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Free generators in free inverse semigroups

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Using the characterization of the free inverse semigroup F on a set X, given by Scheiblich, a necessary and sufficient condition is found for a subset K of an inverse semigroup S to be a set of free generators for the inverse subsemigroup of S generated by K. It is then shown that any non-idempotent element of F generates the free inverse semigroup on one generator and that if $|X| > 2$ then F contains the free inverse semigroup on a countable number of generators. In addition, it is shown that if $|X| = 1$ then F does not contain the free inverse semigroup on two generators.

Let X be a non-empty set. By a free inverse semigroup on X is meant an ordered pair (F, f) where F is an inverse semigroup and f is a mapping of X into F such that for any mapping g of X into an inverse semigroup S there is a unique homomorphism h of F into S with $g = f \circ h$. The inverse semigroup F is then unique to within isomorphism and is referred to as the *free inverse semigroup* on X. In [5], Šaĭn establishes that inverse semigroups form a variety, from which it follows that free inverse semigroups exist. Alternative approaches are discussed by Eberhart and Selden [2] and McAlister [3].

In his recent paper [6], Scheiblich has given a valuable characterization of the free inverse semigroup on X in terms of partial transformations of the power set of the non-identity elements in the free group on X. We begin by describing this characterization. Basic

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information about inverse semigroups and unexplained notation will be found in [1].

Let X be a non-empty set and X' be a set disjoint from X and in one-to-one correspondence with X. Let this correspondence be denoted by $x \leftrightarrow x^{-1}$. We also write $x = (x^{-1})^{-1}$.

A *word* in $Y = X \cup X'$ is any finite sequence, $a_1 \ldots a_n$, of elements from Y and a *reduced word* is any finite sequence, $a_1 \ldots a_n$, of elements from Y such that $a_i \neq a_{i+1}^{-1}$, $i = 1, \ldots, n-1$. The empty sequence, denoted by 1, is also a reduced word. Let G denote the set of reduced words and $R = G\backslash\{1\}$. For any elements $g = a_1 \ldots a_m$, $h = b_1 \ldots b_n$ of G let gh denote the reduced word obtained from $a_1 \ldots a_m b_1 \ldots b_n$ by removing any adjacent pairs yy^{-1}, $y \in Y$ successively until a reduced word is obtained. This defines a binary operation on G with respect to which G is a group - the free group on X. In particular, for any $x \in X$, the elements x and x^{-1} (as sequences in G) are group inverses.

For any $w = a_1 \ldots a_n \in R$, let

$$I(w) = \{a_1, a_1a_2, \ldots, a_1 \ldots a_n\},$$

that is, let $I(w)$ denote the set of initial segments of w.

Let $E = \{A \subseteq R : A \neq \emptyset, A$ is finite and $I(w) \subseteq A$ for all $w \in A\}$. Thus E is the set of subsets of R which are closed with respect to taking initial segments. If $A, B \in E$ then $A \cup B \in E$. Thus, if E is partially ordered by defining $A \leq B$ if $A \supseteq B$, then E is a semilattice such that $A \cup B$ is the greatest lower bound of A and B. For any $A \in E$ we write A^1 for $A \cup \{1\}$.

For each $x \in X$, define a permutation $x\rho$ of R as follows: for $w \in R$,

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$$w(x\rho) = \begin{cases} x^{-1} & \text{if } w = x , \\ x^{-1}w & \text{otherwise.} \end{cases}$$

The mapping $\rho : x \to x\rho$ extends to a homomorphism (in fact, an isomorphism) ρ of G into the group, $P(R)$, of permutations of R. It is convenient to note here that for any $w \in R$, $x \in X$,

$$w\left(x^{-1}\rho\right) = \begin{cases} x & \text{if } w = x^{-1} , \\ xw & \text{otherwise.} \end{cases}$$

This homomorphism induces another homomorphism, which we also denote by ρ, of G into $P(E)$, defined as follows: for any $A \in E$, $g \in G$,

$$A(g\rho) = \{a(g\rho) : a \in A\} .$$

Let $F = \{(A, w) \in E \times G : w \in A^1\}$ and define a binary operation on F as follows:

(1) $$(A, w)(B, v) = \left(A \cup B(w\rho)^{-1}, wv\right) .$$

Define a mapping $f : X \to F$ by $xf = (\{x\}, x)$.

THEOREM 1.1 ([6], Scheiblich). *With respect to the operation defined in* (1), F *is an inverse semigroup and* (F, f) *is the free inverse semigroup on* X.

For any element $U = (A, a)$ in F, let $\Delta'(U) = A$.

The following lemma lists some simple observations from [6] which are used later without further comment.

LEMMA 1.2. *Let* $W = (A, a)$ *and* $W_i = \left(A_i, a_i\right)$, $i = 1, \ldots, n$ *be any elements of* F.

1. W *is an idempotent in* F *if and only if* $a = 1$.

2. $W^{-1} = \left(A(a\rho), a^{-1}\right)$.

3. $WW^{-1} = (A, 1)$, $W^{-1}W = \left(A(a\rho), 1\right)$ *and* $\Delta'(W) = A = \Delta'\left(WW^{-1}\right)$.

4.

$$\Delta'\left(\prod_{i=1}^{n} w_i\right) = A_1 \cup A_2(a_1\rho)^{-1} \cup A_3(a_2\rho)^{-1}(a_1\rho)^{-1} \cup \ldots \cup A_n(a_{n-1}\rho)^{-1} \ldots (a_1\rho)^{-1} .$$

There are some immediate observations that can be made regarding F. For instance, since every idempotent is of the form $(A, 1)$ with $A \in E$ there are $2|X|$ maximal idempotents in F, where $|X|$ denotes the cardinality of X. Also, the following observations can be made regarding Green's relations H, L, R, D and J.

LEMMA 1.3. *Let* $U = (A, a)$, $V = (B, b)$ *be any elements of* F.

(1) $URV \Leftrightarrow A = B$.

(2) $ULV \Leftrightarrow A(a\rho) = B(b\rho)$.

(3) $UHV \Leftrightarrow A = B$ *and* $a = b$.

Thus H *is the identity relation.*

(4) $UDV \Leftrightarrow A(a'\rho) = B(b'\rho)$ *for some* $a' \in A^1$, $b' \in B^1$.

(5) $J = D$.

(6) $|R_U| = |A| + 1 = |L_U|$.

(7) $|D_U| = (|A|+1)^2$.

Proof. Parts (1), (2), (3), (6) and (7) are immediate consequences of Lemma 1.2.

For part (4), first suppose that UDV. Then, for some $W = (C, c)$ we must have URW and WLV. Then $A = C$ and $C(c\rho) = B(b\rho)$. Thus $A(c\rho) = B(b\rho)$ where $c \in C^1 = A^1$, $b \in B^1$.

Conversely, let $A(a'\rho) = B(b'\rho)$ for some $a' \in A^1$, $b' \in B^1$. Let $W = (A, a')$, $Z = (B, b')$. Then URW, WLZ, ZRV. Hence UDV.

One always has $D \subseteq J$ and so, for part (5), we wish to establish the converse inclusion. So let UJV. Then, since $V \in FUF$, there exist elements $W = (C, c)$, $Z = (D, d)$ such that $V = WUZ$. Hence $b = cad$

and

$$B = \Delta'(V) = \Delta'(WUZ)$$
$$= C \cup A(c\rho)^{-1} \cup D(a\rho)^{-1}(c\rho)^{-1} .$$

Hence, $A(c\rho)^{-1} \subseteq B$ and $A \subseteq B(c\rho)$. Similarly, since $U \in FVF$, there exists a $c' \in G$ such that $B \subseteq A(c'\rho)$. Since A and B are finite, since A and $A(c'\rho)$ have the same cardinality and since B and $B(c\rho)$ have the same cardinality we must have $A = B(c\rho)$. Thus $A(a\rho) = B(ca\rho) = B\left(bd^{-1}\rho\right)$. Let $d = d_1 \ldots d_n$. Since $d \in D$ we have $d_1 \in D$. If $bd^{-1} \in I(b)$ then we have the required result. Otherwise $bd^{-1} = ud_1^{-1}$ where either $u = 1$ or ud_1^{-1} is in reduced form. Then $d_1\left(db^{-1}\right)\rho \in D\left(db^{-1}\rho\right) = D(ca\rho)^{-1} \subseteq B$. But

$$d_1\left(db^{-1}\right)\rho = d_1\left(d_1u^{-1}\rho\right) = ud_1^{-1} = bd^{-1} .$$

Thus $A(a\rho) = B\left(bd^{-1}\rho\right)$ where $bd^{-1} \in B$, $a \in A$. Thus $U\mathcal{D}V$.

2. Free generators

For any subset K of an inverse semigroup S let $\prec K \succ$ denote the inverse subsemigroup of S generated by K. If $K = \{U\}$, for some $U \in S$ then we write $\prec K \succ = \prec U \succ$. For any subset K of a group H we denote by $\langle K \rangle$ the subgroup of H generated by K. Let $K \subseteq S$, and let i denote the embedding of K into S. If $(\prec K \succ, i)$ is the free inverse semigroup on K then we say that K is a set of free generators for $\prec K \succ$. For any subset K of S, $K^{-1} = \{k^{-1} : k \in K\}$. For any elements $a_1, \ldots, a_n$ in S we write $\prod_{i=1}^{n} a_i = a_1a_2 \ldots a_n$.

We first observe that the sets which are sets of free generators for F are very restricted.

PROPOSITION 2.1. *Let $W \subseteq F$ and $\prec W \succ = F$. Then W is a set of free generators for F if and only if, $W \subset Xf \cup (Xf)^{-1}$ and, for each*

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$x \in X$, $|W \cap \{xf, (xf)^{-1}\}| = 1$.

Proof. It is clear that if the condition is satisfied then W is a set of free generators. So suppose that W is a set of free generators.

We first show that $W \cup W^{-1} = (Xf) \cup (Xf)^{-1}$. Let $x \in X$. Then, for some $W_1, \ldots, W_n \in W \cup W^{-1}$,

$$(\{x\}, x) = W_1 \ldots W_n .$$

Hence

$$\begin{aligned}\{x\} &= \Delta'(W_1 \ldots W_n) \\ &= \Delta'(W_1) \cup \ldots .\end{aligned}$$

Thus $\Delta'(W_1) = \{x\}$ and so $W_1 = (\{x\}, x) = xf$ or $W_1 = (\{x\}, 1)$. In the latter case, W_1 is an idempotent which is clearly impossible, since W is a set of free generators. Hence $xf = W_1 \in W \cup W^{-1}$. Therefore $(Xf) \cup (Xf)^{-1} \subseteq W \cap W^{-1}$. Since both Xf and W are sets of free free generators for F , we must have $(Xf) \cup (Xf)^{-1} = W \cup W^{-1}$. Thus, for any $x \in X$, we must have $\{xf, (xf)^{-1}\} \cap W \neq \emptyset$. But since W is a set of free generators we cannot have $\{xf, (xf)^{-1}\} \subseteq W$. Hence we have the desired conclusion.

We now give a general criterion for a subset K of an inverse semigroup S to be a set of free generators for $\langle K \rangle$.

THEOREM 2.2. *Let* K *be a subset of an inverse semigroup* S . *Then* K *is a set of free generators for* $\langle K \rangle$ *if and only if the following condition is satisfied:*

(K) *If* $Y \in K \cup K^{-1}$ *and* $YY^{-1} \geq F_1 \ldots F_n$ *where* $F_j = Y_{j1} \ldots Y_{jn(j)} Y_{jn(j)}^{-1} \ldots Y_{j1}^{-1}$ *for some* $Y_{jk} \in K \cup K^{-1}$ *such that* $Y_{jk} \neq Y_{jk+1}^{-1}$ *for* $k = 1, \ldots, n(j)-1$, $j = 1, \ldots, n$, *then* $Y = Y_{j1}$ *for some* j .

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Proof. Let $\theta : X \to K$ be a bijection of some set X onto K. Let (F, f) be the free inverse semigroup on X as described in Section 1. Then θ determines a unique epimorphism of F onto $\langle K \rangle$, which we also denote by θ. Clearly K is a set of free generators for $\langle K \rangle$ if and only if θ is an isomorphism.

First suppose then that θ is an isomorphism. We wish to show that condition (K) is satisfied. Let Y be some element of $K \cup K^{-1}$ such that

$$YY^{-1} \geq F_1 \ldots F_n$$

where, for $j = 1, \ldots, n$,

$$F_j = Y_{j1} \ldots Y_{jn(j)} Y_{jn(j)}^{-1} \ldots Y_{j1}^{-1}$$

for some elements $Y_{jk} \in K \cup K^{-1}$ such that $Y_{jk} \neq Y_{jk+1}^{-1}$, for any $j = 1, \ldots, n$, $k = 1, \ldots, n(j)$. Here $n(j)$ denotes some integer that depends on j.

Let U, U_{jk} be elements of $Xf \cup (Xf)^{-1}$ such that

$$Y\theta^{-1} = U = (\{y\}, y)$$

and

$$Y_{jk}\theta^{-1} = U_{jk} = (\{y_{jk}\}, y_{jk})$$

for some $y, y_{jk} \in X \cup X'$. Let $E_j = F_j\theta^{-1}$, $j = 1, \ldots, n$. Then

$$UU^{-1} \geq E_1 \ldots E_n .$$

Hence

$$\begin{aligned} \Delta'(U) = \Delta'(UU^{-1}) &\subseteq \Delta'(E_1 \ldots E_n) \\ &= \bigcup_{j=1}^{n} \Delta'(E_j) \\ &= \bigcup_{j=1}^{n} \Delta'(U_{j1} \ldots U_{jn(j)}) . \end{aligned}$$

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Therefore, $y \in \Delta'(U_{j1} \ldots U_{jn(j)})$, for some j. Now, since $Y_{jk} \neq Y_{jk+1}^{-1}$, for any j, k, we have that $U_{jk} \neq U_{jk+1}^{-1}$ and hence that $y_{jk} \neq y_{jk+1}^{-1}$. Hence the elements $y_{j1}, y_{j1}y_{j2}, \ldots, y_{j1}y_{j2} \ldots y_{jn(j)}$ are all in reduced form and so, by Lemma 1.2 (4),

$$\begin{aligned}\Delta'(U_{j1} \ldots U_{jn(j)}) &= \Delta'(U_{j1}) \cup \Delta'(U_{j2})(y_{j1}\rho)^{-1} \cup \ldots \\ &= \{y_{j1}, y_{j1}y_{j2}, \ldots, y_{j1}y_{j2} \ldots y_{jn(j)}\} .\end{aligned}$$

Thus the only element of $\Delta'(U_{j1} \ldots U_{jn(j)})$ of length one in reduced form is y_{j1}. Hence $y = y_{j1}$, $U = U_{j1}$ and $Y = Y_{j1}$. Thus (K) is satisfied.

Now suppose that (K) is satisfied. We wish to show that θ is an isomorphism. Since θ is clearly an epimorphism, we need only show that θ is a monomorphism. To this end, we need the following result due to Munn [4].

LEMMA 2.3. *Let T be an inverse semigroup and τ be a congruence on T. Then $\tau \subset H$ if and only if τ is idempotent separating (that is, $(a, b) \in \tau$, $a^2 = a$ and $b^2 = b$ imply that $a = b$).*

If we can show that for any distinct idempotents M, N of F, $M\theta \neq N\theta$ then the congruence $\theta \circ \theta^{-1}$ induced by θ is idempotent separating. By Lemma 2.3 this means that $\theta \circ \theta^{-1} \subseteq H$. But, by Lemma 1.3, H is the identity relation. Consequently θ must be a monomorphism.

So let M, N be distinct idempotents of F such that $M\theta = N\theta$. Then $\Delta'(M) \neq \Delta'(N)$. Without loss of generality, let $z_1 \ldots z_n \in \Delta'(M) \setminus \Delta'(N)$, where $n \geq 1$, $z_1, \ldots, z_n \in X \cup X'$ and $z_\alpha \neq z_{\alpha+1}^{-1}$, $\alpha = 1, \ldots, n-1$. Let $Z_i = (\{z_i\}, z_i)$, $i = 1, \ldots, n$. Since $z_1z_2 \ldots z_n \in \Delta'(M)$,

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$$\Delta'\left(Z_1Z_2 \ldots Z_nZ_n^{-1} \ldots Z_1^{-1}\right) = \Delta'\left(Z_1 \ldots Z_n\right)$$
$$= \{z_1, z_1z_2, \ldots, z_1z_2 \ldots z_n\} \subseteq \Delta'(M) ,$$

and so

$$M \leq Z_1 \ldots Z_nZ_n^{-1} \ldots Z_1^{-1} .$$

Hence,

(2) $$N\theta = M\theta \leq Z_1\theta \ldots Z_n\theta Z_n^{-1}\theta \ldots Z_1^{-1}\theta .$$

Let $\Delta'(N) = \{n_1, \ldots, n_r\}$, for some integer r and, for $i = 1, \ldots, r$, let $n_i = v_{i1} \ldots v_{ik(i)}$ for some $v_{i\alpha} \in X \cup X'$ with $v_{i\alpha} \neq v_{i\alpha+1}^{-1}$, $\alpha = 1, \ldots, k(i)-1$. Let $V_{ij} = \left(\{v_{ij}\}, v_{ij}\right)$, $i = 1, \ldots, r$, $j = 1, \ldots, k(i)$. For each i , let $B_i = I\left(n_i\right)$ and $N_i = \left(B_i, 1\right)$. Then

$$\Delta'(N) = \cup\{B_i : i = 1, \ldots, r\} ,$$

$$N = \prod_{i=1}^{r} N_i$$

and

$$N_i = V_{i1} \ldots V_{in(i)}V_{in(i)}^{-1} \ldots V_{i1}^{-1} .$$

We will show by induction that, for $s = 1, \ldots, n$ there exists an i with $Z_1 = V_{i1}, \ldots, Z_s = V_{is}$.

First let $s = 1$. Since $Z_1\theta Z_1^{-1}\theta \geq Z_1\theta \ldots Z_n\theta Z_n^{-1}\theta \ldots Z_1^{-1}\theta$, we have that $Z_1\theta Z_1^{-1}\theta \geq N\theta$. Let $Z_i\theta = Y_i$ and $V_{ij}\theta = Y_{ij}$. Then $Y_1Y_1^{-1} \geq \prod_{i=1}^{r} \left(Y_{i1} \ldots Y_{i1}^{-1}\right)$. Hence, by (K), $Y_1 = Y_{i1}$, for some i . Since θ is one-to-one on $Xf \cup (Xf)^{-1}$, $Z_1 = V_{i1}$. Now suppose that $Z_1 = V_{i1}, \ldots, Z_{s-1} = V_{is-1}$, for $i = 1, \ldots, k$, but not for

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$i = k+1, \ldots, r$. Then $Y_1 = Y_{i1}, \ldots, Y_{s-1} = Y_{is-1}$, for $i = 1, \ldots, k$, but not for $i = k+1, \ldots, r$. For the sake of brevity, we shall sometimes write A for $Y_1 \ldots Y_{s-1}$ in what follows.

We have, from (2), that

$$Y_1 \ldots Y_s Y_s^{-1} \ldots Y_1^{-1} \geq Y_1 \ldots Y_n Y_n^{-1} \ldots Y_1^{-1} \geq N\theta .$$

Hence

$$A^{-1}AY_sY_s^{-1}A^{-1}A \geq A^{-1}(N\theta)A = \prod_{i=1}^{r} A^{-1}(N_i\theta)A .$$

Thus, since $Y_sY_s^{-1} \geq A^{-1}AY_sY_s^{-1}A^{-1}A$, we have

$$Y_sY_s^{-1} \geq \prod_{i=1}^{r} A^{-1}(N_i\theta)A . \tag{3}$$

If $i \leq k$, then

$$\begin{aligned} A^{-1}(N_i\theta)A &= A^{-1}\left(AY_{is} \ldots Y_{in(i)}Y_{in(i)}^{-1} \ldots Y_{is}^{-1}A^{-1}\right)A \\ &= A^{-1}AY_{is} \ldots Y_{in(i)}Y_{in(i)}^{-1} \ldots Y_{is}^{-1} . \end{aligned} \tag{4}$$

If $k < i$, then for some integer p, dependent upon i, such that $0 \leq p < s-1$, we have

$$\begin{aligned} A^{-1}(N_i\theta)A &= A^{-1}Y_1 \ldots Y_pY_{ip+1} \ldots Y_{in(i)}Y_{in(i)}^{-1} \ldots Y_{ip+1}^{-1}Y_p^{-1} \ldots Y_1^{-1}A \\ &= A^{-1}AY_{s-1}^{-1} \ldots Y_{p+1}^{-1}Y_{ip+1} \ldots Y_{in(i)}Y_{in(i)}^{-1} \ldots \\ &\qquad \ldots Y_{ip+1}^{-1}Y_{p+1} \ldots Y_{s-1} \end{aligned} \tag{5}$$

where $Y_{p+1} \neq Y_{ip+1}$. From (3), (4), (5) and condition (K) we must have $Y_s = Y_{s-1}^{-1}$ or Y_{is} (for some $i = 1, \ldots, k$). But $Y_s = Y_{s-1}^{-1}$ implies that $Z_s = Z_{s-1}^{-1}$, a contradiction. Hence $Y_s = Y_{is}$, for some $i = 1, \ldots, k$. Hence, for some i, $Y_1 = Y_{i1}, \ldots, Y_s = Y_{is}$, and

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$Z_1 = V_{i1}, \ldots, Z_s = V_{is}$. Hence, for some i , $Z_1 = V_{i1}, \ldots, Z_n = V_{in}$, and

$$Z_1 \ldots Z_n Z_n^{-1} \ldots Z_1^{-1} \geq N_i \geq N .$$

Therefore $z_1 z_2 \ldots z_n \in \Delta'(N)$, which contradicts the choice of $z_1 \ldots z_n$. Hence $\theta \circ \theta^{-1}$ is idempotent separating and θ is an isomorphism.

If, in Theorem 2.2, S is actually the free inverse semigroup F on X as described in Section 1, then we can give a reformulation of the condition (K) which is less cumbersome to apply. In F , $YY^{-1} \geq F_1 \ldots F_n$ if and only if $\Delta'(YY^{-1}) \subseteq \Delta'(F_1 \ldots F_n) = \bigcup_{j=1}^{n} \Delta'(F_j)$. Also $\Delta'(YY^{-1}) = \Delta'(Y)$ and $\Delta'(F_j) = \Delta'(Y_{j1} \ldots Y_{jn(j)})$. Thus, in F , condition (K) could have been stated as

(K') *If* $Y \in K \cup K'$ *and* $\Delta'(Y) \subseteq \bigcup_{j=1}^{n} \Delta'(Y_{j1} \ldots Y_{jn(j)})$ *for some* $Y_{jk} \in K \cup K^{-1}$ *such that* $Y_{jk} \neq Y_{jk+1}^{-1}$ *for* $k = 1, \ldots, n(j)-1$, $j = 1, \ldots, n$, *then* $Y = Y_{j1}$, *for some* j .

If we take $|K| = 1$ in Theorem 2.2, then we obtain the following simple criterion for an element in an inverse semigroup to generate the free inverse semigroup on one generator.

COROLLARY 2.4. *Let* U *be an element of the inverse semigroup* S . *Then* $\langle U \rangle$ *is the free inverse semigroup on a single generator if and only if* $UU^{-1} \not\geq U^{-m}U^{m}$ *and* $U^{-1}U \not\geq U^{n}U^{-n}$, *for any positive integers* m, n .

This corollary could also be obtained from the characterization of the θ-classes on a free inverse semigroup with one generator due to Eberhart and Seldon [2].

Once again let F denote the free inverse semigroup on X .

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COROLLARY 2.5. *Let* U *be any non-idempotent of* F . *Then* U *is a free generator of* $\langle U \rangle$.

Proof. Let $U = (A, u)$ where $u \neq 1$. For any $v \in R$, we define a function $d(v, -) : R \to N$, where N is the set of non-negative integers; for any $c \in R$, $d(v, c)$ is the largest integer n such that $c = v^n c'$ for some $c' \in G$, where $v^n c$ is in reduced form, but $c \neq v^{n+1} c''$ in reduced form. Then $d(v, c) \geq 0$, for all $v, c \in R$.

Let $a \in A$ be such that $d(u, a)$ is maximal in $\{d(u, b) : b \in A\}$. Since $u \in A$, $d(u, a) \geq 1$. Clearly $d(u, a) > d(u, b)$, for any $b \in A(u\rho)$, and so $d(u, a) > \max\{d(u, b) : b \in A(u\rho)\}$. Also $\max\{d(u, b) : b \in A(u\rho)^p\} \geq \max\{d(u, b) : b \in A(u\rho)^q\}$ for any positive integers p, q with $1 \leq p \leq q$. Hence, for any integer $m \geq 1$, $d(u, a) > \max\{d(u, b) : b \in A(u\rho)^p, p = 1, \ldots, m\}$. Thus $a \notin A(u\rho) \cup A(u\rho)^2 \cup \ldots \cup A(u\rho)^m = \Delta'(U^{-m}) = \Delta'(U^{-m}U^m)$. Therefore, $UU^{-1} \not\leq U^{-m}U^m$, for any integer $m \geq 1$. Similarly, $U^{-1}U \not\leq U^n U^{-n}$, for any integer $n \geq 1$. By Corollary 2.4, we have the desired result.

Let $\gamma : F \to G$ be such that $(A, u)\gamma = u$. Then clearly γ is an epimorphism, G is the maximal group homomorphic image of F , and $\gamma \circ \gamma^{-1}$ is the minimum group congruence. It follows easily that if $K = \{(A_i, w_i) : i \in I\}$ is a set of free generators for $\langle K \rangle$ then $W = \{w_i : i \in I\}$ is a set of free generators for $\langle W \rangle$.

If $W = \{w_i : i \in I\}$ is a subset of G and a set of free generators for $\langle W \rangle$, it is tempting to conjecture that $K = \{(I(w_i), w_i) : i \in I\}$ will be a set of free generators for $\langle K \rangle$. In general, this will not be the case. Theorem 2.2 can be tailored to this situation as follows.

PROPOSITION 2.6. *Let* $W = \{w_i : i \in I\}$ *be a non-empty subset of* G *with* W *disjoint from* W^{-1} . *Let* $K = \{W_i = (I(w_i), w_i) : i \in I\}$. *Then* K *is a set of free generators for* $\langle K \rangle$ *if and only if the following condition* (K_1) *is satisfied.*

(K_1) *For any subset* $u_1, \ldots, u_{k+1}$ *of* $W \cup W^{-1}$ *with* $u_\alpha \neq u_{\alpha+1}^{-1}$, $\alpha = 1, \ldots, k-1$, *if* $u_k \ldots u_1 \in I(U_{k+1})$ *then* $u_k = u_{k+1}$.

Proof. First suppose that K is a set of free generators. Let $u_1, \ldots, u_{k+1}$ be elements of $W \cup W^{-1}$ such that $u_\alpha \neq u_{\alpha+1}^{-1}$, $\alpha = 1, \ldots, k-1$, and $u_k \ldots u_1 \in I(u_{k+1})$ while $u_k \neq u_{k+1}$. Let k be the smallest positive integer for which there are such elements $u_1, \ldots, u_{k+1}$.

Let $U_i = (I(u_i), u_i)$, $i = 1, \ldots, k+1$. Then $U_i^{-1} = \left(I\left(u_i^{-1}\right), u_i^{-1}\right)$.

First suppose that $k = 1$. Then $u_1 \in I(u_2)$ and so $\Delta'(U_1) \subseteq \Delta'(U_2)$ and, by (K'), $U_1 = U_2$ so that $u_1 = u_2$.

Now suppose that $k \geq 2$. If there is a j such that $2 \leq j \leq k$ and $u_{j-1} \ldots u_1 \in I\left(u_j^{-1}\right)$ then this would contradict the choice of k and the u_α . Hence

$$u_k \ldots u_1 = u_1(u_2\rho)^{-1} \ldots (u_k\rho)^{-1} .$$

Thus

$$\begin{aligned} u_1 &\in I(u_{k+1})(u_k\rho) \ldots (u_2\rho) \\ &= \Delta'(U_{k+1})(u_k\rho) \ldots (u_2\rho) \\ &\subseteq \Delta'\left(U_2^{-1} \ldots U_k^{-1}U_{k+1}\right) \end{aligned}$$

and, by (K'), since $U_{k+1} \neq U_k$, $U_1 = U_2^{-1}$ which is again a contradiction. Hence condition (K_1) must be satisfied.

Now suppose that condition (K_1) is satisfied. Let $Y \in K \cup K^{-1}$ and $YY^{-1} \geq F_1 \ldots F_n$ where $F_j = Y_{j1} \ldots Y_{jn(j)}Y_{jn(j)}^{-1} \ldots Y_{j1}^{-1}$ for some

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$Y_{jk} \in K \cup K^{-1}$ such that $Y_{jk} \neq Y_{jk+1}^{-1}$, $k = 1, \ldots, n(j)-1$.

Let $Y = (I(y), y)$, for some $y \in W \cup W^{-1}$. Then $\Delta'(YY^{-1}) = \Delta'(Y) = I(y)$. Hence $y \in \Delta'(F_j)$, for some j , and so $YY^{-1} \geq F_j$, for some j . For convenience, let

$$F_j = F = Y_1 \ldots Y_n Y_n^{-1} \ldots Y_1^{-1} ,$$

where $Y_i = (I(y_i), y_i)$, $y_i \in W \cup W^{-1}$. Then

$$y \in \Delta'(Y_1) \cup \Delta'(Y_2)(y_1\rho)^{-1} \cup \ldots \cup \Delta'(Y_n)(y_{n-1}\rho)^{-1} \ldots (y_1\rho)^{-1} .$$

If $y \notin \Delta'(Y_1)$, let r be the least positive integer such that

$$y \in \Delta'(Y_r)(y_{r-1}\rho)^{-1} \ldots (y_1\rho)^{-1} .$$

Then $r \geq 2$ and

$$y(y_1\rho) \ldots (y_{r-1}\rho) \in \Delta'(Y_r) .$$

If, for any integer j such that $1 \leq j < r-1$, we have

$$y(y_1\rho) \ldots (y_j\rho) \in I(y_{j+1}) = \Delta'(Y_{j+1})$$

then

$$y \in \Delta'(Y_{j+1})(y_j\rho)^{-1} \ldots (y_1\rho)^{-1} ,$$

contradicting the choice of r . Hence

$$y_{r-1}^{-1} \ldots y_1^{-1}y = y(y_1\rho) \ldots (y_{r-1}\rho) \in \Delta'(Y_r) = I(y_r)$$

and, by (K_1), $y_{r-1}^{-1} = y_r$ which is a contradiction.

Hence $y \in \Delta'(Y_1) = I(y_1)$ and, by (K_1), $y = y_1$ and $Y = Y_1$, as required.

Let W be a subset of G satisfying the conditions of Proposition 2.4. If $w_1, \ldots, w_n \in W \cup W^{-1}$ are such that $w_i \neq w_{i+1}^{-1}$, $i = 1, \ldots, n$

then it is clear from (K_1) that $w_1 \ldots w_n \neq 1$. Hence W is a set of free generators for $\langle W \rangle$. Thus for condition (K_1) to hold it is necessary for W to be a set of free generators of $\langle W \rangle$. To see that this is not sufficient, consider the case where $X = \{x_1, x_2\}$ and $W = \{x_1, x_1x_2\}$. Then W is a set of free generators for G but, since $x_1 \in I(x_1x_2)$, the set $K = \{(\{x_1\}, x_1), (\{x_1, x_1x_2\}, x_1x_2)\}$ is not a set of free generators for $\langle K \rangle$.

We use Proposition 2.6 to show that the free inverse semigroup on two generators contains the free inverse semigroup on a countable number of generators.

COROLLARY 2.7. *Let* $X = \{a, b\}$. *Then there is a countable subset* K *of* F *such that* K *is a set of free generators for* $\langle K \rangle$.

Proof. For each positive integer m, let

$$w_m = a^m b^{-1} a b a^{-m-1}$$

and $K = \{(I(w_m), w_m) : m = 1, 2, \ldots\}$. It is well known [7] that $W = \{w_m : m = 1, 2, \ldots\}$ is a subset of a set of free generators of the derived group of G. We show that W satisfies condition (K_1).

First we note that the $(m+2)$nd term in the reduced expression by which w_m is defined and the $(m+3)$rd term in each w_m^{-1} (that is, the middle "a" or "a^{-1}") is a significant factor in the sense that the reduced form of any expression of the form

$$u_k \ldots u_1$$

where $u_\alpha \in W \cup W^{-1}$ and $u_\alpha \neq u_{\alpha+1}^{-1}$, $\alpha = 1, \ldots, k-1$, will contain the significant factor of each of $u_k, \ldots, u_1$.

Now suppose that, for some $u_1, \ldots, u_{k+1} \in W \cup W^{-1}$,

$$u_k \ldots u_1 \in I(u_{k+1})$$

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where $u_\alpha \neq u_{\alpha+1}^{-1}$, $\alpha = 1, \ldots, k-1$. Let $u_k = w_p \in W$. Then the reduced form of $u_k \ldots u_1$ has an initial segment equal to $a^p b^{-1} a$. Thus

$$u_k \ldots u_1 = a^p b^{-1} a v$$

for some $v \in G$. Now, u_{k+1} will have an initial segment equal to $a^p b^{-1} a$ if and only if $u_{k+1} = w_p$. Hence, $u_{k+1} = w_p = u_k$, as required. The case where $u_k \in W^{-1}$ is treated similarly. Thus condition (K_1) is satisfied.

Finally, we show that any two non-idempotent elements of the free inverse semigroup on a single generator will not be free generators of the inverse subsemigroup that they generate.

PROPOSITION 2.8. *Let* $X = \{x\}$. *Let* U, V *be any two elements in* F . *Then* $\{U, V\}$ *is not a set of free generators for* $H = \langle\{U, V\}\rangle$.

Proof. Since the result is immediate if either U or V is an idempotent, we assume that neither is an idempotent. Clearly U and V are free generators for H if and only if U^ε and V^δ are free generators, for any $\varepsilon, \delta \in \{1, -1\}$. Thus we can assume that $U = (A, x^m)$, $V = (B, x^n)$ where m and n are positive integers.

For any non-zero integers p, q with $p > q$ we shall write

$$[x^p, x^q] = \begin{cases} [x^p, x^{p-1}, \ldots, x^{q+1}, x^q] & \text{if either } p, q > 0 \text{ or } p, q < 0 , \\ [x^p, x^{p-1}, \ldots, x, x^{-1}, \ldots, x^q] & \text{if } q < 0 < p . \end{cases}$$

Then $A = [x^a, x^b]$ and $B = [x^c, x^d]$, for some non-zero integers a, b, c, d with $a \geq m, b$ and $c \geq n, d$. Also $b, c \leq 1$. Now

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$$A(x^n\rho) = \begin{cases} [x^{a-n},\ x^{b-n}] & \text{if } b < 0 < a-n\ , \\ [x^{a-n},\ x^{b-n-1}] & \text{if } 0 < a-n\ ,\ b = 1\ , \\ [x^{a-n-1},\ x^{b-n}] & \text{if } a-n \le 0 \text{ and } b < 0\ , \\ [x^{a-n-1},\ x^{b-n-1}] & \text{if } a-n \le 0\ ,\ b = 1\ . \end{cases}$$

We shall only carry through the argument for one of these cases since the remaining cases may be treated similarly. Let $a-n \le 0$, $b < 0$. Then

$$\begin{aligned} A(x^n\rho)(x^{-m}\rho) &= [x^{a-n-1},\ x^{b-n}](x^{-m}\rho) \\ &= \begin{cases} [x^{a-n-1+m},\ x^{b-n+m}] & \text{if } a-n-1+m < 0\ , \\ [x^{a-n+m},\ x^{b-n+m}] & \text{if } b-n+m < 0 \le a-n-1+m\ , \\ [x^{a-n+m},\ x^{b-n+m+1}] & \text{if } 0 \le b-n+m\ . \end{cases} \end{aligned}$$

Since $b < 0$ and $m \le a$, we have $b-n+m+1 \le a-n$. Thus, for $a-n \le 0$, $b < 0$ we have

$$A(x^n\rho) \cup A(x^n\rho)(x^{-m}\rho) = \begin{cases} [x^{a-n+m},\ x^{b-n}] & \text{if } a-n-1+m \ge 0\ , \\ [x^{a-n+m-1},\ x^{b-n}] & \text{if } a-n-1+m < 0\ . \end{cases}$$

Hence, for sufficiently large r ,

$$A(x^n\rho) \cup A(x^n\rho)(x^{-m}\rho) \cup \ldots \cup A(x^n\rho)(x^{-m}\rho)^r = [x^{a-n+rm},\ x^{b-n}]\ .$$

Let this set be denoted by J . Then, for sufficiently large r ,

$$\begin{aligned} \Delta'(V^{-1}U^r) &= \Delta'(V^{-1}) \cup J \\ &\supseteq \begin{cases} [x^{-1},\ x^{d-n}] \cup J\ , & \text{if either } n \ne 1 \text{ or } d \ne 1\ , \\ \{x^{-1}\} \cup J\ , & \text{if } n = d = 1 \end{cases} \\ &\supseteq [x^{a-n+rm},\ x^{d}]\ . \end{aligned}$$

In particular, if r is such that $a-n+rm > c$, then

$$\Delta'(V^{-1}U^r) \supseteq [x^c,\ x^d] = \Delta'(V)\ .$$

Since $V \ne V^{-1}$, condition (K) is not satisfied. All other possible

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orderings of a, b, m and n produce the same conclusion and so $\{U, V\}$ is not a set of free generators for $\langle\{U, V\}\rangle$.

In conclusion, we observe that if $K = \{U, V\}$ where $U = (\{x, x^2\}, x)$, $V = (\{x^{-1}, x^{-2}\}, x^{-1})$ then $\langle K\rangle$ contains three maximal idempotents, namely, $UU^{-1} = (\{x, x^2\}, 1)$, $U^{-1}U = V^{-1}V = (\{x, x^{-1}\}, 1)$ and $VV^{-1} = (\{x^{-1}, x^{-2}\}, 1)$. Thus not only are U and V not free generators for $\langle K\rangle$ but $\langle K\rangle$ is not a free inverse subsemigroup of F .

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