohyperbolic metric are disjoint.

Following [Lu1] we define the tent spaces $T_r^s(\nu)$, $0 < r \leq +\infty$, $0 < s < +\infty$. If ν is a finite positive Borel measure on $\mathbf{B}^n$, and if $r < +\infty$, let

$$A_{r,\nu}(f)(\zeta) = \left(\int_{D(\zeta)} |f(z)|^r \, d\nu(z) \right)^{\frac{1}{r}}.$$

If $r = +\infty$, let

$$A_{\infty,\nu}(f)(\zeta) = \nu - \operatorname*{ess\,sup}_{z \in D(\zeta)} |f(z)|.$$

When $d\nu(z) = \frac{dV(z)}{(1-|z|)^{n+1}}$, we will skip the subscript ν. The tent space T_r^s is the space of ν-equivalence classes of functions f on $\mathbf{B}^n$ satisfying that

(i) $A_{r,\nu}(f) \in L^s(d\sigma)$, if $r, s < +\infty$,

The space $\tilde{T}_\infty^s(\nu)$ is the space of functions f so that

(ii) $A_{\infty,\nu}(f) \in L^s(d\sigma)$.

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In the particular case that $\nu = \sum_k \delta_k$, with $(z_k)_k$ a separated sequence in $\mathbf{B}^n$, and δ_z the Dirac measure at z, we just will write $T^s_r(z_k)$, and $\tilde{T}^s_\infty(z_k)$ respectively.

We then have the following non-isotropic version of Lemma 3 in [Lu1]:

LEMMA 3.1. *Let λ be a non-negative integer. Let $p > 0$, and if $p < 2$, suppose that $n + \lambda + 1 > \frac{2n}{p}$. Define the linear operator ψ_λ given by*

$$\psi_\lambda\big((c_k)_k\big)(z) = \sum_k c_k \frac{(1-|z_k|)^{n+\lambda+1}}{(1-z\overline{z_k})^{n+\lambda+1}}.$$

Then ψ_λ is a bounded operator from $T^p_2(z_k)$ into H^p. ■

The proof uses a non-isotropic version of the atomic decomposition for the spaces T^p_2, see [CoMeSt] for the isotropic case, due to [Yo].

We want to obtain a bounded operator from $T^p_2(z_k)$ into the Hardy-Sobolev space H^p_α. We proceed as follows: in [CaOr2, Theorem 2.1] it is proved that the linear operator $\Phi\colon H^p \longrightarrow H^p_\alpha$ defined by

$$\Phi(f)(z) = \int_{\mathbf{S}^n} \frac{f(\zeta)}{(1-z\zeta)^{n-\alpha}}\, d\sigma(\zeta)$$

is continuous. Hence the composition $\varphi_\lambda = \Phi \circ \psi_\lambda$ is a continuous operator from $T^p_2(z_k)$ into H^p_α. Cauchy's formula gives then that the operator

$$\varphi_\lambda\big((c_k)_k\big)(z) = \sum_k c_k \frac{(1-|z_k|)^{n+\lambda+1}}{(1-z\overline{z_k})^{n-\alpha+\lambda+1}} \tag{3.1}$$

is bounded from $T^p_2(z_k)$ to H^p_α.

The non-isotropic pseudodistance in $\overline{\mathbf{B}}^n$, will be denoted by $d(z,\omega)$ and is given by

$$d(z,\omega) = |\omega(\overline{z}-\overline{\omega})| + |z-\omega|^2.$$

The corresponding balls in $\overline{\mathbf{B}}^n$ will be denoted by $Q(z,r)$. We recall that $|1 - z\overline{\omega}| \simeq d(z,\omega) + (1-|z|)$, for $z,\omega \in \overline{\mathbf{B}}^n$.

We can now give the following

THEOREM 3.2. *Let μ be a finite positive Borel measure on $\mathbf{B}^n$, and let $1 \le q < p < +\infty$, $n - \alpha p > 0$. Assume that there exists $C > 0$ so that for any $f \in L^p(d\sigma)$,*

$$\int_{\mathbf{B}^n} |P[K_\alpha * f](z)|^q\, d\mu(z) \le C\|f\|^q_p.$$

Then:

(a) If $q \ge 2$ there exists $\delta > 0$ so that

$$\int_{\mathbf{S}^n} \Big(\sup_{z\in D(\zeta)} (1-|z|)^{\alpha q - n} \mu\Big(T\Big(B\big(z_0, \delta(1-|z|)\big)\Big)\Big)\Big)^{\frac{p}{p-q}} d\sigma(\zeta) < +\infty.$$

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(As in the previous section $z_0 = \frac{z}{|z|}$).

(b) If $q < 2$, there exists $\delta > 0$ so that

$$\int_{\mathbf{S}^n}\left(\int_{D(\zeta)}\left(\frac{\mu\Big(Q\big(z,\delta(1-|z|)\big)\Big)}{(1-|z|)^{n-\alpha q}}\right)^{\frac{2}{2-q}}\frac{dV(z)}{(1-|z|)^{n+1}}\right)^{\frac{p}{p-q}\frac{2-q}{2}} d\sigma(\zeta) < +\infty.$$

PROOF OF THEOREM 3.2. Assume μ is a q-Carleson measure for h^p_α. As we have already commented, μ is also a q-Carleson measure for H^p_α, and (3.1) gives then that for any separated sequence $(z_k)_k$,

$$\int_{\mathbf{B}^n}\left|\sum_k c_k\frac{(1-|z_k|)^{n+\lambda+1}}{(1-z\bar{z}_k)^{n-\alpha+\lambda+1}}\right|^q d\mu(z) \le C\left(\int_{\mathbf{S}^n}\Big(\sum_{z_k\in D(\zeta)}|c_k|^2\Big)^{\frac{p}{2}} d\sigma(\zeta)\right)^{\frac{q}{p}}. \tag{3.2}$$

Now, if $\eta_1 < \eta_0$ are small enough and $\varepsilon < 1$ there exists a lattice $(a_{kj})_{kj} \subset \mathbf{B}^n$ so that

(i) $|a_{kj}| = 1-\varepsilon^k$, $j = 1,\dots j_k$.

(ii) $\bigcup_{k,j} E_{\eta_0}(a_{kj}) = \mathbf{B}^n$.

(iii) $E_{\eta_1}(a_{kj}) \cap E_{\eta_1}(a_{k'j'}) \neq \emptyset$ if and only if $k = k'$ and $j = j'$.

If $(a_{kj})_{kj}$ is such lattice,

$$\int_{\mathbf{B}^n}\left|\sum_{k,j} c_{kj}\frac{(1-|a_{kj}|)^{n+\lambda+1}}{(1-z\bar{a}_{kj})^{n-\alpha+\lambda+1}}\right|^q d\mu(z) \le C\left(\int_{\mathbf{S}^n}\Big(\sum_{a_{kj}\in D(\zeta)}|c_{kj}|^2\Big)^{\frac{p}{2}} d\sigma(\zeta)\right)^{\frac{q}{p}}.$$

Applying Khinchine's inequality, (see [Lu1, page 609], we deduce from the previous estimate that

$$\int_{\mathbf{B}^n}\left(\sum_{kj}|c_{kj}|^2\frac{(1-|a_{kj}|)^{2n+2\lambda+2}}{|1-z\bar{a}_{kj}|^{2n-2\alpha+2\lambda+2}}\right)^{\frac{q}{2}} d\mu(z) \le C\|(c_{kj})_{kj}\|^q_{T^p_2(a_{kj})}. \tag{3.3}$$

Assume first that $q \ge 2$, put $\tilde{a}_{kj} = \frac{a_{kj}}{|a_{kj}|}$ and take $\delta > 0$ small enough so that for any $z \in T\big(B(\tilde{a}_{kj},\delta(1-|a_{kj}|)\big)$, we have that $|1-z\bar{a}_{kj}| \simeq 1-|a_{kj}|$. Then we deduce from (3.3) that

$$\int_{\mathbf{B}^n}\left(\sum_{kj}|c_{kj}|^2(1-|a_{kj}|)^{2\alpha}\chi_{T\left(B\left(\tilde{a}_{kj},\delta(1-|a_{kj}|)\right)\right)}\right)^{\frac{q}{2}} d\mu(z) \le C\|(c_{kj})_{kj}\|^q_{T^p_2(a_{kj})}.$$

Since we are assuming that $q \ge 2$, we then obtain that

$$\sum_{kj}|c_{kj}|^q(1-|a_{kj}|)^{\alpha q}\mu(T\Big(B\big(\tilde{a}_{kj},\delta(1-|a_{kj}|)\big)\Big) \le C\|(c_{kj})_{kj}\|^q_{T^p_2(a_{kj})}.$$

Now, putting in the previous estimate $|c_{kj}|^q = b_{kj}$ we obtain that

$$\sum_{kj}b_{kj}(1-|a_{kj}|)^{\alpha q}\mu(T\Big(B\big(\tilde{a}_{kj},\delta(1-|a_{kj}|)\big)\Big) \le C\|(b_{kj})_{kj}\|_{T^{\frac{p}{q}}_{\frac{2}{q}}(a_{kj})},$$

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which also remains true for non-positive $(b_{kj})_{kj}$, and we conclude that

$$\left((1-|a_{kj}|)^{\alpha q-n}\mu\Bigl(T\Bigl(B\bigl(\tilde{a}_{kj},\delta(1-|a_{kj}|)\bigr)\Bigr)\Bigr)\right)_{kj}\in T^{\frac{p}{q}}_{\frac{2}{q}}(a_{kj})^*.$$

But the non-isotropic version of [Lu1, Proposition 3] gives that $T^{\frac{p}{q}}_{\frac{2}{q}}(a_{kj})^* = \tilde{T}^{\frac{p}{p-q}}_{\infty}(a_{kj})$. Equivalently we have that

$$\int_{\mathbf{S}^n}\Bigl\{\sup_{a_{kj}\in D(\zeta)}(1-|a_{kj}|)^{\alpha q-n}\mu(T\Bigl(B\bigl(\tilde{a}_{kj},\delta(1-|a_{kj}|)\bigr)\Bigr)\Bigr\}^{\frac{p}{p-q}}\,d\sigma(\zeta)<+\infty,$$

which gives the discrete version of (a). The continuous version is easily deduced from this one and the properties of the lattice $(a_{kj})_{kj}$.

So we are led to the case $q<2$. By property (iii) of the sequence $(a_{kj})_{kj}$ we deduce that there exists $N>0$ so that any $z\in\mathbf{B}^n$ lies in no more than N non-isotropic balls $Q_{kj}=Q(a_{kj},\delta(1-|a_{kj}|)$, for some $\delta>0$ small enough and depending on η_1. Since for any $z\in Q_{kj}$ we have that $|1-z\bar{a}_{kj}|\simeq(1-|a_{kj}|)$, we deduce from (3.3) that if μ is a q-Carleson measure for h^p_α, then

$$\int_{\mathbf{B}^n}\Bigl(\sum_{kj}|c_{kj}|^2(1-|a_{kj}|)^{2\alpha}\chi_{Q_{kj}}(z)\Bigr)^{\frac{q}{2}}\,d\mu(z)\le C\|(c_{kj})_{kj}\|^q_{T^p_2(a_{kj})}.$$

Now the fact that the balls $(Q_{kj})_{kj}$ don't overlap too much gives that

$$\sum_{kj}|c_{kj}|^q(1-|a_{kj}|)^{\alpha q}\mu(Q_{kj})\le C\|(c_{kj})_{kj}\|^q_{T^p_2(a_{kj})}.$$

An argument like the one used in case (a) shows that

$$\bigl(1-|a_{kj}|)^{\alpha q-n}\mu(Q_{kj})\bigr)_{kj}\in T^{\frac{p}{q}}_{\frac{2}{q}}(a_{kj})^*.$$

This space $T^{\frac{p}{q}}_{\frac{2}{q}}(a_{kj})^*$ equals to $T^{\frac{p}{p-q}}_{\frac{2}{2-q}}\bigl((a_{kj})\bigr)$ (see also [Lu1]). This gives the discrete version of (b), from which we deduce the continuous one, using the properties of the separated sequence $(a_{kj})_{kj}$. ∎

REMARK. In [CaOr2] it is obtained another representation theorem for H^p_α. Namely, if the sequence $(a_{kj})_{kj}$ satisfies properties (i) to (iii), then any f in H^p_α can be written as

$$f(z)=\sum_{kj}c_{kj}\frac{(1-|a_{kj}|)^{n(1-\frac{1}{p})}}{(1-z\bar{a}_{kj})^{n-\alpha}},$$

and

$$\|f\|_{p,\alpha}\le C\sum_{k=1}^{+\infty}\Bigl(\sum_{j=1}^{j_k}|c_{kj}|^p\Bigr)^{\frac{1}{p}}.$$

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From this representation one can obtain, using again Khinchine's inequality the following, in general, weaker necessary condition, which is now valid for all $1 \le q < p < +\infty$. If μ is a q-Carleson measure for H^p_α, then there exist $\delta, C > 0$ so that

$$\sup_r \int_{\mathbf{S}^n} \left(\frac{\mu\Big(T\big(B(\zeta,\delta r)\big)\Big)}{r^{n-\alpha q}} \right)^{\frac{p}{p-q}} d\sigma(\zeta) \le C. \tag{3.4}$$

When $2 \le q$, we can also deduce another necessary condition.

PROPOSITION 3.3. *Let $2 \le q < p$, and assume that μ is a q-Carleson measure for h^p_α. Then*

$$\int_{\mathbf{S}^n} \left(\int_{D(\zeta)} \frac{d\mu(z)}{(1-|z|)^{n-\alpha q}} \right)^{\frac{p}{p-q}} d\sigma(\zeta) < +\infty. \tag{3.5}$$

PROOF OF PROPOSITION 3.3. Let μ be a q-Carleson measure for h^p_α. Let $\zeta \in \mathbf{S}^n$, and consider $B = B(z_0, r)$ be any non-isotropic ball so that $\zeta \in B$, and $r < \delta$, with δ as in Theorem 3.2, part (a). We choose δ small enough so that the point $z = z_0(1 - \frac{r}{\delta})$ is in $D(\zeta)$. Hence

$$\frac{\mu\big(T(B)\big)}{|B|^{1-\frac{\alpha q}{n}}} \preceq \sup_{z\in D(\zeta)} (1-|z|)^{\alpha q - n} \mu\Big(T\Big(B(\zeta, C\delta(1-|z|)\Big)\Big),$$

for some absolute constant $C > 0$.

Now we apply Theorem 3.2 (a), and we obtain

$$\sup_{\zeta\in B} \frac{\mu\big(T(B)\big)}{|B|^{1-\frac{\alpha q}{n}}} \quad \text{is in} \quad L^{\frac{p}{p-q}}(d\sigma). \tag{3.6}$$

The methods in in [CoMeSt, Theorem 3] can be used to show that for any positive measure ν, and $1 < p < +\infty$,

$$\left\| \int_{D(\zeta)} \frac{d\nu(z)}{(1-|z|)^n} \right\|_p \simeq \left\| \sup_{\zeta\in B} \frac{\nu\big(T(B)\big)}{|B|} \right\|_p.$$

In particular applying the above for the measure $d\nu(z) = (1-|z|)^{\alpha q}\, d\mu(z)$

$$\left\| \int_{D(\zeta)} \frac{d\mu(z)}{(1-|z|)^{n-\alpha q}} \right\|_{\frac{p}{p-q}} \simeq \left\| \sup_{\zeta\in B} \frac{1}{|B|} \int_{T(B)} (1-|z|)^{\alpha q}\, d\mu(z) \right\|_{\frac{p}{p-q}},$$

and since if $z \in T(B)$, $1-|z| \preceq |B|^{\frac{1}{n}}$,

$$\begin{aligned}\int_{\mathbf{S}^n} \left(\int_{D(\zeta)} \frac{d\mu(z)}{(1-|z|)^{n-\alpha q}} \right)^{\frac{p}{p-q}} d\sigma(\zeta) &\simeq \int_{\mathbf{S}^n} \left(\sup_{\zeta\in B} \frac{1}{|B|} \int_{T(B)} (1-|z|)^{\alpha q}\, d\mu(z) \right)^{\frac{p}{p-q}} d\sigma(\zeta) \\ &\preceq \int_{\mathbf{S}^n} \left(\sup_{\zeta\in B} \frac{\mu\big(T(B)\big)}{|B|^{1-\frac{\alpha q}{n}}} \right)^{\frac{p}{p-q}} d\sigma(\zeta).\end{aligned}$$

The last estimate together with (3.6) finishes the proof of the proposition. ∎

Now we give a sufficient condition of geometric type. We need some definitions. Suppose $m = n - \alpha p > 0$, and let $1 \leq \tau \leq \frac{n}{m}$. If $\zeta \in \mathbf{S}^n$, $\beta > 0$, let $\Omega_\tau(\zeta)$ be the approach region defined by

$$\Omega_\tau(\zeta) = \Omega_\tau^\beta(\zeta) = \{z \in \mathbf{B}^n \,;\, |1 - z\bar{\zeta}|^\tau < \beta(1 - |z|)\},$$

and if $f\colon \mathbf{B}^n \longrightarrow \mathbf{C}$, let $M_\tau f(\zeta) = \sup_{z \in \Omega_\tau(\zeta)} |f(z)|$. Observe that if $\tau = 1$, $\Omega_1(\zeta) = D(\zeta)$ the admissible approach region, and $M_1 = N$ is the admissible maximal function.

Recall that in Proposition 3.3 we have obtained that any q-Carleson measure for h^p_α, $2 \leq q < p$ satisfies that the function $\int_{D(\zeta)} \frac{d\mu(z)}{(1-|z|)^{n-\alpha q}} \in L^{\frac{p}{p-q}}\big(d\sigma(\zeta)\big)$. In the reverse direction we have the following theorem

THEOREM 3.4. *Let μ be a finite positive Borel measure on $\mathbf{B}^n$ and take $1 \leq q \leq p$, $m = n - \alpha p > 0$. Assume that*

$$\int_{\Omega_{\frac{n}{m}}(\zeta)} \frac{d\mu(z)}{(1 - |z|)^{n-\alpha p}} \in L^{\frac{p}{p-q}}(d\sigma).$$

Then μ is a q-Carleson measure for h^p_α.

PROOF OF THEOREM 3.4. Let $f \in L^p(d\sigma)$ and put $u = P[K_\alpha * f]$, and $m = n - \alpha p$. Then, provided we take $\varepsilon > 0$ small enough we have that

$$\begin{aligned}\int_{\mathbf{B}^n} |u(z)|^q \, d\mu(z) &\simeq \int_{\mathbf{B}^n} \int_{B(z_0, \varepsilon(1-|z|)^{\frac{m}{n}})} d\sigma(\zeta) \frac{|u(z)|^q}{(1 - |z|)^m} \, d\mu(z) \\ &\leq \int_{\mathbf{B}^n} \int_{\{\zeta \in \mathbf{S}^n ; |1 - z\bar{\zeta}|^{\frac{n}{m}} < \beta(1-|z|)\}} d\sigma(\zeta) \frac{|u(z)|^q}{(1 - |z|)^m} \, d\mu(z),\end{aligned}$$

where we have used that we can choose $\varepsilon > 0$ small enough so that $B(z_0, \varepsilon(1 - |z|)^{\frac{m}{n}}) \subset \{\zeta \in \mathbf{S}^n \,;\, |1 - z\bar{\zeta}|^{\frac{n}{m}} < \beta(1 - |z|)\}$. Now Fubini's theorem gives that the previous integral is bounded by

$$\int_{\mathbf{S}^n} \int_{\Omega_{\frac{n}{m}}(\zeta)} \frac{|u(z)|^q}{(1 - |z|)^m} \, d\mu(z) \, d\sigma(\zeta) \leq \int_{\mathbf{S}^n} M_{\frac{n}{m}} u(\zeta)^q \int_{\Omega_{\frac{n}{m}}(\zeta)} \frac{d\mu(z)}{(1 - |z|)^m} \, d\sigma(\zeta).$$

If $q < p$, Hölder's inequality with exponents $\frac{p}{q}$ and $\frac{p}{p-q}$ gives that this is bounded by

$$\left(\int_{\mathbf{S}^n} M_{\frac{n}{m}} u(\zeta)^p \, d\sigma(\zeta)\right)^{\frac{q}{p}} \left(\int_{\mathbf{S}^n} \left(\int_{\Omega_{\frac{n}{m}}(\zeta)} \frac{d\mu(z)}{(1 - |z|)^m}\right)^{\frac{p}{p-q}} d\sigma(\zeta)\right)^{\frac{p-q}{p}}.$$

The estimate follows immediately when $p = q$.

Finally, applying [Su, Theorem 3.8] (see also [CaOr2, Theorem 2.2]) we have that $\|M_{\frac{n}{m}} u\|_p \leq C\|f\|_p$, and we finish the proof of the theorem. ∎

Similar methods to the ones just used in Theorem 3.4 shows that the following slightly more general condition is also sufficient:

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PROPOSITION 3.5. *Let μ be a finite positive Borel measure on $\mathbf{B}^n$, $m = n - \alpha p > 0$, $1 < \tau \leq \frac{n}{m}$, and $1 \leq q \leq p$. Assume there exists a finite positive Borel measure ν on $\mathbf{S}^n$ so that $\nu\big(B(z_0, r)\big) \simeq r^{\tau m}$ for any $r > 0$, $z \in \operatorname{supp}\mu$, $z_0 = \frac{z}{|z|}$ and that it satisfies*

$$\int_{\Omega_\tau(\zeta)} \frac{d\mu(z)}{(1-|z|)^{n-\alpha p}} \in L^{\frac{p}{p-q}}(d\nu).$$

Then μ is a q-Carleson measure for h^p_α. ■

Observe that the previous theorem corresponds to $\tau = \frac{n}{m}$, $\nu = d\sigma$.

REMARK. If we consider the condition

$$\int_{\Omega_\tau(\zeta)} \frac{d\mu(z)}{(1-|z|)^{n-\alpha q-(1-\frac{1}{\tau})\frac{p-q}{p}n}} \in L^{\frac{p}{p-q}}(d\sigma),$$

the necessary condition of Proposition 3.3 corresponds to $\tau = 1$, whereas the sufficient condition of Theorem 3.4 corresponds to $\tau = \frac{n}{m}$. We do not know whether the intermediate conditions with $1 < \tau < \frac{n}{m}$ are still sufficient.

The remark after Theorem 3.2 says that if μ is a q-Carleson measure for h^p_α then

$$\left(\int_{\mathbf{S}^n} \mu\Big(T\big(B(\zeta, \delta r)\big)\Big)^{\frac{p}{p-q}} d\sigma(\zeta)\right)^{\frac{p-q}{p}} = O(r^{n-\alpha q}).$$

Our next result gives a result in the reverse direction. We obtain a sufficient condition in terms of the growth of an $L^{\frac{p}{p-q}}$-modulus of continuity of the measure μ. Recall that $T^\beta(E) = \mathbf{B}^n \setminus \bigcup_{\zeta\notin E} D^\beta(\zeta)$ denotes the admissible tent over E.

THEOREM 3.6. *Let μ be a finite positive Borel measure on $\mathbf{B}^n$, and let $1 \leq q < p$, $n - \alpha p > 0$, $\beta > 1$. Define*

$$g(r) = g^\beta(r) = \left(\int_{\mathbf{S}^n} \mu\Big(T^\beta\big(B(\zeta, r)\big)\Big)^{\frac{p}{p-q}} d\sigma(\zeta)\right)^{\frac{p-q}{p}},$$

and assume that for some $k_0 < 1$,

$$\int_0^{k_0} \left(\frac{g(r)}{r^{n-\alpha q}}\right)^{\frac{1}{q}} \log\frac{1}{r}\frac{dr}{r} < +\infty.$$

Then μ is a q-Carleson measure for h^p_α.

PROOF OF THEOREM 3.6. The proof is based in the following proposition, which gives a sufficient condition for μ so that the weighted measure $(1-|z|)^{\alpha q}\, d\mu(z)$ is a q-Carleson measure for the space h^p of Poisson-Szegö transforms of L^p functions in the unit sphere, with an explicit control of the norm of the operator from H^p into $L^q\big((1-|z|)^{\alpha q}\, d\mu(z)\big)$.

PROPOSITION 3.7. *Let μ be a finite positive Borel measure on $\mathbf{B}^n$, $1 \leq q < p$, and $m = n - \alpha p > 0$. Suppose that there exists $0 < K < 1$, $\beta > 1$ so that*

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(1) $\sup \mu \subset \{z \in \mathbf{B}^n \,;\, 1-|z| < K\}$.

(2) The function $g^\beta(r)$ *satisfies that for some* $k_0 < 1$,

$$\int_0^{k_0} \frac{g^\beta(r)}{r^{n-\alpha q}}^{\frac{1}{q}} \frac{dr}{r} < +\infty.$$

Then there exist $M, P > 0$ *(not depending on* K*) so that for any* $\mathcal{M}$*-harmonic function* f *on* $\mathbf{B}^n$,

$$\int_{\mathbf{B}^n} (1-|z|)^{\alpha q} |f(z)|^q \, d\mu(z) \le P\left(\int_0^{KM} \frac{g^\beta(r)}{r^{n-\alpha q}} \frac{dr}{r}\right) \|Nf\|_p^q.$$

PROOF OF PROPOSITION 3.7. For any $k \in \mathbf{Z}^+$, let $C_k = \{z \in \mathbf{B}^n \,;\, \frac{1}{2^{k+1}} < 1-|z| \le \frac{1}{2^k}\}$. We then have

$$\int_{\mathbf{B}^n} (1-|z|)^{\alpha q} |f(z)|^q \, d\mu(z) \simeq \sum_{\frac{1}{2^k} < K} 2^{-k\alpha q} \int_{C_k} |f(z)|^q \, d\mu(z). \tag{3.7}$$

Mean-value inequality together with Hölder's inequality gives that for any $z \in C_k$, $\varepsilon_0 < 1$,

$$|f(z)|^q \le \int_{D_{\varepsilon_0}(z)} |f(\omega)|^q \frac{dV(\omega)}{(1-|\omega|)^{n+1}}.$$

We fix $\varepsilon_0 < 1$ which will be chosen later. There exists $C > 0$ so that if $\varepsilon = C\varepsilon_0$, then $D_{\varepsilon_0}(z) = \phi_z(\varepsilon_0 \mathbf{B}^n) \subset Q\big(z, \varepsilon(1-|z|)\big)$. Thus, from (3.7) we obtain

$$\int_{\mathbf{B}^n} (1-|z|)^{\alpha q} |f(z)|^q \, d\mu(z) \preceq \sum_{\frac{1}{2^k} < K} 2^{-k\alpha q} \int_{C_k} \int_{Q\left(z, \varepsilon(1-|z|)\right)} |f(\omega)|^q \frac{dV(\omega)}{(1-|\omega|)^{n+1}} \, d\mu(z).$$

Now, if ε_0 is small enough, ε is also small, and we have that if $\omega \in Q\big(z, \varepsilon(1-|z|)\big)$, then $1-|z| \simeq 1-|\omega|$. Hence, if $z \in C_k$, we deduce that there exists $C > 0$ so that for any $\omega \in Q\big(z, \varepsilon(1-|z|)\big)$, $\omega \in \tilde{C}_k = \{\omega \,;\, \frac{1}{C}\frac{1}{2^{k+1}} \le 1-|\omega| < \frac{C}{2^k}\}$. Consequently, Fubini's theorem gives that the previous sum can be estimated by

$$\begin{aligned} &\sum_{\frac{1}{2^k} < K} 2^{k(n+1-\alpha q)} \int_{\tilde{C}_k} \int_{Q(z, A(1-|\omega|)} d\mu(z) |f(\omega)|^q \, dV(\omega) \\ &= \sum_{\frac{1}{2^k} < K} 2^{k(n+1-\alpha q)} \int_{\tilde{C}_k} \mu\Big(Q\big(z, A(1-|\omega|)\big)\Big) |f(\omega)|^q \, dV(\omega). \end{aligned} \tag{3.8}$$

Next there exists $M > 0$, so that if $\omega_0 = \frac{\omega}{|\omega|}$, and $1-|\omega| \simeq \frac{1}{2^k}$, then $Q\big(\omega, A(1-|\omega|)\big) \subset T^\beta\big(B(\omega_0, \frac{M}{2^k})\big)$. Indeed it is enough to show that if $z \in Q\big(\omega, A(1-|\omega|)\big)$ and $z \in D(\zeta)$, $\zeta \in \mathbf{S}^n$, then $\zeta \in B(\omega_0, \frac{M}{2^k})$.

Hence we obtain that (3.8) is bounded by

$$\sum_{\frac{1}{2^k} < K} 2^{k(n+1-\alpha q)} \int_{\tilde{C}_k} |f(\omega)|^q \mu\left(T^\beta\left(B\left(\omega_0, \frac{M}{2^k}\right)\right)\right) dV(\omega)$$

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$$\leq \sum_{\frac{1}{2^k}<K} 2^{k(n+1-\alpha q)} \int_{\tilde{C}_k} |Nf(\omega_0)|^q \mu\left(T^\beta\left(B\left(\omega_0, \frac{M}{2^k}\right)\right)\right) dV(\omega)$$

$$\leq \sum_{\frac{1}{2^k}<K} 2^{k(n-\alpha q)} \int_{\mathbf{S}^n} |Nf(\omega_0)|^q \mu\left(T^\beta\left(B\left(\omega_0, \frac{M}{2^k}\right)\right)\right) d\sigma(\omega_0),$$

where in the last estimate we have integrated in polar coordinates. Hölder's inequality with exponents $\frac{p}{q}$ and $\frac{p-q}{p}$ gives that the previous sum is bounded by

$$\sum_{\frac{1}{2^k}<K} 2^{k(n-\alpha q)} \|Nf\|_p^q \left(\int_{\mathbf{S}^n} \mu\left(T^\beta\left(B\left(\omega_0, \frac{M}{2^k}\right)\right)\right)^{\frac{p}{p-q}} d\sigma(\omega_0)\right)^{\frac{p-q}{p}}$$

$$= \left(\sum_{\frac{1}{2^k}<K} 2^{k(n-\alpha q)} g^\beta\left(\frac{M}{2^k}\right)\right) \|Nf\|_p^q \simeq \left(\int_0^{MK} \frac{g^\beta(r)}{r^{n-\alpha q}} \frac{dr}{r}\right) \|Nf\|_p^q,$$

which finishes the proof of the proposition. ∎

We now go back to the proof of the theorem which goes in a similar way to Theorem 2.3 in [CaOr1], with Proposition 2.2 there substituted by Proposition 3.7.

We need to show that there exists $C > 0$ so that for any $f \in L^p_+(d\sigma)$,

$$\int_{\mathbf{B}^n} \left(\int_{\mathbf{S}^n} \frac{f(\zeta)}{|1 - z\bar{\zeta}|^{n-\alpha}} d\sigma(\zeta)\right)^q d\mu(z) \leq C\|f\|_p^q.$$

Since (see theorem 2.3 in [CaOr1]),

$$\int_{\mathbf{S}^n} \frac{f(\zeta)}{|1 - z\bar{\zeta}|^{n-\alpha}} d\sigma(\zeta) \preceq \int_0^1 (1-t)^{\alpha-1} F(tz)\, dt,$$

with $F = P[f]$, we just need to check that

$$\int_{\mathbf{B}^n} \left(\int_0^1 (1-t)^{\alpha-1} F(tz)\, dt\right)^q d\mu(z) \leq C\|f\|_p^q, \tag{3.9}$$

with $F = P[f]$.

We break the integral from 0 to 1 in (3.9) in two parts, from 0 to δ_0 and from δ_0 to 1 ($\delta_0 < 1$ to be chosen), and denote them by F_1 and F_2 respectively. The fact that the L^∞-norm of F over each compact in $\mathbf{B}^n$ is bounded by the L^p-norm of f implies that the contribution of F_1 to (3.9) is the desired one, and that without loss of generality we may assume that $\sup \mu \subset \{z \in \mathbf{B}^n \,;\, 1 - |z| < \varepsilon_0\}$, with $\varepsilon_0 << 1$ fixed that will be chosen later.

Let $z \in \mathbf{B}^n$ so that $1 - |z| \leq \varepsilon_0$ and choose $1 = t_0 > t_1 > \cdots > t_l$, where $l = l(z) \in \mathbf{Z}^+$ is the first integer satisfying that $2^l(1 - |z|) > \frac{1}{\beta}$, and where $1 - t_k|z| = 2^k(1 - |z|)$, $k \leq l$. Now, if $\varepsilon_0 < \frac{1}{\beta}$ and $\delta_0 = (1 - \frac{2}{\beta})(1 - \frac{1}{\beta})^{-1}$, $t_l \leq \delta_0$, and consequently,

(3.10)

$$F_2(z) \leq \int_{t_1}^1 (1 - t|z|)^\alpha F(tz) \frac{(1-t)^{\alpha-1}}{(1-t|z|)^\alpha} dt + \sum_{k=1}^{l-1} \int_{t_{k+1}}^{t_k} (1 - t|z|)^\alpha F(tz) \frac{dt}{1-t} := \sum_{k=0}^{l-1} F_{2,k}(z).$$

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We define the measures μ_k on $\mathbf{B}^n$, $0 \le k \le l-1$ given by:

$$\mu_0(h) = \mu\Big(\int_{t_1}^1 h(tz)\frac{(1-t)^{\alpha-1}}{(1-t|z|)^\alpha}\,dt\Big),$$
$$\mu_k(h) = \mu\Big(\int_{t_{k+1}}^{t_k} h(tz)\frac{dt}{1-t}\Big), \quad 1 \le k \le l,\ h \in C(\mathbf{B}^n).$$

Then, see [CaOr1], each μ_k is a finite measure on the unit ball. Furthermore, $\|\mu_k\| = O(1)$ and $\sup \mu_k \subset \{z \in \mathbf{B}^n\ ;\ 1-|z| \le \frac{\varepsilon_0}{2^k}\}$.

We now check that the measures μ_k are in the conditions of Proposition 3.7 with constants $K = \frac{\varepsilon_0}{2^k}$. If $tz \in T^\beta\big(B(\zeta,r)\big)$, $t \le 1$, then $z \in T^\beta\big(B(\zeta,r)\big)$. Thus

$$\mu_0(T^\beta\big(B(\zeta,r)\big) = \mu\Big(\int_{t_1}^1 \chi_{T^\beta(B(\zeta,r))}(tz)\frac{(1-t)^{\alpha-1}}{(1-t|z|)^\alpha}\,dt\Big) \preceq \mu\Big(T^\beta\big(B(\zeta,r)\big)\Big),$$

and similarly, $\mu_k\Big(T^\beta\big(B(\zeta,r)\big)\Big) \preceq \mu\Big(T^\beta\big(B(\zeta,r)\big)\Big)$, $1 \le k \le l-1$.

Now Hölder's inequality followed by Proposition 3.7 applied to each μ_k, together with (3.11) give that

$$\|F_2\|_{L^q(d\mu)} \le \sum_{k\ge0} \|F_{2,k}\|_{L^q(d\mu)} \le \sum_{k\ge0}\Big(\int_{\mathbf{B}^n}(1-|z|)^{\alpha q}F(z)^q\,d\mu_k(z)\Big)^{\frac1q}$$
$$\preceq \sum_{k\ge0}\Big(\int_0^{M\frac{\varepsilon_0}{2^k}} \frac{g(r)}{r^{n-\alpha q}}\frac{dr}{r}\Big)^{\frac1q}\|NF\|_p^q.$$

The fact that $\frac1q \le 1$ together with Fubini's theorem give that the previous sum is bounded by

$$\sum_{k\ge0}\int_0^{M\frac{\varepsilon_0}{2^k}}\Big(\frac{g(r)}{r^{n-\alpha q}}\Big)^{\frac1q}\frac{dr}{r}\|f\|_p^q \preceq \int_0^{k_0}\Big(\frac{g(r)}{r^{n-\alpha q}}\Big)^{\frac1q}\log\frac1r\frac{dr}{r}\|f\|_p^q,$$

provided we take ε_0 small enough. ∎

Now that we have obtained two different sufficient conditions in terms of moduli of continuity and the geometric type one, we would like to compare them. We will construct measures that satisfy only one of the conditions, and deduce that none of the sufficient conditions are, in general necessary. We will begin with an example of a measure in the conditions of Theorem 3.4 but does not satisfies the hypothesis of Theorem 3.6, and in the other sense, we will construct a measure satisfying the conditions of Theorem 3.6 but not the conditions of Theorem 3.4.

EXAMPLE 1. Assume $q = 1$ and $p = 2$. There exists a finite positive Borel measure μ in $\mathbf{B}^n$ such that

$$\int_{\Omega_{\frac{n}{m}}(\zeta)}\frac{d\mu(z)}{(1-|z|)^{n-\alpha}} \in L^2(d\sigma),$$

but if

$$g(r) = \Big(\int_{\mathbf{S}^n}\mu\Big(T\big(B(\zeta,r)\big)\Big)^2\,d\sigma(\zeta)\Big)^{\frac12},$$

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then for each $0 < k_0 < 1$,

$$\int_0^{k_0} \frac{g(r)}{r^{n-\alpha}} \log \frac{1}{r} \frac{dr}{r} = +\infty.$$

Let δ_{ζ_0} be the Dirac measure at a point $\zeta_0 \in \mathbf{S}^n$. We will construct a measure $d\mu(r\zeta) = h(1-r)\,dr\,\delta_{\zeta_0}$, with h a positive function in $(0,\eta)$, $h(Cr) \preceq h(r)$, for some $C > 1$, and for r small enough, and $\eta < 1$. We first check how the two type of sufficient conditions look like for these special measures.

We start with the geometric condition. If $d\mu(r\zeta) = h(1-r)\,dr\delta_{\zeta_0}$,

$$\int_{\Omega_{\frac{n}{m}}(\zeta)} \frac{d\mu(z)}{(1-|z|)^{n-\alpha p}} = \int_0^1 \chi_{\Omega_{\frac{n}{m}}(\zeta)}(t\zeta_0) \frac{h(1-t)}{(1-t)^{n-\alpha p}}\, dt,$$

with $\chi_{\Omega_{\frac{n}{m}}(\zeta)}$ the characteristic function of the set $\Omega_{\frac{n}{m}}$. Since for any $t\zeta_0 \in \Omega_{\frac{n}{m}}(\zeta)$ we have that $|1-\zeta\overline{\zeta_0}|^{\frac{n}{m}} \preceq (1-t)$, and conversely if $\varepsilon > 0$ is small enough and $|1-\zeta\overline{\zeta_0}|^{\frac{n}{m}} \leq \varepsilon(1-t)$ we obtain that $t\zeta_0 \in \Omega_{\frac{n}{m}}(\zeta)$, we conclude that the integral is different from zero only when $c|1-\zeta\overline{\zeta_0}|^{\frac{n}{m}} \leq 1-t \leq 1$, for some constant $c > 0$. These facts together with the doubling property of h give that the convergence of the integral

$$\int_{\mathbf{S}^n} \left(\int_{\Omega_{\frac{n}{m}}(\zeta)} \frac{d\mu(z)}{(1-|z|)^{n-\alpha p}} \right)^{\frac{p}{p-q}} d\sigma(\zeta)$$

is equivalent to the convergence of

$$\int_{\mathbf{S}^n} \left(\int_{|1-\zeta\overline{\zeta_0}|^{\frac{n}{m}}}^{k} \frac{h(t)}{t^{n-\alpha p}}\, dt \right)^{\frac{p}{p-q}} d\sigma(\zeta).$$

The convergence of the above integral is equivalent to the convergence of

$$\sum_j \int_{\frac{1}{2^{j+1}} < |1-\zeta\overline{\zeta_0}| \leq \frac{1}{2^j}} \left(\int_{(\frac{1}{2^j})^{\frac{n}{m}}}^{k} \frac{h(t)}{t^{n-\alpha p}} \right)^{\frac{p}{p-q}} d\sigma(\zeta),$$

which in turn is equivalent to the convergence of

$$\int_0^k x^{n-1} \left(\int_{x^{\frac{n}{m}}}^{k} \frac{h(t)}{t^{n-\alpha p}} \right)^{\frac{p}{p-q}} dx.$$

The change of variables $x^{\frac{n}{m}} = y$ gives that the last integral equals to

$$\int_0^k y^{n-\alpha p-1} \left(\int_y^k \frac{h(t)}{t^{n-\alpha p}}\, dt \right)^{\frac{p}{p-q}} dy. \tag{3.11}$$

Next, let us compute $g(r)$ for these measures. A similar argument to the one we have just used gives that the convergence of

$$\int_{\mathbf{S}^n} \mu\Big(T\big(B(\zeta,r)\big)\Big)^{\frac{p}{p-q}} d\sigma(\zeta) \simeq \int_{|1-\zeta\overline{\zeta_0}|<kr} \left(\int_0^{kr-|1-\zeta\overline{\zeta_0}|} h(s)\,ds \right)^{\frac{p}{p-q}} d\sigma(\zeta)$$

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is equivalent to the convergence of

$$\int_0^{kr} l^{n-1}\Big(\int_0^{kr-l} h(s)\,ds\Big)^{\frac{p}{p-q}}\,dl.$$

Thus the condition on g is equivalent to the fact that the function

$$\tilde g(r)=\Big(\int_0^r l^{n-1}\Big(\int_0^{r-l} h(s)\,ds\Big)^{\frac{p}{p-q}}\,dl\Big)^{\frac{p-q}{p}},$$

satisfies that

$$\int_0^k \Big(\frac{\tilde g(r)}{r^{n-\alpha q}}\Big)^{\frac{1}{q}}\log\frac{1}{r}\frac{dr}{r}<+\infty. \tag{3.12}$$

Since we are considering $p=2$, and $q=1$, and for any smooth function F so that $F(0)=0$, $F(x)=\int_0^x F'(t)\,dt$, we can rewrite the integral in (3.11) as

$$\begin{aligned}\int_0^k y^{n-\alpha p-1}\Big(\int_y^k \frac{h(t)}{t^{n-\alpha p}}\,dt\Big)^2\,dy &\simeq \int_0^k y^{n-\alpha p-1}\int_y^k\int_\tau^k \frac{h(r)}{r^{n-\alpha p}}\,dr\frac{h(\tau)}{\tau^{n-\alpha p}}\,d\tau\,dy\\ &=\int_0^k\int_0^\tau y^{n-\alpha p-1}\,dy\int_\tau^k \frac{h(r)}{r^{n-\alpha p}}\,dr\frac{h(\tau)}{\tau^{n-\alpha p}}\,d\tau\\ &\simeq \int_0^k\int_\tau^k \frac{h(r)h(\tau)}{r^{n-\alpha p}}\,dr\,d\tau\\ &=\int_0^k\int_0^r h(\tau)\,d\tau\frac{h(r)}{r^{n-\alpha p}}\,dr,\end{aligned} \tag{3.13}$$

where we have used Fubini's theorem in the second and the forth equivalence.

A similar argument shows that the function $\tilde g$ can be rewritten in this case as

$$\tilde g(r)=\Big(\int_0^r\int_0^\tau (r-\tau)^n h(s)h(\tau)\,ds\,d\tau\Big)^{\frac12}. \tag{3.14}$$

Now we can construct our example. Take $h(r)=r^{\frac{m}{2}-1}(\log\frac{1}{r})^{-\lambda}$, $r>0$ small enough and $\lambda>0$ to be chosen later, where $m=n-\alpha p$. Since $\int_0^r h(\tau)\,d\tau\simeq r^{\frac{m}{2}}(\log\frac{1}{r})^{-\lambda}$, we have that (3.13) is equivalent to

$$\int_0^k\int_0^r h(\tau)\,d\tau\frac{h(r)}{r^{n-\alpha p}}\,dr\simeq\int_0^k\frac{dr}{r(\log\frac{1}{r})^{2\lambda}},$$

which is finite provided we take $\lambda>\frac12$.

A similar argument shows that

$$\tilde g(r)\succeq\Big(\int_0^r (r-\tau)^n\tau^{m-1}\frac{1}{(\log\frac{1}{\tau})^{2\lambda}}\,d\tau\Big)^{\frac12}\succeq\frac{r^{\frac{n+m}{2}}}{(\log\frac1r)^\lambda}=\frac{r^{n-\alpha}}{(\log\frac1r)^\lambda}.$$

Finally, for each $k<1$,

$$\int_0^k\frac{\tilde g(r)}{r^{n-\alpha}}\log\frac1r\frac{dr}{r}\succeq\int_0^k\frac{dr}{r(\log\frac1r)^{\lambda-1}}=+\infty,$$

provided $\lambda\le 2$. Thus if we choose $\frac12<\lambda\le 2$ we obtain the desired example.

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EXAMPLE 2. Assume $n = 2, q = 1, p = 2$, and $\alpha = \frac{1}{2}$. Then there exists a finite positive Borel measure μ on $\mathbf{B}^2$ so that μ satisfies the sufficient condition in Theorem 3.6:

$$\int_0^{k_0} \frac{g(r)}{r^{\frac{3}{2}}} \log \frac{1}{r} \frac{dr}{r} < +\infty,$$

for some $k_0 < 1$, but it does not satisfy the sufficient condition of Theorem 3.4, that is

$$\int_{\Omega_2(\zeta)} \frac{d\mu(z)}{(1-|z|)} \notin L^2(d\sigma).$$

We define the measure μ on $\mathbf{B}^2$ by

$$\mu(g) = \int_{\frac{1}{2}}^1 \int_0^{2\pi} f'(1-\tau) g(\tau e^{i\theta}, 0)\, d\theta\, d\tau,$$

where $g \in \mathcal{C}(\mathbf{B}^2)$, and $f(\tau) = (\log \frac{1}{\tau})^{1-p_1}$, with $p_1 > 3$.

We will first show that μ satisfies the condition on the moduli of continuity. We recall that we can substitute the admissible tents by non-isotropic Carleson "boxes" $V^k\big(B(\zeta, r)\big) = \{z \in \mathbf{B}^2 ; 1 - |z| \le r, |1 - z_0\bar{\zeta}| \le kr\}$, with $z_0 = \frac{z}{|z|}$.

Suppose that $(\tau e^{i\theta}, 0) \in V^k\big(B(\zeta, r)\big)$, $\zeta = (\zeta_1, \zeta_2) \in \mathbf{S}^2$. We have that $1 - \tau \le r$ and if we write $\zeta_1 = |\zeta_1| e^{i\theta_1}$, then $|\theta - \theta_1| \preceq r$ and $1 - |\zeta_1| \preceq r$. Consequently for any $\zeta \in \mathbf{S}^2$, $\zeta = (\zeta_1, \zeta_2)$ with $1 - |\zeta_1| \preceq r$,

$$\mu\Big(V^k\big(B(\zeta, r)\big)\Big) \preceq \int_0^r \int_{|\theta-\theta_1| \le cr} d\theta f'(x)\, dx \preceq r f(r) = r\Big(\log \frac{1}{r}\Big)^{1-p_1}.$$

Now

$$\begin{aligned}\int_{1-|\zeta_1| \le cr} \mu\Big(V^k\big(B(\zeta, r)\big)\Big)^2 d\sigma(\zeta) &\preceq \int_{1-|\zeta_1| \le cr} r^2 \Big(\log \frac{1}{r}\Big)^{2-2p_1} d\sigma(\zeta) \\ &\preceq r^2 (\log \frac{1}{r})^{2-2p_1} \int_0^{2\pi} \int_{1-s \le cr} s\, ds\, d\theta \simeq r^3 \Big(\log \frac{1}{r}\Big)^{2-2p_1},\end{aligned}$$

and

$$g(r) = \left(\int_{\mathbf{S}^2} \mu\Big(V^k\big(B(\zeta, r)\big)\Big)^2 d\sigma(\zeta)\right)^{\frac{1}{2}} \preceq r^{\frac{3}{2}} \Big(\log \frac{1}{r}\Big)^{1-p_1}.$$

Finally,

$$\int_0^{k_0} \frac{g(r)}{r^{\frac{3}{2}}} \log \frac{1}{r} \frac{dr}{r} \preceq \int_0^{k_0} \Big(\log \frac{1}{r}\Big)^{2-p_1} \frac{dr}{r},$$

which is convergent provided $p_1 > 3$.

So we are led to show that μ does not satisfy the geometric condition. If $\varepsilon > 0$ is small enough, $\zeta = (\zeta_1, \zeta_2) \in \mathbf{S}^2$, with $\zeta_1 = |\zeta_1| e^{i\theta_1}$, and $(1 - |\zeta_1| + |\theta - \theta_1|)^2 \le \varepsilon(1-t)$, we easily deduce that $(te^{i\theta}, 0) \in \Omega_2(\zeta)$, and then

$$\begin{aligned}\int_{\Omega_2(\zeta)} \frac{d\mu(z)}{1-|z|} &= \int_0^1 \int_0^{2\pi} \frac{\chi_{\Omega_2(\zeta)}(te^{i\theta}, 0)}{1-t} f'(1-t)\, d\theta\, dt \\ &\ge \int \int_{(1-|\zeta_1|) \le \varepsilon t^{\frac{1}{2}}, |\theta-\theta_1| \le \varepsilon t^{\frac{1}{2}}} \frac{f'(t)}{t}\, dt\, d\theta \simeq \int_{(1-|\zeta_1|) \le \varepsilon t^{\frac{1}{2}}} \frac{f'(t)}{t^{\frac{1}{2}}}\, dt \\ &\simeq \frac{(\log \frac{1}{1-|\zeta_1|})^{-p_1}}{1-|\zeta_1|}.\end{aligned}$$

Altogether they give that

$$\int_{\mathbf{S}^2}\Big(\int_{\Omega_2(\zeta)}\frac{d\mu(z)}{1-|z|}\Big)^2 d\sigma(\zeta) \succeq \int_{1-|\zeta_1|<\delta}\frac{(\log\frac{1}{1-|\zeta_1|})^{-2p_1}}{(1-|\zeta_1|)^2}\,d\sigma(\zeta)$$
$$\simeq \int_{1-\delta}^{1}\frac{(\log\frac{1}{1-s})^{-2p_1}}{(1-s)^2}s\,ds = +\infty.$$

Finally we will see that the geometric sufficient condition is, in a certain sense, sharp.

PROPOSITION 3.8. *Let $m = n-\alpha p > 0$, $1 \le q < p < +\infty$. Assume ψ be a non-decreasing positive function defined in $(0,k)$, $k << 1$, so that there exists $C > 1$ and $\psi(Cr) \preceq \psi(r)$, for r small enough, and*

(i) $\lim_{t\to 0+}\psi(t) = 0$.
(ii) $\frac{t^{n-\alpha p}}{\psi(t)} = o(1)$, *as* $t \longrightarrow 0+$.
(iii) $\int_0^k \frac{t^{n-\alpha p}}{\psi(t)}\frac{dt}{t} < +\infty$.

Then there exists a measure μ on $\mathbf{B}^n$ so that

$$\int_{\mathbf{S}^n}\Big(\int_{\Omega_{\frac{n}{m}}(\zeta)}\frac{d\mu(z)}{\psi(1-|z|)}\Big)^{\frac{p}{p-q}} d\sigma(\zeta) < +\infty,$$

but μ is not a q-Carleson measure for h^p_α.

PROOF OF PROPOSITION 3.8. Let $d\mu(r\zeta) = (\frac{q}{p}-1)\psi(r)^{\frac{q}{p}-1}\psi'(r)\,dr\,\delta_{\zeta_0}$, where $\zeta_0 \in \mathbf{S}^n$. An argument like the one used in Example 2.1 shows that the convergence of

$$\int_{\mathbf{S}^n}\Big(\int_{\Omega_{\frac{n}{m}}(\zeta)}\frac{d\mu(z)}{\psi(1-|z|)}\Big)^{\frac{p}{p-q}} d\sigma(\zeta)$$

is equivalent to the convergence of

$$\int_0^k t^{n-\alpha p-1}\Big(\int_t^k \psi(r)^{\frac{q}{p}-2}\psi'(r)\,dr\Big)^{\frac{p}{p-q}} dt \le \int_0^k t^{n-\alpha p-1}\big(k_1+\psi(t)^{\frac{q}{p}-1}\big)^{\frac{p}{p-q}} dt$$
$$\preceq 1+\int_0^k \frac{t^{n-\alpha p}}{\psi(t)}\frac{dt}{t},$$

which is finite by (iii). But μ does not satisfy the necessary condition

$$\mu\big(T(A)\big) \preceq C_{\alpha p}(A)^{\frac{q}{p}}.$$

Indeed, suppose it does satisfy this condition, and take $B(\zeta_0, r)$, $r > 0$. Since $\mu\Big(T\big(B(\zeta_0,r)\big)\Big) \succeq \psi(r)^{\frac{q}{p}}$, we deduce that $\psi(r) \preceq r^{n-\alpha p}$, which is contradictory with (ii). ∎

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