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REGULAR BISIMPLE RINGS

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We characterize regular bisimple rings in terms of some perspectivity conditions on their lattices of principal right ideals. We also show that, if S is the multiplicative subsemigroup generated by all the idempotents of a regular bisimple ring R, then

(i) if R does not have an identity, then $S=R$ and has depth 2;

(ii) if R does have an identity but is not a division ring, then $S=\{a\in R: a$ is neither left nor right invertible$\}\cup\{1\}$ and has depth 3.

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Introduction

A ring R is said to be *bisimple* if it has more than one element, and satisfies the following conditions:

(B1) for any $a\in R$ we have $a\in aR\cap Ra$, and

(B2) for any nonzero $a,b\in R$ there is some $c\in R$ such that $aR=cR$ and $Rc=Rb$.

These rings were introduced by Munn [**9**] in 1981, and are precisely those rings whose multiplicative semigroups are 0–bisimple [**9**, Appendix]. Bisimple rings are always simple and are (von Neumann) regular as long as they contain a nonzero idempotent [**9**, Lemmas 1.2 and 1.3].

Let R be a regular ring and $L(R)$ its lattice of principal right ideals. Thus $L(R)$ is a relatively complemented modular lattice. Munn showed that if R is bisimple, then any two intervals in $L(R)$ are lattice-isomorphic [**9**, Theorem 2.1], and he went on to ask whether every relatively complemented, modular lattice L with this property can be viewed as $L(R)$ for some regular bisimple ring R. In Section 1 of this paper we give an example (Example 1.1) of such a lattice L which is not isomorphic to $L(R)$ for any regular bisimple ring R, thus answering Munn's question in the negative. We then find two lattice-theoretic characterizations of regular bisimple rings (Theorem 1.3), and use these to identify the complemented modular lattices which can arise as $L(R)$ for some regular bisimple ring R with identity (Corollary 1.6).

Munn's paper essentially discusses the ring-theoretic consequences of imposing the semigroup condition, 0–bisimplicity, on the multiplicative semigroup of a ring. In Section 2 we take another point of view and consider some semigroup-theoretic consequences. Specifically, we study the multiplicative subsemigroup S generated by all the idempotents of a regular bisimple ring R. We show, in Proposition 2.2, that if R does not have an identity element, then every element of R is a product of two idempotents (so that $S=R$ and has depth 2). On the other hand, if R does have an identity but is not a division ring, then (Proposition 2.3) every element of R which is neither left nor right invertible is a product of three idempotents (so that S has depth exactly 3). These results may be compared with the situation where R is an arbitrary regular 0–bisimple semigroup: in that setting it is possible for S to have arbitrarily large depth.

Preliminaries

All rings in this paper are associative, but they need not have an identity element. If a is an element of a ring R, then we write $r(a)$ for its *right annihilator* $\{s\in R: as=0\}$. Similarly $l(a)$ denotes the left annihilator of a in R.

A ring R is (*von Neumann*) *regular* if for any $a\in R$ there is some $x\in R$ with $a=axa$. Notice that if $a=axa$ in a ring R then $e=ax$ and $f=xa$ are idempotents such that $aR=eR$ and $Ra=Rf$. Furthermore, $l(a)=R(1-e)$ and $r(a)=(1-f)R$, (if R does not have an identity, we interpret $R(1-e)$ as $\{s-se: s\in R\}$, and similarly for $(1-f)R$). Thus if a,b are elements of a regular ring R, we have $Ra=Rb$ if and only if $r(a)=r(b)$. In particular, if R is a regular ring with an identity element and $a\in R$, then a is left invertible if and only if $r(a)=0$. Similarly, a is right invertible if and only if $l(a)=0$.

A regular ring R is *abelian* if all its idempotents are central in R. For basic results about regular rings and for any unexplained notation we refer the reader to Goodearl's book [**1**].

We say that a modular lattice L is *relatively complemented* if, for any $A,B\in L$ with $A\leqq B$, there is some $C\in L$ with $A+C=B$ and $A\cap C=0$ (that is, C is a complement of A in B). If L has a greatest element, then L is relatively complemented if and only if it is complemented. If L is a relatively complemented modular lattice, we say that $A,B\in L$ are *perspective* in L if A and B have a common complement in $A+B$; that is, if there is some $C\in L$ such that

$$A+C=B+C=A+B$$

and

$$A\cap C=0=B\cap C.$$

If L has a greatest element (and so is complemented), this definition agrees with that in [**11**] because of [**11**, Theorem 3.1, p. 17]. If L is in fact the lattice $L(R)$ of principal right ideals of a regular ring R, then any pair of perspective elements A,B in $L(R)$ are isomorphic as right R-modules, since if C is as above, then

$$A\cong\frac{A+C}{C}=\frac{B+C}{C}\cong B.$$

For other basic facts about lattices we refer the reader to [**11**]. Notice however that, because we shall always have in mind the lattices $L(R)$, we shall use $+$ and $\cap$ to indicate the lattice operations. Furthermore, if A and B are *independent* elements of a lattice (that is, if $A \cap B = 0$), we shall write their supremum in the lattice as $A \oplus B$ rather than simply $A+B$, (this agrees with the usage for $L(R)$ in [**1**]).

Finally for any unexplained terminology or basic results about semigroups, see Howie's book [**5**].

1. Lattice characterizations of regular bisimple rings

We begin this section with the example mentioned in the introduction.

Example 1.1. There is a complemented modular lattice L in which any two intervals are lattice-isomorphic, but for which there is no regular bisimple ring R such that $L \cong L(R)$.

Proof. We construct L as the lattice of principal right ideals of a suitable regular ring T. Choose any field F, and for each positive integer n, set $F_n = F$. Consider the direct product $S = \prod_n F_n$. The direct sum $I = \oplus_n F_n$ is an ideal of the ring S, and so the ring $T = S/I$ is a commutative regular ring with an identity element. Let L be the lattice $L(T)$ so that L is complemented and modular.

To see that any two intervals in L are lattice-isomorphic, it is enough (as in [**9**, Theorem 2.1]) to show that all intervals of the form $[0, A]$ are isomorphic, where $0 \neq A \in L$. So let A be a nonzero principal ideal of T. Then A is of the form $(S'+I)/I$, where $S' = \prod_J F_n$ and J is an infinite subset of N. Thus $A \cong S'/(S' \cap I)$ and so A and T are isomorphic as rings. Since A is a ring-direct-summand of T, it follows that the intervals $[0, A]$ and $[0, T]$ in L are lattice-isomorphic. Hence any two intervals in L are lattice-isomorphic.

Now suppose that R is a regular bisimple ring such that $L(R) \cong L$. Since T is a commutative (and so abelian) regular ring, [**1**, Theorem 3.4] shows that $L = L(T)$ is a distributive lattice. Conversely, since $L(R) \cong L$ is distributive, [**1**, Theorem 3.4] shows that the ring R is abelian. Inasmuch as R is also a simple ring (by [**9**, Lemma 1.2]), we see that R must be a division ring (by [**1**, Theorem 3.2]). But then $L(R) = \{0, R\}$ which is a contradiction, since $L(R) \cong L$ and $|L| > 2$. □

The construction of the above example shows that regular bisimple rings R are not characterized by the property that any two intervals in $L(R)$ are lattice-isomorphic. The following lemma pinpoints the property of regular bisimple rings which was missing in the ring T in the example.

Lemma 1.2. *Let R be a regular ring. Then R is bisimple if and only if any two nonzero principal right ideals of R are isomorphic as right R-modules.*

Proof. Munn showed that bisimple rings have this property in [**9**, Lemma 1.2]. For the converse, suppose that any two nonzero principal right ideals of R are isomorphic. Let a, b be nonzero elements of R. By hypothesis, there is an isomorphism $\theta: bR \to aR$. Then $c = \theta(b)$ satisfies $aR = cR$ and $r(c) = r(b)$. Since R is regular, it follows that $Rc = Rb$, and so R is bisimple. □

This characterization shows that regular bisimple rings are closely related to the strongly prime rings used by Goodearl and Handelman in [**2**]. A ring R with identity is said to be (*right*) *strongly prime with bound 1* (or SP(1) for short) if for each nonzero $a \in R$ there is some $x \in R$ with $r(ax) = 0$. Goodearl and Handelman show that R is an SP(1) ring if and only if $R_R \lesssim aR$ for each nonzero $a \in R$. Thus by Lemma 1.2, a regular bisimple ring is always an SP(1) ring. It is not known whether a regular SP(1) ring R need be bisimple (see [**1**, Open Problem 51]), but [**2**, Theorem 2.1] shows that R is bisimple in the special case where R is also right self-injective.

Viewed in terms of Lemma 1.2, Example 1.1 shows that a lattice isomorphism between the intervals $[0, A]$ and $[0, B]$ (in $L(R)$) need not force an R-module isomorphism between the principal right ideals A and B. This problem can be overcome if, instead, we insist that A and B be *perspective* in $L(R)$. However, we cannot expect all nonzero pairs in $L(R)$ to be perspective since, if $A \subset B$, then clearly A and B cannot have a common complement. Condition (b) in the following theorem shows that, if we just insist that all *independent* nonzero pairs A and B be perspective, then we capture all regular bisimple rings, but that rings of 2×2 matrices over division rings also get caught in the net. On the other hand, condition (c) gives a perspectivity condition that can be imposed on all nonzero pairs in $L(R)$, and no extra rings are caught this time.

Theorem 1.3. *Let R be a regular ring and $L(R)$ its lattice of principal right ideals. The following conditions are equivalent:*

(a) *R is bisimple;*

(b) (i) *if $L(R)$ has a largest element, then this element is not of the form $A_1 \oplus A_2$ where A_1, A_2 are distinct atoms in $L(R)$, and*

(ii) *any two nonzero independent elements of $L(R)$ are perspective in $L(R)$;*

(c) *for any nonzero $A, B \in L(R)$ there are decompositions $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$ such that, for $i = 1, 2$, the elements A_i and B_i are perspective in $L(R)$.*

Proof. (a)⇒(b). Assume that R is bisimple. Suppose first that $L(R)$ has a largest element which has the form $A_1 \oplus A_2$ where A_1, A_2 are distinct atoms in $L(R)$. Then $R = A_1 \oplus A_2$ and so R is Artinian. Since R is bisimple, it follows from [**9**, Theorem 1.4] that R must be a division ring. Thus R is itself an atom in $L(R)$. This contradiction shows that (b)(i) holds.

Now let A, B be nonzero principal right ideals of R such that $A \cap B = 0$. Since R is bisimple, there are elements $a, b \in R$ such that $A = aR$ and $B = bR$ while $Ra = Rb$. We shall show that $C = (a+b)R$ is a common complement for A and B in $A + B$. Clearly $A + C = B + C = A + B$. To see that $A \cap C = 0$ suppose $x, y \in R$ with $ax = (a+b)y$. Then

$a(x-y)=by\in A\cap B=0$ and so $by=0$. As $Ra=Rb$, we get $ay=0$ too. Hence $A\cap C=0$. Similarly $B\cap C=0$, and so A and B are perspective in $L(R)$.

(b)⇒(c). Assume that (b) holds. Let A,B be nonzero principal right ideals of R, and consider first the case where $A\subset B$.

We begin by showing that A cannot be an atom in $L(R)$. Suppose, on the contrary, that A is an atom and let C be its complement in B. By hypothesis, A and C are perspective (since $A\cap C=0$), and so C is also an atom. Thus B is the join of two distinct atoms, and so cannot be the largest element of $L(R)$. Since $L(R)$ is relatively complemented and modular, it follows that there is some nonzero $D\in L(R)$ with $B\cap D=0$. But now D is perspective to both A and B. Since A is an atom and B is the join of two distinct atoms, we have the desired contradiction.

Hence there is a decomposition $A=A_1\oplus A_2$ where A_1 and A_2 are nonzero elements of $L(R)$. Let B_1 be a complement of A_1 in B and let $B_2=A_1$ so that $B=B_1\oplus B_2$. Since $A_1\cap B_1=0$ and $A_2\cap B_2=0$, both pairs of elements A_i,B_i $(i=1,2)$ are perspective in $L(R)$ as required.

A similar decomposition works if $B\subset A$, so suppose finally that $A\not\subset B$ and $B\not\subset A$. Then there are decompositions

$$A=(A\cap B)\oplus A_1$$

$$B=(A\cap B)\oplus B_1$$

where A_1,B_1 are nonzero elements of $L(R)$. Notice that $A\cap B\cap(A_1+B_1)=0$. Since $A_1\cap B_1=0$ our hypothesis implies that A_1 and B_1 are perspective in $L(R)$. We claim that A and B are also perspective in $L(R)$. Indeed, let C be a common complement of A_1 and B_1 in A_1+B_1. Then clearly $A+C=B+C=A+B$. Also

$$\begin{aligned}A\cap C&=A\cap C\cap(A_1+B_1)\\&=C\cap[A_1+(A\cap B)]\cap(A_1+B_1)\\&=C\cap\{A_1+[(A\cap B)\cap(A_1+B_1)]\}\\&=C\cap A_1\\&=0\end{aligned}$$

and similarly $B\cap C=0$. Thus A and B are indeed perspective, and so in this case we can use the decompositions $A=A\oplus 0$ and $B=B\oplus 0$ to complete the proof that (c) holds.

(c)⇒(a). This follows from Lemma 1.2 since perspective modules are isomorphic. □

Remark 1.4. We cannot omit condition (i) in (b) of the above theorem. Indeed if R is the ring of 2×2 matrices over a division ring, then $L(R)$ satisfies (b)(ii), but R is not a

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bisimple ring (because of Lemma 1.2). It is easy to see that this is the only extra class of regular rings which satisfy (b)(ii). □

Remark 1.5. Let R be a regular bisimple ring. Then the proof of (b)⇒(c) in Theorem 1.3 shows that a sort of trichotomy law holds in $L(R)$: for any $A, B \in L(R)$ we have either $A \subset B$ or $B \subset A$ or A and B are perspective in $L(R)$. □

Conditions (b) and (c) are essentially lattice-theoretic conditions on the lattice $L(R)$, and indeed the proof that (b)⇒(c) would carry over for any relatively complemented modular lattice L. Conversely, we could use [**11**, Theorem 3.6, p. 21] to show that (c)⇒(b) for such an L. If L has a greatest element, then the following result shows how von Neumann's co-ordinatization theorem can be used to find a regular bisimple ring R such that $L(R) \cong L$. If you like, this gives a partial affirmative answer to the obvious modification of Munn's question [**9**, p. 185].

Corollary 1.6. *Let L be a complemented modular lattice such that*

(i) *the largest element of L is not the join of two distinct atoms, and*

(ii) *any two nonzero independent elements of L are perspective in L.*

Then there is a regular bisimple ring R (with identity) such that $L(R) \cong L$. *Conversely, if R is a regular bisimple ring with identity, then L(R) is a complemented modular lattice satisfying (i) and (ii).*

Proof. If L has just two elements $0, 1$ then any division ring R will give $L(R) \cong L$. So suppose L has more than two elements. To use von Neumann's co-ordinatization theorem [**11**, Theorem 14.1, p. 208] we need to show that L has order (at least) 4. That is, we need to find 4 independent, pairwise perspective elements A_1, A_2, A_3, A_4 in L such that $A_1 \oplus A_2 \oplus A_3 \oplus A_4 = 1$, the largest element of L.

Let $A \in L$ with $A \neq 0, 1$ and let A' be a complement of A. As in the proof of Theorem 1.3(b)⇒(c), we see that $A \subset 1$ implies that A is not an atom in L. Similarly A' cannot be an atom. Hence there are decompositions $A = A_1 \oplus A_2$ and $A' = A_3 \oplus A_4$ where all the A_i are nonzero. Since the A_i are independent, condition (ii) ensures that L has order 4. By the co-ordinatization theorem there is a (unique) regular ring R such that $L(R) \cong L$. By Theorem 1.3 this ring must be bisimple. The converse result is contained in Theorem 1.3. □

I do not know whether this result is also true for relatively complemented modular lattices.

2. Products of idempotents

Let R be a regular bisimple ring. In this section we consider the multiplicative subsemigroup S of R generated by all the idempotents of R. We obtain a simple characterization of the elements of S, and calculate the depth of S. This continues a theme studied in [**10**], [**3**] and [**4**], where corresponding results were obtained in the cases where R is a unit-regular ring, or where R is regular and right self-injective. For regular bisimple rings R, we get an interesting dichotomy depending on whether or not R has an identity element.

For the case where R does not have an identity element, we need the following well-known lemma.

Lemma 2.1. *Let R be a regular ring. For any $a \in R$ there is an idempotent $g \in R$ such that $a \in gRg$.*

Proof. There is some $x \in R$ such that $a = axa$. Then $e = ax$ is idempotent and $ea = a$. Again there is some $y \in R$ such that $a - ae = (a - ae)\,y(a - ae)$, and $f = (y - ey)(a - ae)$ is an idempotent. As e and f are orthogonal, $g = e + f$ is idempotent and it is easy to see that $a \in gRg$. □

Proposition 2.2. *Let R be a regular bisimple ring which does not have an identity element. Then every element of R is a product of two idempotents, and so R is a semiband of depth* 2.

Proof. Let $a \in R$. By Lemma 2.1, there is an idempotent $g \in R$ such that $a \in gRg$. Since g is not an identity element for R, there is a nonzero idempotent $h \in R$ such that $gh = hg = 0$ (for example, if $z \in R$ with $z \neq gz$ then there is some $x \in R$ with $(z - gz)x(z - gz) = z - gz \neq 0$ and so $h = (z - gz)(x - xg)$ is such an idempotent). By Lemma 1.2 we have $hR \cong gR$ and so (by [7, Proposition 4, p. 51]) there are elements $u \in gRh$ and $v \in hRg$ such that $g = uv$. Then

$$a = [g + u][g + v(a - g)]$$

is a product of two idempotents. (See [**4**, Example 2.15] for a matrix picture of this factorization.)

If every element of R were idempotent, then R would be commutative and so, by [**9**, Lemma 1.2(i)], would be a field. As this is impossible, R must have depth exactly 2. □

There is no direct parallel for this result in [**10**], [**3**] or [**4**] because the rings considered there all have identity elements. Munn gives some examples of regular bisimple rings without an identity in [**9**, 1.1].

On the other hand, if R is a regular bisimple ring with an identity element, Proposition 2.3 below gives a characterization and depth for the semigroup S which is similar to those found in [**3**, Theorem 2.8] (or [**10**, Corollary 11]) and [**4**, Remark 1.7] for the case where R is a regular right self-injective ring of type III. There is some overlap in these results: a simple, right self-injective ring of type III is always bisimple (this follows from Lemma 1.2 because of [**2**, Theorem 2.1] and [**1**, Corollary 10.17]). But there are many regular bisimple rings with identity which are not right self-injective.

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For example, most of the countable bisimple rings constructed by Munn in [9, Theorem 3.4] are not right self-injective, since a countable right self-injective ring is Artinian [8], and so a bisimple one must be a division ring by [9, Theorem 1.4].

Proposition 2.3. *Let R be a regular bisimple ring with identity and suppose that R is not a division ring. Let S be the subsemigroup of R generated by all the idempotents of R. Then*

$$S=\{a\in R: a \text{ is neither left nor right invertible}\}\cup\{1\}$$

and S has depth 3.

Proof. Clearly a product of proper ($\neq 1$) idempotents cannot be left or right invertible. Suppose conversely that a is a nonzero element of R which is neither left nor right invertible. As R is regular, we have $l(a)\neq 0$ and $r(a)\neq 0$. Also there are idempotents $e,f\in R$ such that $aR=eR$ and $Ra=Rf$. Thus $a\in eRf$ while $l(a)=R(1-e)$ and $r(a)=(1-f)R$. We shall use [3, Lemma 2.6] to show that a is a product of three idempotents in R. Hence we just have to find some $g\in R$ such that $eR\cap gR=0$ and $fR\cap gR=0$, yet $fR\lesssim gR$.

By Remark 1.5 we have either $eR\subset fR$, or $fR\subset eR$, or else eR and fR are perspective. Suppose firstly that $eR\subset fR$, and consider $g=1-f$. Since $r(a)\neq 0$, Lemma 1.2 shows that $fR\cong gR$. Also $eR\cap gR\subseteq fR\cap gR=0$ and so we are finished in this case. Similarly, if $fR\subset eR$ then $g=1-e$ will do the trick. Finally suppose that eR and fR are perspective, and let gR be their common complement in R. Then gR is nonzero and so Lemma 2.1 again gives an isomorphism $fR\cong gR$. Thus in all three cases [3, Lemma 2.6] shows that a is a product of three idempotents. Hence $a\in S$ and S has depth at most 3.

Now since R is not a division ring, there is a decomposition $R=A_1\oplus A_2$ with A_1,A_2 both nonzero principal right ideals. By Lemma 2.1 we have $A_1\cong A_2\cong R$, and so $R\cong R\oplus R$ as right R-modules. Hence [3, Example 2.2] shows that R contains an element which is a product of three idempotents but no fewer. Thus S has depth exactly 3. □

Remark 2.4. In terms of the depth of the semigroup generated by all the idempotents of R, Propositions 2.2 and 2.3 both represent much "better" behaviour than is observed in arbitrary regular 0–bisimple semigroups. For example, let X be the finite set $\{1,2,\ldots,n\}$ and consider the semigroup $\mathscr{T}(X)$ of all transformations on X. Let

$$J=\{\alpha\in\mathscr{T}(X): \text{rank } \alpha\leqq n-1\}$$

and

$$I=\{\alpha\in\mathscr{T}(X): \text{rank } \alpha< n-1\}$$

which are both ideals of $\mathscr{T}(X)$. Then the Rees quotient semigroup $R=J/I$ is regular and 0–bisimple. By Howie's results in [6] it follows that R is itself idempotent-generated and has depth $[\frac{3}{2}(n-1)]$.

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