COMPOSITIO MATHEMATICA

The J-invariant over splitting fields of Tits algebras

Maksim Zhykhovich

Compositio Math. **160** (2024), 2100–2114.

doi:10.1112/S0010437X24007255

FOUNDATION COMPOSITIO MATHEMATICA

LONDON MATHEMATICAL SOCIETY
EST. 1865

https://doi.org/10.1112/S0010437X24007255 Published online by Cambridge University Press

Compositio Math. 160 (2024) 2100–2114
doi:10.1112/S0010437X24007255

The J-invariant over splitting fields of Tits algebras

Maksim Zhykhovich

Abstract

We describe the J-invariant of a semisimple algebraic group G over a generic splitting field of a Tits algebra of G in terms of the J-invariant over the base field. As a consequence we prove a 10-year-old conjecture of Quéguiner-Mathieu, Semenov, and Zainoulline on the J-invariant of groups of type D_n. In the case of type D_n we also provide explicit formulas for the first component and in some cases for the second component of the J-invariant.

1. Introduction

Chow motives were introduced by Grothendieck, and since then they have become a fundamental tool for investigating the structure of algebraic varieties. The study of Chow motives and motivic decompositions has several outstanding applications to other topics. For example, Voevodsky's proof of the Milnor conjecture relied on Rost's computation of the motivic decomposition of a Pfister quadric. In [Kar10] Karpenko established the relation between the motivic decomposition and the canonical dimension of a projective homogeneous variety, which allowed the canonical dimension to be computed in many cases.

In [PSZ08] Petrov, Semenov, and Zainoulline investigated the structure of the motives of generically split projective homogeneous varieties and introduced a new invariant of an algebraic group G, called the *J-invariant*. In the case of quadratic forms the J-invariant was introduced previously by Vishik in [Vis05]. For a fixed prime number p the J-invariant of G modulo p is a discrete invariant consisting of several non-negative integer components $(j_1,\dots,j_r)$ with degrees $1\le d_1\le\cdots\le d_r$. The integers r and $d_1,\dots d_r$ depend only on the type of G and are known for all types (see the table in [PSZ08, § 4.13]). The J-invariant encodes the motivic decomposition of the variety X of Borel subgroups in G. More precisely, it turns out that the Chow motive of X with coefficients in $\mathbb{F}_p$ decomposes into a direct sum of Tate twists of an indecomposable motive $\mathcal{R}_p(G)$ and the Poincaré polynomial of $\mathcal{R}_p(G)$ over a splitting field of G equals

$$\prod_{i=1}^{r}\frac{t^{d_ip^{j_i}}-1}{t^{d_i}-1}\in\mathbb{Z}[t]. \tag{1.1}$$

The J-invariant proved to be an important tool for solving several long-standing problems. For example, it plays an important role in the progress on the Kaplansky problem about possible values of the u-invariant, see [Vis09]. Another example is the proof of a conjecture of Serre about

Received 3 March 2023, accepted in final form 30 January 2024, published online 11 September 2024.
2020 Mathematics Subject Classification 20G15, 14C15 (primary).
Keywords: linear algebraic groups, Tits algebras, algebras with involution, twisted flag varieties, Chow motives.

groups of type E_8 and its finite subgroups, where the J-invariant plays a crucial role (see [GS10, Sem16]). More recently, Petrov and Semenov generalized the J-invariant for groups of inner type to arbitrary oriented cohomology theories in the sense of Levine and Morel [LM07] satisfying some axioms (see [PS21]).

Let (A, σ) be a central simple algebra of even degree $2n$, endowed with an involution of orthogonal type and trivial discriminant. Let $G = \mathrm{PGO}^+(A, \sigma)$ be the connected component of the automorphism group of (A, σ). The group G is adjoint of type D_n. Denote by $J(G) = (j_1, \dots, j_r)$ the J-invariant of G modulo $p = 2$. It is known that the first component j_1 is zero if the algebra A is split. In particular, j_1 becomes zero over the function field F_A of the Severi–Brauer variety of A, which is a generic splitting field of A.

In [QSZ12] Quéguiner-Mathieu, Semenov, and Zainoulline stated a conjecture that the remaining components do not change after generic splitting of A.

CONJECTURE 1.2 [QSZ12, Remark 7.3]. *If $J(G) = (j_1, \dots, j_r)$, then $j_i = (j_i)_{F_A}$ for $i = 2, \dots, r$.*

Note that, in the settings of Conjecture 1.2 the central simple algebra A is a Tits algebra of the algebraic group $\mathrm{PGO}^+(A, \sigma)$. In the present paper we prove Conjecture 1.2 and, moreover, generalize it to the case of an arbitrary semisimple algebraic group G of inner type. Let A be a Tits algebra of G. The main result of the paper (Theorem 4.1) describes the connection between the J-invariant of G over a generic splitting field of A and the J-invariant over a base field. In particular, we prove that all components of the J-invariant of G of degree greater than 1 do not change after extending to a generic splitting field F_A of A. Moreover, the main theorem provides some control on how the components of degree 1 can change over the field F_A. In the case $G = \mathrm{PGO}^+(A, \sigma)$ we improve this control in Proposition 5.1, which together with the main theorem allows us to prove Conjecture 1.2 (see Corollary 5.3).

The main result of the paper allows us to split a Tits algebra of an algebraic group without losing much information on the J-invariant of the group (only components of degree one may be affected). This may be a useful tool to compute the J-invariant, since the algebraic groups with trivial Tits algebras are considered as 'less complex objects' compared with those groups with non-trivial Tits algebras. For example, in the settings of Conjecture 1.2 the group $\mathrm{PGO}^+(A, \sigma)$ over the field F_A becomes isomorphic to $\mathrm{PGO}^+(q_\sigma)$, where q_σ is the respective quadratic form adjoint to the split algebra with involution $(A, \sigma)_{F_A}$. Hence, Conjecture 1.2 allows the computation of the J-invariant (except the first component) of algebras with orthogonal involution to be reduced to the case of quadratic forms. Note that recently a similar approach was used to investigate the motivic equivalence of algebras with involutions, see [DQZ22].

Section 5 of the paper is devoted to the computation of the first two components j_1 and j_2 of the J-invariant of the group $\mathrm{PGO}^+(A, \sigma)$ of type D_n, where (A, σ) is a central simple algebra with orthogonal involution. This question was already investigated in [QSZ12]. Namely, in [QSZ12, Corollary 5.2] the upper bounds for j_1 and j_2 were provided in terms of 2-adic valuations i_A, i_+, and i_- of indices of algebras A, C_+, and C_-, respectively, where C_+ and C_- are the components of the Clifford algebra of (A, σ). We improve the upper bound for j_1 and then show that it is, in fact, the exact value of j_1 (see Theorem 5.9). More precisely, we obtain the following formula

$$j_1 = \min\{k_1, i_A, \max\{i_+, i_-\}\}, \tag{1.3}$$

where k_1 denotes the 2-adic valuation of n.

Note that the formula for j_1 and Conjecture 1.2 (Corollary 5.3) allow us to *completely* reduce the computation of the J-invariant of the group $\mathrm{PGO}^+(A, \sigma)$ to the case of quadratic forms.

https://doi.org/10.1112/S0010437X24007255 Published online by Cambridge University Press

Moreover, in Proposition 5.14 we also provide an explicit formula for the second component j_2 in some cases.

Note that recently Henke in his PhD thesis [Hen22] applied the main result of this paper to investigate motivic decompositions of projective homogeneous varieties for groups of type E_7.

The proofs in this paper rely on the computations of rational cycles, the properties of generically split varieties, the theory of upper motives and the index reduction formula.

2. Preliminaries and notation

2.1 Chow motives

Let F be a field. In the present paper we work in the category of the Grothendieck–Chow motives over F with coefficients in $\mathbb{F}_p$ for a fixed prime number p (see [EKM08]).

For a smooth projective variety X over F we denote by $M(X)$ the motive of X in this category. We consider the Chow ring $\mathrm{CH}(X)$ of X modulo rational equivalence and we write $\mathrm{Ch}(X)$ for the Chow ring with coefficients in $\mathbb{F}_p$.

For a motive M over F and a field extension E/F we denote by M_E the extension of scalars. A motive M is called *split* (respectively, *geometrically split*), if it is isomorphic to a finite direct sum of Tate motives (respectively, if M_E is split over some field extension E/F).

Let G be a semisimple algebraic group of inner type. Let X be a projective homogeneous variety under the action of G. Note that the motive of X splits over any field extension, over which the group G splits (in the sense of algebraic groups). By $\overline{X}$ we denote the variety X_E over a splitting field E of the group G. The Chow ring $\mathrm{CH}(\overline{X})$ does not depend on the choice of E and, therefore, we do not specify the splitting fields in the formulas below. By $\overline{\mathrm{CH}}(X)$ we denote the image of the restriction homomorphism $\mathrm{CH}(X) \to \mathrm{CH}(\overline{X})$. We say that a cycle from $\mathrm{CH}(\overline{X})$ is F-rational if it belongs to $\overline{\mathrm{CH}}(X)$.

The *Poincaré polynomial* $P(X,t)$ of X is defined as $\sum_{i\geq 0} \dim \mathrm{Ch}^i(\overline{X})t^i$. Similarly, for direct motivic summand M of X we define the Poincaré polynomial $P(M,t)$ by replacing $\overline{X}$ by $\overline{M}$ in the formula, where $\overline{M}$ denotes the motive M over a splitting field of G.

Recall that the *Krull–Schmidt principle* holds for any motivic direct summand M of a projective homogeneous variety X. Namely, M decomposes in a unique way in a finite direct sum of indecomposable motives, see [CM06]. The *upper motive* $U(X)$ of X is defined as an indecomposable summand of $M(X)$ with the property that the Chow group $\mathrm{Ch}^0(U(X))$ is non-zero. It follows by the Krull–Schmidt principle that the isomorphism class of $U(X)$ is uniquely determined by X.

Given two projective homogeneous varieties X_1 and X_2 (under possibly different algebraic groups) over F, the upper motives of the varieties X_1 and X_2 satisfy the following isomorphism criterion.

Proposition 2.1 [Kar13, Corollary 2.15]. *The upper motives $U(X_1)$ and $U(X_2)$ are isomorphic if and only of the varieties X_1 and X_2 possesses 0-cycles of degree 1 modulo p over $F(X_2)$ and $F(X_1)$, respectively.*

2.2 Tits algebras and the Picard group

Let G_0 be a split semisimple algebraic group of inner type of rank n over F. We fix a split maximum torus T in G_0 and a Borel subgroup B of G_0 containing T. Let $\Pi = \{\alpha_1, \dots, \alpha_n\}$ be a set of simple roots with respect to B and let $\{\omega_1, \dots, \omega_n\}$ be the respective set of fundamental weights. Enumeration of roots and weight follows Bourbaki.

https://doi.org/10.1112/S0010437X24007255 Published online by Cambridge University Press

Denote by Λ_ω the respective weight lattice and by Λ_ω^+ the cone of dominant weights. There is a natural one-to-one correspondence between the isomorphism classes of the irreducible finite-dimensional representations of G_0 and the elements of Λ_ω^+. This correspondence associates with an irreducible representation of G_0 its highest weight.

Now let G be an arbitrary (not necessarily split) semisimple algebraic group over F of inner type which is a twisted form of G_0. With each element $\omega \in \Lambda_\omega^+$ one can associate a unique central simple algebra A_ω such that there exists a group homomorphism $\rho : G \to \mathrm{GL}_1(A)$ having the property that the representation $\rho \otimes F_{\mathrm{sep}}$ of the split group $G \otimes F_{\mathrm{sep}}$ is the representation with the highest weight ω. The algebra A_ω is called the *Tits algebra* of G corresponding to ω. In particular, to any fundamental weight ω_i corresponds a Tits algebra A_{ω_i}.

Any projective homogeneous G-variety X is the variety of parabolic subgroups in G of some fixed type, where the type corresponds to a subset Θ of the set of simple roots Π. The Picard group $\mathrm{Pic}(\overline{X})$ can be identified with a free $\mathbb{Z}$-module generated by ω_i, $i \in \Pi \backslash \Theta$. Consider the group homomorphism $\alpha_X : \mathrm{Pic}(\overline{X}) \to \mathrm{Br}(F)$ sending ω_i to the Brauer class of the Tits algebra A_{ω_i} corresponding to the fundamental representation with the highest weight ω_i.

By [MT95, § 2] the following sequence of groups is exact

$$0 \to \mathrm{Pic}(X) \xrightarrow{\mathrm{res}} \mathrm{Pic}(\overline{X}) \xrightarrow{\alpha_X} \mathrm{Br}(F), \tag{2.2}$$

where res is the scalar extension to a splitting field of G.

This sequence allows us to express the group $\mathrm{Pic}(X)$ in terms of the Tits algebras of G.

3. J-invariant

The J-invariant of a semisimple algebraic group was introduced in [PSZ08] by Petrov, Semenov, and Zainoulline. In this section we briefly recall the definition and the main properties of the J-invariant following [PSZ08].

Let G_0 be a split semisimple algebraic group over a field F and B a Borel subgroup of G_0. An explicit presentation of $\mathrm{Ch}^*(G_0)$ in terms of generators and relations is known for all groups and all primes p. Namely, by [Kac85, Theorem 3]

$$\mathrm{Ch}^*(G_0) \simeq \mathbb{F}_p[e_1, \dots, e_r]/(e_1^{p^{k_1}}, \dots, e_r^{p^{k_r}}) \tag{3.1}$$

for some non-negative integers r, k_i and some homogeneous generators $e_1, \dots, e_r$ with degrees $1 \le d_1 \le \cdots \le d_r$ coprime to p. A complete list of numbers r, k_i, and d_i is provided in [PSZ08, p. 21] for any split group G_0.

We introduce an order on the set of additive generators of $\mathrm{Ch}^*(G_0)$, i.e. on the monomials $e_1^{m_1} \dots e_r^{m_r}$. To simplify the notation, we denote the monomial $e_1^{m_1} \dots e_r^{m_r}$ by e^M, where M is an r-tuple of integers $(m_1, \dots, m_r)$. The codimension (in the Chow ring) of e^M is denoted by $|M|$. Note that $|M| = \sum_{i=1}^r d_i m_i$.

Given two r-tuples $M = (m_1, \dots, m_r)$ and $N = (n_1, \dots, n_r)$ we say $e^M \le e^N$ (or, equivalently, $M \le N$) if either $|M| < |N|$ or $|M| = |N|$ and $m_i \le n_i$ for the greatest i such that $m_i \ne n_i$. This gives a well-ordering on the set of all monomials (r-tuples).

Now let $G = {}_\xi G_0$ be an inner twisted form of G_0 given by a cocycle $\xi \in Z^1(F, G_0)$ and let $X = {}_\xi(G/B)$ be the variety of Borel subgroups in G. Since X and G_0/B are isomorphic over any splitting field of G, we identify the Chow groups $\mathrm{Ch}(\overline{X})$ and $\mathrm{Ch}(G_0/B)$.

We consider the following composite map:

$$\mathrm{Ch}^*(X) \xrightarrow{\mathrm{res}} \mathrm{Ch}^*(G_0/B) \xrightarrow{\pi} \mathrm{Ch}^*(G_0), \tag{3.2}$$

https://doi.org/10.1112/S0010437X24007255 Published online by Cambridge University Press

where π is the surjective pullback of the canonical projection $G_0 \to G_0/B$ and res is the scalar extension to a splitting field of G.

Definition 3.3 [PSZ08, Definition 4.6]. For each i, $1 \le i \le r$, set j_i to be the smallest non-negative integer such that the image of the composite map $\pi \circ \mathrm{res}$ contains an element a with the greatest monomial $e_i^{p^{j_i}}$ with respect to the order on $\mathrm{Ch}^*(G_0)$ as above, i.e., of the form

$$a = e_i^{p^{j_i}} + \sum_{e^M < e_i^{p^{j_i}}} c_M e^M, \quad c_M \in \mathbb{F}_p.$$

The r-tuple of integers $(j_1, \ldots, j_r)$ is called the *J-invariant of G modulo p* and is denoted by $J_p(G)$.

Remark 3.4. Note that the J-invariant of G up to a permutation of some components may depend on the choice of the cocycle ξ (see [QSZ12, § 3]). By considering the J-invariant of a group G in the following we also fix a cocycle ξ.

Remark 3.5. According to [GZ12, Proposition 5.1] and [KM06, Theorem 6.4] the sequence (3.2) of graded rings is exact in the middle term if and only if the cocycle ξ is generic.

In [PSZ08] the following motivic interpretation of the J-invariant was provided.

Proposition 3.6 [PSZ08, Theorem 5.13]. *Let X be the variety of Borel subgroups in G. Then the Chow motive of X with coefficients in $\mathbb{F}_p$ decomposes into a direct sum*

$$M(X) \simeq \bigoplus_{i \in I} \mathcal{R}_p(G)\{i\} \tag{3.7}$$

of twisted copies of an indecomposable motive $\mathcal{R}_p(G)$ for some finite multiset I of non-negative integers. Moreover, the Poincaré polynomial of $\mathcal{R}_p(G)$ is given by

$$P(\mathcal{R}_p(G), t) = \prod_{i=1}^{r} \frac{1 - t^{d_i p^{j_i}}}{1 - t^{d_i}}. \tag{3.8}$$

Remark 3.9. Note that the J-invariant allows us to compute not only the Poincaré polynomial of $\mathcal{R}_p(G)$ by formula (3.8) but also all twisting numbers in the motivic decomposition (3.7) of X. Namely, we have

$$\frac{P(X,t)}{P(\mathcal{R}_p(G), t)} = \sum_{i \ge 0} a_i t^i,$$

where a_i is the number of the copies of $\mathcal{R}_p(G)$ with the twisting number i in the motivic decomposition (3.7). Note that the Poincaré polynomial $P(X, t)$ can be explicitly computed by the Solomon formula (see [Car72, 9.4 A]).

The motivic decomposition from Proposition 3.6 holds for a more general class of varieties. Namely, a projective homogeneous G-variety X is called *generically split* if the group G splits over the generic point of X. In particular, the variety of Borel subgroups in G is generically split. Other examples include Pfister quadrics and Severi–Brauer varieties. By [PSZ08, Theorem 5.17] the Chow motive of any generically split G-variety with coefficients in $\mathbb{F}_p$ decomposes into a direct sum of twisted copies of the motive $\mathcal{R}_p(G)$.

PROPOSITION 3.10 [PS10, Theorem 5.5]. *Let X be a generically split G-variety and recall that $\overline{\mathrm{Ch}}(X)$ denotes the subring of F-rational cycles in* $\mathrm{Ch}(X)$*. Then*

$$\frac{P(X,t)}{P(\mathcal{R}_p(G),t)} = P(\overline{\mathrm{Ch}}(X),t).$$

In particular, the number of copies of the motive $\mathcal{R}_p(G)\{i\}$ in the complete motivic decomposition of X is equal to $\dim_{\mathbb{F}_p} \overline{\mathrm{Ch}}^i(X)$.

Remark 3.11. The above proposition shows that the subring $\overline{\mathrm{Ch}}(X) \subset \mathrm{Ch}(\overline{X})$ of F-rational cycles encodes the complete motivic decomposition of a generically split variety X over F. In contrast, in order to find the complete motivic decomposition of a projective homogeneous G-variety X in general, one usually needs to describe F-rational projectors in the Chow group of the product $\overline{X \times X}$.

Remark 3.12. Let X be a generically split G-variety. Let $\{b_1,\dots,b_r\}$ and $\{a_1,\dots,a_n\}$ be homogeneous bases for $\mathbb{F}_p$-vector spaces $\overline{\mathrm{Ch}}(X)$ and $\mathrm{Ch}(\overline{X})$, respectively. For every $k=1,\dots,n$ let $\alpha_k \in \mathrm{Ch}(X\times X)$ be a preimage of a_k under the surjective flat pull-back

$$\mathrm{Ch}(X\times X) \twoheadrightarrow \mathrm{Ch}(X_{F(X)}) \simeq \mathrm{Ch}(\overline{X})$$

along the morphism induced by the generic point of the first factor X. Note that $\bar{\alpha}_k = 1 \times a_k + \sum_{i\in I} c_i \times d_i$, where $\operatorname{codim} c_i > 0$ for all $i \in I$. Then the set $\mathcal{B} = \{(b_i\times 1)\cdot\bar{\alpha}_k \mid i\in[1,r],\ k\in[1,n]\}$ forms a basis of the $\mathbb{F}_p$-vector space $\overline{\mathrm{Ch}}(X\times X)$.

Indeed, by [PSZ08, Theorem 3.7], the motive $M(X\times X)$ is isomorphic to a direct sum of twisted copies of the motive $M(X)$. It follows that

$$\dim_{\mathbb{F}_p} \overline{\mathrm{Ch}}(X\times X) = \dim_{\mathbb{F}_p} \overline{\mathrm{Ch}}(X)\cdot \dim_{\mathbb{F}_p} \mathrm{Ch}(\overline{X}) = rn.$$

It remains to observe that there are exactly rn elements in $\mathcal{B}$ and they are linearly independent.

We finish this section with several observations, which will be useful later in this paper.

LEMMA 3.13. *Let $\mathfrak{X},\mathcal{Y}$ be two projective homogeneous varieties over a field F. Assume that $\mathcal{Y}$ possesses a zero-cycle of degree 1 over the function field $F(\mathfrak{X})$. Then the cycle $a\in\mathrm{Ch}(\overline{\mathfrak{X}})$ is F-rational if and only if the cycle $a\times 1 \in \mathrm{Ch}(\overline{\mathfrak{X}\times\mathcal{Y}})$ is F-rational.*

Proof. The direct implication is clear. To show the inverse implication we assume that $a\times 1\in \mathrm{Ch}(\overline{\mathfrak{X}\times\mathcal{Y}})$ is F-rational.

Let $\alpha\in\mathrm{Ch}(\mathfrak{X}\times\mathcal{Y})$ be a cycle, such that $\overline{\alpha} = a\times 1$. Let $x\in \mathrm{Ch}_0(\mathcal{Y}_{F(\mathfrak{X})})$ be a zero-cycle of degree 1. Let $\beta \in \mathrm{CH}(X\times\mathcal{Y})$ be a preimage of x under the flat pull-back

$$\mathrm{Ch}(\mathfrak{X}\times\mathcal{Y}) \twoheadrightarrow \mathrm{Ch}(\mathcal{Y}_{F(\mathfrak{X})})$$

along the morphism induced by the generic point of $\mathfrak{X}$. Since $\overline{\beta} = 1\times[\mathbf{pt}] + \sum_{i\in I} a_i\times b_i$, where $\dim b_i > 0$ for all $i\in I$, we have

$$\overline{(pr_{\mathfrak{X}})_*(\alpha\beta)} = (pr_{\mathfrak{X}})_*(\overline{\alpha\beta}) = (pr_{\mathfrak{X}})_*(a\times[\mathbf{pt}] + \sum_{i\in I}(aa_i\times b_i)) = a,$$

where $pr_{\mathfrak{X}}$ is the projection $\mathfrak{X}\times\mathcal{Y}\to\mathfrak{X}$ on the first factor.

It follows from the above equality that the cycle a is F-rational. □

DEFINITION 3.14. Let G be a semisimple algebraic group of inner type, A a Tits algebra of G, and X a projective homogeneous variety such that $U(X)\simeq R_p(G)$. Let $w\in\mathrm{Ch}^1(\overline{X})$ be a cycle such that for some lifting $\tilde{w}\in\mathrm{CH}^1(\overline{X})$ of w holds $\alpha_X(\tilde{w}) = l[A]\in\mathrm{Br}(F)$, where l is an integer

https://doi.org/10.1112/S0010437X24007255 Published online by Cambridge University Press

coprime to p. We define $j_{G,A}$ to be the smallest integer $j \geq 0$ such that the cycle $w^{p^j} \in \mathrm{Ch}(\overline{X})$ is F-rational.

Remark 3.15. Note that the definition of $j_{G,A}$ does not depend on the choice of the variety X and the cycle w. Indeed, if Y and v is another choice of a variety and a cycle satisfying the above conditions, then using the exact sequence (2.2) for the variety $X \times Y$ we obtain an F-rational cycle $w \times 1 - \lambda(1 \times v) \in \mathrm{Ch}(\overline{X \times Y})$ for some $\lambda \in \mathbb{F}_p^*$. Then for every $j \geq 0$ we have

$$(w \times 1 - \lambda(1 \times v))^{p^j} = w^{p^j} \times 1 - \lambda(1 \times v^{p^j}) \in \overline{\mathrm{Ch}}(X \times Y).$$

Hence, we get the following equivalences

$$w^{p^j} \in \overline{\mathrm{Ch}}(X) \iff w^{p^j} \times 1 \in \overline{\mathrm{Ch}}(X \times Y) \iff 1 \times v^{p^j} \in \overline{\mathrm{Ch}}(X \times Y) \iff v^{p^j} \in \overline{\mathrm{Ch}}(Y),$$

where the first and the last equivalences hold by Lemma 3.13.

4. Main result

Let F be a field and p a prime number. Let G be a semisimple algebraic group over a field F of inner type. Let $J(G) = (j_1, \ldots, j_r)$ be the J-invariant of G modulo p and let $d_1 \leq \cdots \leq d_r$ be the respective degrees of the components. Denote by $J^1(G) = (j_1, \ldots, j_l)$, $l \leq r$, the family of all components of degree 1, that is $d_l = 1$ and $d_{l+1} > 1$.

Let A be a Tits algebra of G. Denote by F_A the function field of the Severi–Brauer variety $\mathrm{SB}(A)$. The main theorem of this paper describes the J-invariant of G_{F_A} in terms of the J-invariant of G over a base field F.

Theorem 4.1. *Let $J(G_{F_A}) = (j'_1, \ldots, j'_r)$. Then the following hold:*

(i) *$j_i = j'_i$ for every component of degree > 1;*
(ii) *$J^1(G) \cup \{0\} = J^1(G_{F_A}) \cup \{j_{G,A}\}$ as multisets;*
(iii) *in particular, if $J(G) \neq J(G_{F_A})$, then $j_k \neq j'_k = 0$ for some component j_k of degree 1.*

Let X be a generically split G-variety and let A be an algebra which splits over a function field of X (in particular, this is the case for a Tits algebra of G). Before proving the main theorem we need to investigate the relation between F-rational cycles on the variety $\overline{X \times \mathrm{SB}(A)}$ and F_A-rational cycles on the variety $\overline{X}$. Note that $\overline{\mathrm{SB}(A)} \simeq \mathbb{P}^{\deg(A)-1}$ and denote by h the hyperplane class in $\mathrm{Ch}^1(\overline{\mathrm{SB}(A)})$.

Lemma 4.2. *Let $y = a_k \times h^k + \sum_{i>k} a_i \times h^i$ be a homogeneous element in $\mathrm{Ch}(\overline{X \times \mathrm{SB}(A)})$, where $a_i \in \mathrm{Ch}(\overline{X})$. If y is rational over F, then:*

(i) *$1 \times h^k$ is rational over F;*
(ii) *a_k is rational over F_A.*

Proof. We first prove the second statement. We have

$$(pr_1)_*(y \cdot (1 \times h^i)) = (pr_1)_*(a_k \times [pt]) = a_k,$$

where $i = \deg(A) - 1 - k$, $[pt]$ is the class of a rational point in $\mathrm{Ch}(\overline{\mathrm{SB}(A)})$ and $pr_1 : \overline{X \times \mathrm{SB}(A)} \to \overline{X}$ is the projection on the first factor. Since y and $1 \times h^i$ are both rational over F_A, it follows that a_k is also rational over F_A.

We now prove the first statement. Consider the surjective pullback

$$f : \mathrm{Ch}(X \times (X \times \mathrm{SB}(A))) \twoheadrightarrow \mathrm{Ch}(X_{F(X \times \mathrm{SB}(A))}).$$

Since X is split over $F(X \times \mathrm{SB}(A))$, there exists a cycle a_k^* in $\mathrm{Ch}(X_{F(X\times \mathrm{SB}(A))})$, such that $\deg(a_k \cdot a_k^*) = 1$. Let α be the image in $\mathrm{Ch}(\overline{X \times (X \times \mathrm{SB}(A))})$ of some lifting of a_k^* via f. Let $\beta = (pr_{1,3})^*(y)$, where $pr_{1,3} : \overline{X \times X \times \mathrm{SB}(A)} \to \overline{X \times \mathrm{SB}(A)}$ is the projection on the product of the first and third factors. We have

$$\beta = a_k \times 1 \times h^k + \sum_{i>k} a_i \times 1 \times h^i$$

and

$$\alpha = a_k^* \times (1 \times 1) + \sum_{j\in I} b_j \times (c_j),$$

where $\operatorname{codim} c_j > 0$ and $\operatorname{codim} b_j < \operatorname{codim} a_k^*$ for all $j \in I$. Then

$$\alpha \cdot \beta = [pt] \times 1 \times h^k + \sum_{j\in I} b'_j \times (c'_j),$$

where $b'_j \in \mathrm{Ch}(\overline{X})$ and $\dim b'_j > 0$ for all $j \in I$. Hence, $(pr_{2,3})_*(\alpha \cdot \beta) = 1 \times h^k$, where $pr_{2,3} : \overline{X \times (X \times \mathrm{SB}(A))} \to \overline{X \times \mathrm{SB}(A)}$ is the projection on the product of second and third factors. Since both cycles α and β are F-rational, the cycle $1 \times h^k$ is also F-rational. □

Let $\mathcal{A} = \{a_1, \dots, a_s\}$ be a basis of $\mathbb{F}_p$-vector subspace of F_A-rational cycles in $\mathrm{Ch}(\overline{X})$. For every $i = 1, \dots, s$ we fix an F-rational lifting y_i of a_i via the surjective pullback:

$$\mathrm{Ch}(\overline{X \times \mathrm{SB}(A)}) \twoheadrightarrow \mathrm{Ch}(\overline{X_{F_A}}).$$

Consider the set

$$J = \{0 \le j < \deg A \mid 1 \times h^j \text{ is } F\text{-rational in } \mathrm{Ch}(\overline{X \times \mathrm{SB}(A)})\}. \tag{4.3}$$

Recall that we denote by $\overline{\mathrm{Ch}}(X \times \mathrm{SB}(A))$ the subring of F-rational cycles in $\mathrm{Ch}(\overline{X \times \mathrm{SB}(A)})$. We can now describe a basis of the $\mathbb{F}_p$-vector space $\overline{\mathrm{Ch}}(X \times \mathrm{SB}(A))$ in terms of $\mathcal{A}$ and J.

PROPOSITION 4.4. *The set $\mathcal{B} = \{y_i \cdot (1 \times h^j) \mid i \in [1, s], j \in J\}$ forms a basis of the $\mathbb{F}_p$-vector space $\overline{\mathrm{Ch}}(X \times \mathrm{SB}(A))$.*

Proof. Since a_i, $i \in [1, s]$ are linearly independent, the same holds for the elements from $\mathcal{B}$. Denote by $\langle\mathcal{B}\rangle$ the $\mathbb{F}_p$-vector subspace in $\mathrm{Ch}(\overline{X \times \mathrm{SB}(A)})$ generated by the elements from $\mathcal{B}$. Our goal is to show that $\langle\mathcal{B}\rangle = \overline{\mathrm{Ch}}(X \times \mathrm{SB}(A))$.

Clearly $\langle\mathcal{B}\rangle \subset \overline{\mathrm{Ch}}(X \times \mathrm{SB}(A))$. Assume that the subspaces are not equal and among the homogeneous elements in $\overline{\mathrm{Ch}}(X \times \mathrm{SB}(A)) \setminus \langle\mathcal{B}\rangle$ choose an element

$$y = a \times h^k + \sum_{i>k} a_i \times h^i, b_i \in \mathrm{CH}(\overline{X}), a \neq 0,$$

with the maximal $k \geq 0$. By Lemma 4.2 the cycle a is rational over F_A and $k \in J$. Hence, we can write $a = \lambda_1 a_1 + \cdots + \lambda_s a_s$ for some $\lambda_i \in \mathbb{F}_p$. Consider the cycle

$$y' = y - (\lambda_1 y_1 + \cdots + \lambda_s y_s) \cdot (1 \times h^k).$$

By our assumption on y we see that $y' = 0$ and we get a contradiction. It follows that $\overline{\mathrm{Ch}}(X \times \mathrm{SB}(A)) = \langle\mathcal{B}\rangle$ and, therefore, $\mathcal{B}$ is indeed a basis of $\overline{\mathrm{Ch}}(X \times \mathrm{SB}(A))$. □

We are now ready to prove the main theorem.

Proof of Theorem 4.1. Recall that we work with Chow groups and motives modulo p. Without loss of generality we can pass to a field extension of degree coprime to p and assume that the

https://doi.org/10.1112/S0010437X24007255 Published online by Cambridge University Press

index of A is a power of p. Since in this case the algebra A and its p-primary component have same splitting fields, we can also assume that the degree n of A is a power of p.

Let X be a generically split G-variety. Then the variety $X \times \mathrm{SB}(A)$ is also generically split for the group $G' = G \times \mathrm{PGL}_1(A)$. Moreover, by Proposition 2.1 and our assumption on algebra A, we have $\mathcal{R}_p(G) \simeq \mathcal{R}_p(G')$. Applying Proposition 3.10 for the generically split varieties $X \times \mathrm{SB}(A)$ and X_{F_A} we obtain

$$P(X \times \mathrm{SB}(A), t) = P(\mathcal{R}_p(G), t) \cdot P(\overline{\mathrm{Ch}}(X \times \mathrm{SB}(A)), t)$$

and

$$P(X_{F_A}, t) = P(\mathcal{R}_p(G_{F_A}), t) \cdot P\big(\overline{\mathrm{Ch}}(X_{F_A}), t\big).$$

Now let us compare the left- and right-hand sides of these polynomial equalities. We have $P(X \times \mathrm{SB}(A), t) = P(X, t) \cdot P(\mathrm{SB}(A), t) = P(X, t) \cdot P(\mathbb{P}^{n-1}, t) = P(X, t)((t^n - 1)/(t-1))$. Note also that $P(X, t) = P(X_{F_A}, t)$. The first factor on the right-hand sides of the equalities can be expressed in terms of the corresponding J-invariant using formula (3.8). Note that the last factor in the second equality divides the last factor in the first equality by Proposition 4.4 and the quotient is given by the polynomial

$$Q(t) = \sum_{i \in J} t^i,$$

where the set J is defined in (4.3).

Now dividing the first polynomial equality by the second equality we get

$$\frac{t^n - 1}{t - 1} = \prod_{i=1}^{r} \frac{t^{d_i p^{j_i}} - 1}{t^{d_i p^{j'_i}} - 1} \cdot Q(t). \tag{4.5}$$

The first statement of the theorem follows directly from the above polynomial equality. Indeed, if $j_i > j'_i$ for some component $i \in \{1, \dots, r\}$ with $d_i > 1$, then the primitive $d_i p^{j_i}$-root of unity $\zeta \in \mathbb{C}$ is a complex root of the right-hand-side polynomial from equality (4.5). However, ζ is not a complex root of the left-hand-side polynomial in (4.5), since n is a power of p, $d_i > 1$ and p does not divide d_i.

Before proving statements (ii) and (iii) of the theorem we will first find the explicit form of the polynomial $Q(t) = \sum_{i \in J} t^i$, which is defined by the set J. It follows from polynomial equality (4.5), that $Q(t) = t^{\deg Q} Q(1/t)$. Hence, the set J is symmetric with respect to its midpoint.

Another property

$$x \in J,\, y \in J \quad \Longrightarrow \quad x + y \in J$$

follows from the definition of the set J, since the product of two F-rational cycles is F-rational.

Let $m = \min J \setminus \{0\}$ (we set $m = n$, if $J = \{0\}$). Using the two above-mentioned properties of the set J it is easy to check that

$$Q(t) = 1 + t^m + t^{2m} + \dots + t^{(k-1)m} = \frac{t^{km} - 1}{t^m - 1}$$

for some integer $k \geq 1$, such that $(k-1)m < n \leq km$.

Finally, it follows from (4.5), that km is a power of p and, thus, is equal to n. Moreover, m is also a power of p and we write $m = p^j$. Therefore, the polynomial $Q(t)$ has the following form $(t^n - 1)/(t^{p^j} - 1)$.

https://doi.org/10.1112/S0010437X24007255 Published online by Cambridge University Press

Using the description of $Q(t)$ and the statement (i) of the theorem and formula (4.5) we get

$$(t^{p^j}-1)\prod_{i=1}^{l}(t^{p^{j'_i}}-1)=(t-1)\prod_{i=1}^{l}(t^{p^{j_i}}-1).$$

It follows from the above polynomial equality that $J^1(G_{F_A})\cup\{j\}=J^1(G)\cup\{0\}$ as multisets. Moreover, by the definition of j we have $j=j_{G,A}$ and we obtain the statement (ii) of the theorem.

In the case $J(G_{F_A})\neq J(G)$, by statement (ii), we have $j\neq 0$. Hence, $j_k\neq j'_k=0$ for some $k\in\{1,\dots,r\}$, which proves the statement (iii) of the theorem. □

With the same notation as in the above theorem, we get the following corollary.

COROLLARY 4.6. *Assume that there exists a component j_i of degree one of $J(G)$, such that all other components of degree one are zero (in particular, this is the case when there is only one component of degree one). Assume also that j_i becomes zero over F_A. Then $j_i=j_{G,A}$.*

5. J-invariant of algebras with involutions

Let (A,σ) be a degree $2n$ central simple algebra over F, endowed with an involution of orthogonal type and trivial discriminant. We refer to [KMRT98] for definitions and classical facts on algebras with involution. Recall, that the Clifford algebra of (A,σ) splits as a direct product $C(A,\sigma)=C_+\times C_-$ of two central simple algebras over F.

Let $G=\mathrm{PGO}^+(A,\sigma)$ be the connected component of the automorphism group of (A,σ). Since (A,σ) has trivial discriminant, the group G is an inner twisted form of $G_0=\mathrm{PGO}^+_{2n}$. Both groups G and G_0 are adjoint of type D_n. Let $\{\omega_1,\dots,\omega_n\}$ be the respective set of fundamental weights. Note that A is a Tits algebra A_{ω_1} of G. We fix fundamental weights ω_+ and ω_-, which are a permutation of ω_{n-1} and ω_n, in such a way that the Tits algebras A_{ω_+} and A_{ω_-} are the components C_+ and C_-, respectively, of the Clifford algebra $C(A,\sigma)$.

Note that in this section we assume $p=2$, which is the only torsion prime of the group G.

Let X be the variety of Borel subgroups in G. Recall, that the Picard group $\mathrm{Pic}(\overline{X})$ can be identified with a free $\mathbb{Z}$-module generated by ω_i, $i=1,\dots,n$. We denote by w_i the images of ω_i in $\mathrm{Ch}^1(\overline{X})=\mathrm{CH}^1(\overline{X})\otimes\mathbb{F}_2$.

In [QSZ12] Quéguiner-Mathieu, Semenov, and Zainoulline introduced the notion of the J-invariant of algebras with orthogonal involutions. The J-invariant of (A,σ) is denoted by $J(A,\sigma)$. By definition, $J(A,\sigma)$ is the J-invariant of the respective group $G=\mathrm{PGO}^+(A,\sigma)$, where in the definition of $J(G)$ we take a cocycle whose class corresponds to (A,σ) and a designation of the components C_+ and C_- (note that the choice of the designation does not affect the value of $J(G)$; see [QSZ12, § 3]).

The J-invariant $J(A,\sigma)$ is an r-tuple $(j_1,\dots,j_r)$, where $r=m+1$ if $n=2m$ or $n=2m+1$ (see the table in [PSZ08, § 4.13]). Note that the first two components j_1 and j_2 are of degree 1 and $d_i=2i-3$ for $i\geq 2$. For every component j_i we also have an explicit upper bound k_i (see the table in [PSZ08, § 4.13]). In particular, $j_1\leq k_1=v_2(n)$, where $v_2(-)$ denotes the 2-adic valuation. According to [QSZ12, § 3] one can take $e_1=\pi(w_1)$ and $e_2=\pi(w_+)$ for the generators in $\mathrm{Ch}(G_0)$ corresponding to the components j_1 and j_2, respectively, where $\pi:\mathrm{Ch}(\overline{X})\to\mathrm{Ch}(G_0)$ is the pullback map (see § 3).

The goal of this section is to prove Conjecture 1.2 and to explicitly compute j_1.

By [QSZ12, Corollary 5.2] the first component j_1 of $J(A,\sigma)$ is zero if the algebra A is split. We denote by F_A the function field of the Severi–Brauer variety of A, which is a generic splitting field of A. By Theorem 4.1 we have $J(A,\sigma)_{F_A}=(0,j'_2,j_3,\dots,j_r)$, that is all components of degree >1 does not change over F_A. However, to prove Conjecture 1.2 we still need to check that $j_2 = j_2'$. In order to show this we will use the following proposition.

Proposition 5.1. *The first component j_1 of the $J(A,\sigma)$ is equal to $j_{G,A}$, where $G = \mathrm{PGO}^+(A,\sigma)$.*

Proof. Let X be the variety of Borel subgroups in G. Assume n is odd. By the fundamental relation we have $[A] = 2[C_+] = 2[C_-] \in \mathrm{Br}(F)$, where C_+ and C_- are two components of the Clifford algebra of (A,σ). Using the exact sequence (2.2) we see that $w_1 \in \overline{\mathrm{Ch}}(X)$. Hence, by definition, $j_1 = j_{G,A} = 0$.

Assume now that n is even. Observe that in Definition 3.3 of j_1 we can assume $a = e_1^{2^{j_1}}$. Indeed, $e_1^{2^{j_1}}$ is the smallest monomial of codimension 2^{j_1} with respect to the order defined at the beginning of § 3. Note also that $\pi(w_1^{2^{j_1}}) = e_1^{2^{j_1}}$. It follows from Definition 3.3 that j_1 is the smallest non-negative integer, such that there exists an F-rational homogeneous cycle of the form $w_1^{2^{j_1}} + \delta \in \mathrm{Ch}(\overline{X})$, where δ is an element of codimension 2^{j_1} from the kernel of the map $\pi : \mathrm{Ch}(\overline{X}) \to \mathrm{Ch}(G_0)$. Since n is even, we have $[A] + [C_+] + [C_-] = 0 \in \mathrm{Br}(F)$. Hence, the cycle $w_1 + w \in \mathrm{Ch}^1(\overline{X})$ is F-rational, where we set $w = w_+ + w_-$. Therefore, in the definition of j_1 we can replace w_1 by w.

Recall that the kernel of π is the ideal in $\mathrm{Ch}(\overline{X}) \simeq \mathrm{Ch}(G_0/B)$ generated by the rational elements of positive codimension for the twisted form ${}_{\xi}(G_0/B)$ given by a generic cocycle ξ (see Remark 3.5). Our goal is to show that we can assume $\delta = 0$ in the definition of j_1. To do this we first show that, instead of X, we can consider a 'smaller' variety, where the cycle w is also defined and where there are 'few' rational cycles in the generic case.

Namely, let Y be the variety of isotropic right ideals in (A,σ) of reduced dimension $n-1$ (it corresponds to the parabolic subgroup $P \subset G_0$ of type $\{1,2,\dots,n-2\}$). We have a natural projection $f : X \to Y$. Since Y is generically split, the projection f is a cellular fibration. Therefore, by [PSZ08, Theorem 3.7] the motive of X decomposes into a direct sum

$$M(X) \simeq \bigoplus_{l=0}^{m} M(Y)\{i_l\} \tag{5.2}$$

of twisted copies of the motive $M(Y)$ with some twisting numbers $i_0,\dots,i_m$, where $i_0 = 0$ and $i_l > 0$ for $l \in [1,m]$. More precisely, we have a decomposition of the diagonal class $\Delta_X \in \mathrm{Ch}(X \times X)$ into a sum of pairwise orthogonal projectors $\rho_0,\dots,\rho_m$, such that $(X,\rho_l) \simeq M(Y)\{i_l\}$ for every $l \in [0,m]$, and, moreover, the Chow group of $(X,\rho_0) \simeq M(Y)$ as a subgroup of $\mathrm{Ch}(X)$ can be identified with $\mathrm{Ch}(Y)$ via the pull-back f^*. Thus, we can assume $\mathrm{Ch}(Y) \subset \mathrm{Ch}(X)$.

Note that decomposition (5.2) also holds for a generic cocycle ξ (that is, for ${}_{\xi}(G_0/B)$ and ${}_{\xi}(G_0/P)$ instead of X and Y, respectively). Thus, we can assume that the projectors $\bar\rho_0,\dots,\bar\rho_m$ are rational for the generic cocycle. Since $\mathrm{Ch}^0(\overline{X},\bar\rho_l) = 0$ for $l > 0$, the projector $\bar\rho_l$ can be written, by Remark 3.12, as a linear combination of cycles of the type $(b \times 1)\alpha$, where $b \in \mathrm{Ch}(\overline{X})$, $\alpha \in \mathrm{Ch}(\overline{X \times X})$ are rational in the generic case and $\operatorname{codim} b > 0$. It follows that for $l > 0$ the Chow group of $(\overline{X},\bar\rho_l)$ lies in the kernel of π, that is $\mathrm{Ch}(\overline{X},\bar\rho_l) = (\bar\rho_l)^* \mathrm{Ch}(\overline{X}) \subset \operatorname{Ker}\pi$.

Since the cycle w (and also $w^{2^{j_1}}$) is defined in $\mathrm{Ch}(\overline{Y})$, the projection of the F-rational cycle $w^{2^{j_1}} + \delta \in \mathrm{Ch}(\overline{X})$ on the subgroup $\mathrm{Ch}(\overline{Y})$ is also F-rational and has the form $w^{2^{j_1}} + \tilde\delta$, where $\tilde\delta = (\bar\rho_0)^*(\delta)$. Note that $\tilde\delta = \delta - \sum_{l=1}^{m} (\bar\rho_l)^*(\delta) \in \operatorname{Ker}\pi$.

Recall that the varieties Y and G_0/P are isomorphic over any splitting field of G and consider the pull-back $\tilde\pi : \mathrm{Ch}(\overline{Y}) \to \mathrm{Ch}(G_0)$ induced by the quotient map. Note that $\tilde\pi = \pi \circ f^*$. Hence, $\tilde\delta \in \operatorname{Ker}\tilde\pi$.

https://doi.org/10.1112/S0010437X24007255 Published online by Cambridge University Press

Denote by $\overline{\mathrm{Ch}}(Y)_{\mathrm{gen}}$ the subring of rational cycles in $\mathrm{Ch}(\overline{Y}) \simeq \mathrm{Ch}(G_0/P)$ for the twisted form ${}_\xi(G_0/P)$ given by a generic cocycle ξ (note that $\overline{\mathrm{Ch}}(Y)_{\mathrm{gen}} \subseteq \overline{\mathrm{Ch}}(Y)$). Let $\mathcal{I}$ be the ideal in $\mathrm{Ch}(\overline{Y})$ generated by all cycles of positive codimension from $\overline{\mathrm{Ch}}(Y)_{\mathrm{gen}}$. Note that the parabolic subgroup $P \subset G_0$ is special (see [Kar18, § 8]) and recall that $\tilde{\delta} \in \operatorname{Ker} \tilde{\pi}$. Applying the exact sequence from [PS17, Lemma 7.1] (for $G = G_0$, $S = P$ and a split torsor E) and using [KM06, Theorem 6.4] we get that $\tilde{\delta} \in \mathrm{Ch}(\overline{Y})$ lies in the ideal $\mathcal{I}$.

We claim that the minimal non-zero codimension of a cycle in $\overline{\mathrm{Ch}}(Y)_{\mathrm{gen}}$ is 2^{k_1}. It would follow from the claim that any non-zero cycle in $\mathcal{I}$ has codimension at least 2^{k_1} and any cycle of codimension 2^{k_1} in $\mathcal{I}$ lies in $\overline{\mathrm{Ch}}(Y)_{\mathrm{gen}}$. In particular, since $\tilde{\delta} \in \mathcal{I}$ and $\operatorname{codim} \tilde{\delta} = 2^{j_1} \leq 2^{k_1}$, we obtain that $\tilde{\delta} \in \overline{\mathrm{Ch}}(Y)_{\mathrm{gen}}$ (more precisely, $\tilde{\delta} = 0$ if $j_1 < k_1$). Recall that $\overline{\mathrm{Ch}}(Y)_{\mathrm{gen}} \subseteq \overline{\mathrm{Ch}}(Y)$. Then the cycle $\tilde{\delta} \in \mathrm{Ch}(\overline{Y})$ is also F-rational and $w^{2^{j_1}} \in \overline{\mathrm{Ch}}(X)$. Since $\alpha_X(\omega_+ + \omega_-) = [A]$, by the definition of $j_{G,A}$ we have $j_{G,A} = j_1$.

It remains to show the above claim. Note that the Poincaré polynomial of $P(Y,t)$ is known and can be computed by Solomon's formula (see [Car72, 9.4 A]). Using the explicit formula (3.8) for the Poincaré polynomial $P(\mathcal{R}_2({}_\xi G_0), t)$, where ξ is a generic cocycle, and applying Proposition 3.10 we get

$$P(\overline{\mathrm{Ch}}(Y)_{\mathrm{gen}}, t) = \frac{P(Y,t)}{P(\mathcal{R}_2({}_\xi G_0), t)} = \frac{t^n - 1}{t^{2^{k_1}} - 1} = 1 + t^{2^{k_1}} + \cdots .$$

It follows that in the generic case a rational cycle in $\mathrm{Ch}(\overline{Y})$ has codimension either zero or at least 2^{k_1}. This proves the claim. □

COROLLARY 5.3. *Conjecture 1.2 holds.*

Proof. The Conjecture follows from Theorem 4.1 and the above proposition. □

Let $J(A,\sigma) = (j_1, j_2, \ldots, j_r)$. Then by the above corollary we have $J(A,\sigma)_{F_A} = (0, j_2, \ldots, j_r)$. Note that the group $\mathrm{PGO}^+(A,\sigma)$ over the field F_A becomes isomorphic to $\mathrm{PGO}^+(q_\sigma)$, where q_σ is the respective quadratic form adjoint to the split algebra with involution $(A,\sigma)_{F_A}$. Hence, we can reduce the computation of the components $j_2, \ldots, j_r$ of $J(A,\sigma)$ to the case of quadratic forms. However, over the field F_A we lose information about the first component j_1. Our next goal is to find an explicit formula for j_1.

We know that $j_1 \leq k_1 = v_2(n)$. Thus, $j_1 = 0$ if n is odd. Starting from now, in this section we assume that n is a positive even integer.

Note that in this case all three algebras A, C_+, and C_- have exponent 2. Therefore, the indices of these algebras are powers of 2. We denote by i_A (respectively, by i_+, i_-) the 2-adic valuation of the index of A (respectively, of C_+, C_-).

We start the computation of j_1 by collecting upper bounds for j_1. Clearly we have

$$j_1 \leq k_1. \tag{5.4}$$

By [QSZ12, Corollary 5.2] we have another upper bound for j_1

$$j_1 \leq i_A. \tag{5.5}$$

Finally, the lemma below shows that

$$j_1 \leq \max\{i_+, i_-\}. \tag{5.6}$$

LEMMA 5.7. *Inequality (5.6) holds.*

https://doi.org/10.1112/S0010437X24007255 Published online by Cambridge University Press

Proof. Let X be a variety of Borel subgroups in $\mathrm{PGO}^+(A,\sigma)$. Let $i = \max\{i_+, i_-\}$. Since $[A] + [C_+] + [C_-] = 0 \in \mathrm{Br}(F)$, the cycle $w_1 + w_+ + w_- \in \mathrm{Ch}^1(\overline{X})$ is F-rational. We have

$$(w_1 + w_+ + w_-)^{2^i} = w_1^{2^i} + w_+^{2^i} + w_-^{2^i}.$$

By [PS10, Proposition 4.2] the cycles $w_+^{2^i}$ and $w_-^{2^i}$ are F-rational and, hence, $w_1^{2^i}$ is also F-rational. It follows from the very definition of j_1 that $j_1 \le i$. □

Combining upper bounds (5.4), (5.5), and (5.6) we get

$$j_1 \le \min\{k_1, i_A, \max\{i_+, i_-\}\}. \tag{5.8}$$

It appears that the above upper bound is the exact value of j_1.

Theorem 5.9. *The following formula holds for the first component j_1 of $J(A,\sigma)$:*

$$j_1 = \min\{k_1, i_A, \max\{i_+, i_-\}\}. \tag{5.10}$$

Proof. Since inequality (5.8) holds, it remains to show that

$$j_1 \ge \min\{k_1, i_A, \max\{i_+, i_-\}\}.$$

Let X be the variety of totally isotropic ideals in (A,σ) of reduced dimension n (the variety X is the twisted form of the variety of maximal totally isotropic subspaces of a quadratic form). Since the discriminant of σ is trivial, the variety X has two connected components X^+ and X^- corresponding to the components C_+ and C_-, respectively, of the Clifford algebra $C(A,\sigma)$. The varieties X^+ and X^- are projective homogeneous under the action of the group $\mathrm{PGO}^+(A,\sigma)$.

Recall that $\overline{\mathrm{SB}(A)} \simeq \mathbb{P}^{2n-1}$ and denote by h the hyperplane class in $\mathrm{Ch}^1(\overline{\mathrm{SB}(A)})$. Note that the variety $X = X^+ \times \mathrm{SB}(A)$ and the cycle $1 \times h \in \mathrm{Ch}^1(\overline{X^+ \times \mathrm{SB}(A)})$ satisfy the conditions in Definition 3.14 for the group $G = \mathrm{PGO}^+(A,\sigma)$ and central simple algebra A.

Therefore, by Proposition 5.1, j_1 is equal to the smallest integer j, such that $1 \times h^{2^j} \in \mathrm{Ch}(\overline{X^+ \times \mathrm{SB}(A)})$ is F-rational. If j is such a smallest integer, then $h^{2^j} \in \mathrm{Ch}(\overline{\mathrm{SB}(A)})$ is $F(X^+)$-rational. It follows that $2^j \ge \operatorname{ind} A_{F(X^+)}$. On the other hand, by the index reduction formula [MPW96, p. 594] we have

$$\operatorname{ind} A_{F(X^+)} = \min\{\operatorname{ind}(A), 2^{k_1}\operatorname{ind}(A \otimes_F A), \operatorname{ind}(A \otimes_F C_+), 2^{k_1}\operatorname{ind}(A \otimes_F C_-)\}.$$

Recall that the algebras A and C_- have exponent 2 and $[A] + [C_+] + [C_-] = 0 \in \mathrm{Br}(F)$. Hence, $A \otimes_F A$ is split and the algebras $A \otimes_F C_+$ and C_- are Brauer equivalent. We get

$$\operatorname{ind} A_{F(X^+)} = \min\{\operatorname{ind}(A), 2^{k_1}, \operatorname{ind}(C_-)\} = \min\{2^{i_A}, 2^{k_1}, 2^{i_-}\}.$$

Therefore, the following inequality holds

$$j_1 \ge \min\{k_1, i_A, i_-\}. \tag{5.11}$$

Repeating the same arguments but now for the variety $X^- \times \mathrm{SB}(A)$ we get another inequality

$$j_1 \ge \min\{k_1, i_A, i_+\}. \tag{5.12}$$

It follows from inequalities (5.11) and (5.12) that $j_1 \ge \min\{k_1, i_A, \max\{i_+, i_-\}\}$ and we conclude that formula (5.10) holds. □

Corollary 5.13. *Assume that (A,σ) is half-spin, that is one of the components of the Clifford algebra $C(A,\sigma)$ is split. Then $j_1 = \min\{k_1, i_A\}$.*

Proof. Since $[A] + [C_+] + [C_-] = 0 \in \mathrm{Br}(F)$ and one of the components C_+ or C_- is split, we have $\{i_+, i_-\} = \{i_A, 0\}$. Then the formula $j_1 = \min\{k_1, i_A\}$ follows from Theorem 5.9. □

https://doi.org/10.1112/S0010437X24007255 Published online by Cambridge University Press

The J-invariant of (A,σ) contains two components j_1 and j_2 of degree one. The formula for j_1 is computed in Theorem 5.9. By [QSZ12, Corollary 5.2] we have $j_2 \le \min\{i_+, i_-\}$. It appears that this upper bound is the exact value of j_2 in the following case.

PROPOSITION 5.14. *If* $\min\{i_+, i_-\} < \min\{k_1, i_A\}$, *then* $j_2 = \min\{i_+, i_-\}$.

Proof. Denote by j_i^+ (respectively, j_i^-) the ith component of the J-invariant of (A,σ) over the function field of $\mathrm{SB}(C_+)$ (respectively, of $\mathrm{SB}(C_-)$). Note that over such a function field the Clifford invariant of (A,σ) is trivial. By Corollary 5.13 and using the index reduction formula [MPW96, p. 592] we have

$$j_1^+ = \min\{k_1, v_2(\operatorname{ind} A_{F(\mathrm{SB}(C_+))})\} = \min\{k_1, i_A, i_-\}.$$

Similarly, we get $j_1^- = \min\{k_1, i_A, i_+\}$.

Recall that at least one of the numbers i_+ or i_- is less than $\min\{k_1, i_A\}$. Hence, $j_1^+ = i_-$ or $j_1^- = i_+$. Let ε be a symbol $+$ or $-$, such that $j_1^\varepsilon = \min\{i_+, i_-\}$. Observe that $j_2^\varepsilon = 0$. Therefore, by Theorem 4.1 we have $j_1^\varepsilon \in \{j_1, j_2\}$. If $i_+ \ne i_-$, then by Theorem 5.9 we have

$$j_1 = \min\{k_1, i_A, \max\{i_+, i_-\}\} > \min\{i_+, i_-\} = j_1^\varepsilon$$

and, hence, $j_2 = j_1^\varepsilon$. Assume now $i_+ = i_-$, then $j_1 = j_1^\varepsilon$. It follows from Theorem 4.1 that $j_2 = j_{G,C_\varepsilon}$. Considering the variety $X^{-\varepsilon} \times \mathrm{SB}(C_\varepsilon)$ and applying the index reduction formula in the same way as in the proof of Theorem 5.9 one can check that $j_{G,C_\varepsilon} \ge i_+ = i_-$. Therefore, in this case we also have $j_2 = \min\{i_+, i_-\}$. □

ACKNOWLEDGEMENTS

I am grateful to Anne Quéguiner-Mathieu for introducing me to Conjecture 1.2 and for useful discussions. I would like to thank Nikita Semenov for numerous useful discussions on the subject of this paper.

CONFLICTS OF INTEREST

None.

FINANCIAL SUPPORT

The author gratefully acknowledges the support of the DFG project ZH 918/2-1.

JOURNAL INFORMATION

Compositio Mathematica is owned by the Foundation Compositio Mathematica and published by the London Mathematical Society in partnership with Cambridge University Press. All surplus income from the publication of *Compositio Mathematica* is returned to mathematics and higher education through the charitable activities of the Foundation, the London Mathematical Society and Cambridge University Press.

REFERENCES

Car72 R. Carter, *Simple groups of Lie type* (John Wiley & Sons, 1972).

CM06 V. Chernousov and A. Merkurjev, *Motivic decomposition of projective homogeneous varieties and the Krull–Schmidt theorem*, Transform. Groups **11** (2006), 371–386.

DQZ22 C. De Clercq, A. Quéguiner-Mathieu and M. Zhykhovich, *Critical varieties and motivic equivalence for algebras with involution*, Trans. Amer. Math. Soc. **375** (2022), 7529–7552.

EKM08 R. Elman, N. Karpenko and A. Merkurjev, *The algebraic and geometric theory of quadratic forms*, Colloquium Publications, vol. 56 (American Mathematical Society, Providence, RI, 2008).

GS10 S. Garibaldi and N. Semenov, *Degree 5 invariant of E_8*, Int. Math. Res. Not. IMRN **2010** (2010), 3746–3762.

GZ12 S. Gille and K. Zainoulline, *Equivariant pretheories and invariants of torsors*, Transform. Groups **17** (2012), 471–498.

Hen22 A. Henke, *Chow motives of projective, homogeneous E_7-varieties*, Dissertation, LMU München (2022).

Kac85 V. Kac, *Torsion in cohomology of compact Lie groups and Chow rings of reductive algebraic groups*, Invent. Math. **80** (1985), 69–79.

Kar10 N. Karpenko, *Canonical dimension,* in *Proceedings of the International Congress of Mathematicians*, vol. II (Hindustan Book Agency, New Delhi, 2010), 146–161.

Kar13 N. Karpenko, *Upper motives of algebraic groups and incompressibility of Severi–Brauer varieties*, J. Reine Angew. Math. **677** (2013), 179–198.

Kar18 N. Karpenko, *On generically split generic flag varieties*, Bull. Lond. Math. Soc. **50** (2018), 496–508.

KM06 N. Karpenko and A. Merkurjev, *Canonical p-dimension of algebraic groups*, Adv. Math. **205** (2006), 410–433.

KMRT98 M-A. Knus, A. Merkurjev, M. Rost and J-P. Tignol, *The book of involutions*, Colloquium Publications, vol. 44 (American Mathematical Society, 1998).

LM07 M. Levine and F. Morel, *Algebraic cobordism*, Springer Monographs in Mathematics (Springer, Berlin, Heidelberg, 2007).

MPW96 A. Merkurjev, I. Panin and A. Wadsworth, *Index reduction formulas for twisted flag varieties. I*, K-Theory **6** (1996), 517–596.

MT95 A. Merkurjev and J-P. Tignol, *The multipliers of similitudes and the Brauer group of homogeneous varieties*, J. Reine Angew. Math. **461** (1995), 13–47.

PS10 V. Petrov and N. Semenov, *Generically split projective homogeneous varieties*, Duke Math. J. **152** (2010), 155–173.

PS17 V. Petrov and N. Semenov, *Rost motives, affine varieties, and classifying spaces*, J. Lond. Math. Soc. (2) **95** (2017), 895–918.

PS21 V. Petrov and N. Semenov, *Hopf-theoretic approach to motives of twisted flag varieties*, Compos. Math. **157** (2021), 963–996.

PSZ08 V. Petrov, N. Semenov and K. Zainoulline, *J-invariant of linear algebraic groups*, Ann. Sci. Éc. Norm. Supér. (4) **41** (2008), 1023–1053.

QSZ12 A. Quéguiner-Mathieu, N. Semenov and K. Zainoulline, *The J-invariant, Tits algebras and triality*, J. Pure Appl. Algebra **216** (2012), 2614–2628.

Sem16 N. Semenov, *Motivic construction of cohomological invariants*, Comment. Math. Helv. **91** (2016), 163–202.

Vis05 A. Vishik, *On the Chow groups of quadratic Grassmannians*, Doc. Math. **10** (2005), 111–130.

Vis09 A. Vishik, *Fields of u-invariant $2^r + 1$*, in *Algebra, arithmetic, and geometry: in honor of Yu. I. Manin*, volume II, Progress in Mathematics, vol. 270 (Birkhäuser, 2009), 661–685.

Maksim Zhykhovich zhykhovich@math.lmu.de
Mathematisches Institut, Ludwig-Maximilians-Universität München, Theresienstr. 39, D-80333 München, Germany

https://doi.org/10.1112/S0010437X24007255 Published online by Cambridge University Press