Glasgow Math. J. **47A** (2005) 99–107. © 2005 Glasgow Mathematical Journal Trust.
doi:10.1017/S0017089505002326. Printed in the United Kingdom

SOLUTIONS OF A DERIVATIVE NONLINEAR SCHRÖDINGER HIERARCHY AND ITS SIMILARITY REDUCTION

SABURO KAKEI
Department of Mathematics, Rikkyo University, Nishi-ikebukuro, Toshima-ku, Tokyo 171-8501, Japan
e-mail: kakei@rkmath.rikkyo.ac.jp

and TETSUYA KIKUCHI
Mathematical Institute, Tohoku University, Aoba, Sendai 980-8578, Japan
e-mail: tkikuchi@math.tohoku.ac.jp

(Received 30 November, 2003; accepted 27 February, 2004)

Abstract. The hierarchy structure of a derivative nonlinear Schrödinger equation is investigated in terms of the Sato-Segal-Wilson formulation. Special solutions are constructed as ratios of Wronski determinants. Relations to the Painlevé IV and the discrete Painlevé I are discussed by applying a similarity reduction.

2000 *Mathematics Subject Classification.* 35Q51, 35Q55, 37K10.

1. Introduction. The nonlinear Schrödinger (NLS) equation,

$$\mathrm{i}q_T = \frac{1}{2}q_{XX} + 4|q|^2 q, \tag{1.1}$$

is one of the most celebrated examples of soliton equations. (The subscripts denote partial differentiation with respect to the corresponding variables.) There are several modified versions of the NLS equation that have integrability properties. Among those equations, we consider a derivative NLS (∂NLS) equation [**1**, **2**],

$$\mathrm{i}q_T = \frac{1}{2}q_{XX} + 2\mathrm{i}q^2\bar{q}_X + 4|q|^4 q. \tag{1.2}$$

Hereafter we will forget the complex structure of (1.2) and consider nonlinear coupled equations,

$$\begin{cases} q_{t_2} = \frac{1}{2}q_{t_1t_1} - 2q^2 r_{t_1} - 4q^3 r^2, \\ r_{t_2} = -\frac{1}{2}r_{t_1t_1} - 2r^2 q_{t_1} + 4r^3 q^2. \end{cases} \tag{1.3}$$

We note that (1.3) is reduced to (1.2) under the condition $r = \bar{q}$, $X = \mathrm{i}t_1$, $T = \mathrm{i}t_2$.

Recently the authors developed a generalization of the Drinfel'd-Sokolov hierarchy from the viewpoint of affine Lie groups [**3**, **4**]. The ∂NLS equation (1.3) is a typical example of the equations that can be treated using our approach. The aim of the present article is to give a method of constructing determinant solutions for the ∂NLS hierarchy without using affine Lie groups. Since the ∂NLS equation is related to the fourth Painlevé equation [1, 4],

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{2y}\left(\frac{\partial y}{\partial x}\right)^2 + \frac{3}{2}y^3 + 4xy^2 + 2(x^2 - A)y + \frac{B}{y}, \tag{1.4}$$

our method provides a class of solutions also to the Painlevé IV equation (1.4). Furthermore, we can show that the symmetry of the affine Weyl group $W(A_1^{(1)})$ can lead to the following difference equation,

$$X_{n-1} + X_n + X_{n+1} = -2x + \frac{\kappa_1 n + \kappa_2 + (-1)^n \kappa_3}{X_n}, \tag{1.5}$$

where κ_1, κ_2, κ_3 are constants. We note that equation (1.5) is referred to as the asymmetric discrete Painlevé I in [5], and as discrete Painlevé II in [6].

2. Hierarchy structure. We first describe the hierarchy structure associated with the ∂NLS equation (1.2). We introduce a formal series ("formal Baker-Akhiezer function"),

$$\boldsymbol{\Psi}(\lambda) = \boldsymbol{W}(\lambda)\boldsymbol{\Psi}_0(\lambda), \tag{2.1}$$

$$\boldsymbol{W}(\lambda) = \sum_{n=0}^{\infty} \boldsymbol{w}_n(t_1, t_2, \ldots)\lambda^{-n}, \tag{2.2}$$

$$\boldsymbol{\Psi}_0(\lambda) = \begin{bmatrix} e^{\lambda t_1 + \lambda^2 t_2 + \cdots} & 0 \\ 0 & e^{-(\lambda t_1 + \lambda^2 t_2 + \cdots)} \end{bmatrix}, \tag{2.3}$$

with $\boldsymbol{w}_n(t_1, t_2, \ldots)$ being (2×2)-matrix-valued functions:

$$\boldsymbol{w}_n(t_1, t_2, \ldots) = \begin{bmatrix} w_n^{(11)}(t_1, t_2, \ldots) & w_n^{(12)}(t_1, t_2, \ldots) \\ w_n^{(21)}(t_1, t_2, \ldots) & w_n^{(22)}(t_1, t_2, \ldots) \end{bmatrix}. \tag{2.4}$$

We assume that the 0-th coefficient $\boldsymbol{w}_0(t_1, t_2, \ldots)$ is of the form,

$$\boldsymbol{w}_0(t_1, t_2, \ldots) = \begin{bmatrix} 1 & 0 \\ w_0^{(21)}(t_1, t_2, \ldots) & 1 \end{bmatrix}. \tag{2.5}$$

A hierarchy associated with (1.3) is defined by the following evolution equations:

$$\frac{\partial \boldsymbol{\Psi}(\lambda)}{\partial t_n} = \boldsymbol{B}_n(\lambda)\boldsymbol{\Psi}(\lambda), \tag{2.6}$$

$$\boldsymbol{B}_n(\lambda) = \left[\lambda^n \boldsymbol{W}(\lambda)\, \boldsymbol{Q} \boldsymbol{W}(\lambda)^{-1}\right]_{\geq 0}, \qquad \boldsymbol{Q} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{2.7}$$

where we have used the notation $[\sum_{n\in\mathbb{Z}} a_n\lambda^n]_{\geq 0} = \sum_{n\geq 0} a_n\lambda^n$. The compatibility conditions for (2.6) give rise to the zero-curvature (or Zakharov-Shabat) equations,

$$\frac{\partial \boldsymbol{B}_m}{\partial t_n} - \frac{\partial \boldsymbol{B}_n}{\partial t_m} + [\boldsymbol{B}_m, \boldsymbol{B}_n] = 0, \qquad m, n = 1, 2, \ldots, \tag{2.8}$$

which gives a hierarchy of soliton equations.

By straightforward calculations, we can obtain expressions for $\boldsymbol{B}_1(\lambda)$ and $\boldsymbol{B}_2(\lambda)$:

$$\boldsymbol{B}_1(\lambda) = \begin{bmatrix} 1 & 0 \\ 2r & -1 \end{bmatrix}\lambda + \begin{bmatrix} 2qr & -2q \\ 0 & -2qr \end{bmatrix}, \tag{2.9}$$

$$\begin{aligned} \boldsymbol{B}_2(\lambda) &= \begin{bmatrix} 1 & 0 \\ 2r & -1 \end{bmatrix}\lambda^2 + \begin{bmatrix} 2qr & -2q \\ -r_{t_1} & -2qr \end{bmatrix}\lambda \\ &\quad + \begin{bmatrix} q_{t_1}r - qr_{t_1} - 2q^2r^2 & -q_{t_1} \\ 0 & qr_{t_1} - q_{t_1}r + 2q^2r^2 \end{bmatrix}, \end{aligned} \tag{2.10}$$

where $q = w_1^{(12)}$ and $r = w_0^{(21)}$. These matrices gives a Lax pair for the ∂NLS equation, i.e., the zero-curvature equation (2.8) with $m = 1, n = 2$ gives the ∂NLS equation (1.2). From this fact, we refer to the family of the nonlinear equations induced by (2.8) as the ∂NLS hierarchy.

3. Wronskian solutions. We apply a modified version of Date's method [**7**, **8**] to construct a special class of solutions for (2.6), which we shall seek in the form

$$\widetilde{\boldsymbol{\Psi}}(\lambda) = \widetilde{\boldsymbol{W}}_N(\lambda)\boldsymbol{\Psi}_0(\lambda), \tag{3.1}$$
$$\widetilde{\boldsymbol{W}}_N(\lambda) = \tilde{\boldsymbol{w}}_0 + \tilde{\boldsymbol{w}}_1\lambda^{-1} + \cdots + \tilde{\boldsymbol{w}}_N\lambda^{-N} \tag{3.2}$$

with $\tilde{\boldsymbol{w}}_n = \tilde{\boldsymbol{w}}_n(t_1, t_2, \ldots)$ being (2×2)-matrix-valued functions. The 0-th coefficient $\tilde{\boldsymbol{w}}_0$ and the N-th coefficient $\tilde{\boldsymbol{w}}_N$ are assumed to be of the form,

$$\tilde{\boldsymbol{w}}_0 = \begin{bmatrix} 1 & 0 \\ \tilde{w}_0^{(21)} & 1 \end{bmatrix}, \quad \tilde{\boldsymbol{w}}_N = \begin{bmatrix} \tilde{w}_N^{(11)} & \tilde{w}_N^{(12)} \\ 0 & \tilde{w}_N^{(22)} \end{bmatrix}. \tag{3.3}$$

As the data for the solution constructed below, let us consider a formal series

$$\boldsymbol{\Xi}(\lambda, t) = \sum_{j \in \mathbb{Z}} \boldsymbol{\xi}_j(t)\lambda^{-j}, \tag{3.4}$$

where $\boldsymbol{\xi}_j(t) = \boldsymbol{\xi}_j(t_1, t_2, \ldots)$ are $(2 \times 2N)$-matrix-valued functions of the form,

$$\boldsymbol{\xi}_j(t) = \begin{bmatrix} f_1^{(j)}(t) & \cdots & f_{2N}^{(j)}(t) \\ g_1^{(j)}(t) & \cdots & g_{2N}^{(j)}(t) \end{bmatrix}. \tag{3.5}$$

Here we assume

$$\det \begin{bmatrix} f_1^{(0)} & \cdots & f_1^{(N-1)} & g_1^{(0)} & \cdots & g_1^{(N-1)} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ f_{2N}^{(0)} & \cdots & f_{2N}^{(N-1)} & g_{2N}^{(0)} & \cdots & g_{2N}^{(N-1)} \end{bmatrix} \neq 0. \tag{3.6}$$

We furthermore impose the following conditions for $\boldsymbol{\Xi}(\lambda, t)$:

$$\frac{\partial}{\partial t_n}\boldsymbol{\Xi}(\lambda, t) = \lambda^n \boldsymbol{Q}\boldsymbol{\Xi}(\lambda, t) + \boldsymbol{\Xi}(\lambda, t)\boldsymbol{\beta}_n \quad (n = 1, 2, \ldots), \tag{3.7}$$
$$\lambda\boldsymbol{\Xi}(\lambda, t) = \boldsymbol{\Xi}(\lambda, t)\boldsymbol{\gamma}, \tag{3.8}$$

where $\boldsymbol{\beta}_n$, $\boldsymbol{\gamma}$ are $(2N \times 2N)$-matrices.

The polynomial (3.2) is characterized uniquely by the linear equation,

$$\oint \frac{d\lambda}{2\pi i}\lambda^{N-1}\widetilde{\boldsymbol{W}}_N(\lambda)\Xi(\lambda) = 0. \tag{3.9}$$

Solving (3.9) by Cramer's formula, we have

$$\tilde{w}_0^{(21)}(t) = (-1)^N \frac{|0,1,\ldots,N-2,N-1,N\,;\,0,1,\ldots,N-2|}{|0,1,\ldots,N-2,N-1\,;\,0,1,\ldots,N-2,N-1|}, \tag{3.10}$$

$$\tilde{w}_1^{(12)}(t) = (-1)^{N+1} \frac{|1,2,\ldots,N-1\,;\,0,1,\ldots,N-2,N-1,N|}{|1,2,\ldots,N-1,N\,;\,0,1,\ldots,N-2,N-1|}, \tag{3.11}$$

where we have used the notation due to Freeman and Nimmo [**9**, **10**]:

$$|k_1,\ldots,k_m;l_1,\ldots,l_n| \stackrel{\text{def}}{=} \begin{vmatrix} f_1^{(k_1)} & \cdots & f_1^{(k_m)} & g_1^{(l_1)} & \cdots & g_1^{(l_n)} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ f_{2N}^{(k_1)} & \cdots & f_{2N}^{(k_m)} & g_{2N}^{(l_1)} & \cdots & g_{2N}^{(l_n)} \end{vmatrix}. \tag{3.12}$$

PROPOSITION 1. *If* $\widetilde{\boldsymbol{W}}_N(\lambda)$ *satisfies* (3.9), *then the corresponding Baker-Akhiezer function* (3.1) *solves the linear equations* (2.4).

Proof. From (3.8), we obtain

$$\oint \frac{d\lambda}{2\pi i}\lambda^{N+n-1}\widetilde{\boldsymbol{W}}_N(\lambda)\Xi(\lambda) = 0, \tag{3.13}$$

for any non-negative integer n. Differentiating (3.9) with respect to t_n and applying (3.7), we have

$$\oint \frac{d\lambda}{2\pi i}\lambda^{N-1}\left\{\frac{\partial \widetilde{\boldsymbol{W}}_N(\lambda)}{\partial t_n} + \lambda^n \widetilde{\boldsymbol{W}}_N(\lambda)\boldsymbol{Q}\right\}\Xi(\lambda) = 0. \tag{3.14}$$

There exist polynomials $\boldsymbol{B}_n(\lambda)$ and $\boldsymbol{R}(\lambda)$ such that

$$\frac{\partial \widetilde{\boldsymbol{W}}_N(\lambda)}{\partial t_n} + \lambda^n \widetilde{\boldsymbol{W}}_N(\lambda)\boldsymbol{Q} = \boldsymbol{B}_n(\lambda)\widetilde{\boldsymbol{W}}_N(\lambda) + \boldsymbol{R}(\lambda), \tag{3.15}$$

where the degree of $\boldsymbol{R}(\lambda)$ is at most $N-1$. In view of (3.13) and (3.14), we obtain $\oint \boldsymbol{R}(\lambda)\Xi(\lambda)\,d\lambda = 0$. The condition (3.6) implies $\boldsymbol{R}(\lambda) = 0$ and that $\widetilde{\boldsymbol{W}}_N(\lambda)$ satisfies

$$\frac{\partial \widetilde{\boldsymbol{W}}_N(\lambda)}{\partial t_n} = \boldsymbol{B}_n(\lambda)\widetilde{\boldsymbol{W}}_N(\lambda) - \lambda^n \widetilde{\boldsymbol{W}}_N(\lambda)\boldsymbol{Q}, \tag{3.16}$$

$$\boldsymbol{B}_n(\lambda) = \left[\lambda^n \widetilde{\boldsymbol{W}}_N(\lambda)\boldsymbol{Q}\widetilde{\boldsymbol{W}}_N(\lambda)^{-1}\right]_{\geq 0}, \tag{3.17}$$

which gives the desired result. □

Thus we can obtain special solutions of the ∂NLS equation (1.3) by setting $q(t) = \tilde{w}_1^{(12)}(t)$ and $r(t) = \tilde{w}_0^{(21)}(t)$. The solutions obtained are expressed in terms of "double Wronskians" (3.12). We remark that the double Wronskian solutions for the ∂NLS equation have been obtained in [**11**] by using Hirota's bilinear formulation.

4. Affine Weyl group symmetry. Let $\widetilde{W} = \langle s_0, s_1, \pi \rangle$ be the extended affine Weyl group of $A_1^{(1)}$-type. The generators s_0, s_1 and π obey the relations $s_0^2 = s_1^2 = 1$, $\pi s_0 = s_1 \pi$. A realization of $\widetilde{W}$ is given by (2×2)-matrices as follows:

$$s_0 \mapsto \boldsymbol{S}_0 \overset{\text{def}}{=} \begin{bmatrix} 0 & \lambda^{-1} \\ \lambda & 0 \end{bmatrix}, \quad s_1 \mapsto \boldsymbol{S}_1 \overset{\text{def}}{=} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \pi \mapsto \boldsymbol{\Pi} \overset{\text{def}}{=} \begin{bmatrix} 0 & 1 \\ \lambda & 0 \end{bmatrix}. \tag{4.1}$$

We define an action of $\widetilde{W}$ on $\Xi(\lambda, t)$ of 3.4:

$$s_0 : \Xi(\lambda, t) \mapsto \boldsymbol{S}_0 \Xi(\lambda, -t) \tag{4.2}$$

$$s_1 : \Xi(\lambda, t) \mapsto \boldsymbol{S}_1 \Xi(\lambda, -t) \tag{4.3}$$

$$\pi : \Xi(\lambda, t) \mapsto \boldsymbol{\Pi} \Xi(\lambda, -t) \tag{4.4}$$

Note that the form of the differential relation (3.7) is unchanged under the transformations above. These transformations on $\Xi(\lambda, t)$ induce an action of $\widetilde{W}$ on $\widetilde{\boldsymbol{W}}_N$ though the relation (3.9).

PROPOSITION 2. *The action of $\widetilde{W}$ on the variables $q(t)$, $r(t)$ are written as follows:*

$$s_0 : q(t) \mapsto \frac{1}{q(-t)}, \quad r(t) \mapsto -q(-t)^2 r(-t) + \frac{1}{2} q_{t_1}(-t), \tag{4.5}$$

$$s_1 : q(t) \mapsto -q(-t) r(-t)^2 - \frac{1}{2} r_{t_1}(-t), \quad r(t) \mapsto \frac{1}{r(-t)}, \tag{4.6}$$

$$\pi : q(t) \mapsto r(-t), \quad r(t) \mapsto q(-t). \tag{4.7}$$

Proof. Here we prove the s_0-case only. The cases of s_1 and π can be proved in similar manner.

Define $\hat{\boldsymbol{W}}_N(\lambda, t)$ as

$$\hat{\boldsymbol{W}}_N(\lambda, t) \overset{\text{def}}{=} \begin{bmatrix} 1/\tilde{w}_1^{(12)}(-t) & 0 \\ -\lambda & -\tilde{w}_1^{(12)}(-t) \end{bmatrix} \widetilde{\boldsymbol{W}}_N(\lambda, -t) \boldsymbol{S}_0. \tag{4.8}$$

It may be shown in a straightforward way that

$$\oint \frac{d\lambda}{2\pi i} \lambda^{N-1} \hat{\boldsymbol{W}}_N(\lambda) \boldsymbol{S}_0 \Xi(\lambda, -t) = 0, \tag{4.9}$$

and that $\hat{\boldsymbol{W}}_N(\lambda, t)$ is of the form,

$$\hat{\boldsymbol{W}}_N(\lambda, t) = \begin{bmatrix} 1 & 0 \\ \left(\tilde{w}_1^{(12)}\right)_{t_1} \Big/ 2 - \left(\tilde{w}_1^{(12)}\right)^2 \tilde{w}_0^{(21)} & 1 \end{bmatrix} + \begin{bmatrix} * & 1/\tilde{w}_1^{(12)} \\ * & * \end{bmatrix} \lambda^{-1} + \cdots. \tag{4.10}$$

By the same line of argument as Proposition 1, one can show that $\hat{\boldsymbol{\Psi}}(\lambda, t) \overset{\text{def}}{=} \hat{\boldsymbol{W}}_N(\lambda, t) \boldsymbol{\Psi}_0(\lambda, t)$ solves the linear equations (2.6). Namely, the transformation (4.5) maps a solution to another one. □

REMARK. The transformations (4.5), (4.6) are different from s_0^{L}, s_1^{L} of **[4]** because of different choice of $\boldsymbol{S}_0$ and $\boldsymbol{S}_1$.

https://doi.org/10.1017/S0017089505002326 Published online by Cambridge University Press

5. Similarity reduction to Painlevé IV. We consider the following condition on $\boldsymbol{W}(\lambda, t)$ of 2.2:

$$\boldsymbol{W}(k\lambda, t) = k^{\alpha \boldsymbol{Q}} \boldsymbol{W}(\lambda, \tilde{t}) k^{-\alpha \boldsymbol{Q}}, \quad \tilde{t} = (kt_1, k^2 t_2, k^3 t_3, \ldots). \tag{5.1}$$

This induces the similarity conditions for $q(t)$, $r(t)$:

$$q(\tilde{t}) = k^{-1-2\alpha} q(t), \quad r(\tilde{t}) = k^{2\alpha} r(t). \tag{5.2}$$

Proposition 3. *Define $y(x)$ by*

$$y(x) \stackrel{\text{def}}{=} 2q(t)r(t)|_{t_1=x,\, t_2=1/2,\, t_3=t_4=\cdots=0}\,. \tag{5.3}$$

Under the condition (5.2), *$y(x)$ satisfies the Painlevé IV equation* (1.4) *with $A = 4\alpha + 3C + 1$, $B = -2C^2$, where C is an integration constant.*

Proof. Differentiating (5.2) with respect to k and setting $k = 1$, we have

$$t_1 q_{t_1}(t) + 2t_2 q_{t_2}(t) + \cdots = -(1+2\alpha)q(t), \quad t_1 r_{t_1}(t) + 2t_2 r_{t_2}(t) + \cdots = 2\alpha r(t). \tag{5.4}$$

We introduce another dependent variable $\varphi(x)$ defined by

$$\varphi(x) \stackrel{\text{def}}{=} \left\{q_{t_1}(t)r(t) - q(t)r_{t_1}(t)\right\}\big|_{t_1=x,\, t_2=1/2,\, t_3=t_4=\cdots=0}\,. \tag{5.5}$$

From these relations and (1.3), we can show that the variables $y(x)$ and $\varphi(x)$ satisfy the following relations:

$$\varphi - \frac{1}{2}y^2 + xy = C, \tag{5.6}$$

$$\frac{y''}{2y} - \left(\frac{y'}{2y}\right)^2 + \left(\frac{\varphi}{y}\right)^2 + 2\varphi - 2y^2 + \frac{2x\varphi}{y} + 1 + 4\alpha = 0, \tag{5.7}$$

where $' = d/dx$ and C is an integration constant. Eliminating φ, we obtain the Painlevé IV (1.4). $\square$

The Weyl group action still works under the conditions (5.2).

Lemma 1. *The Weyl group action on the parameters α, C are given by*

$$s_0 : \alpha \mapsto -\alpha - 1, \quad C \mapsto -C + 2\alpha + 1, \tag{5.8}$$

$$s_1 : \alpha \mapsto -\alpha, \quad C \mapsto -C + 2\alpha, \tag{5.9}$$

$$\pi : \alpha \mapsto -\alpha - \frac{1}{2}, \quad C \mapsto C. \tag{5.10}$$

Proof. Define $q_0 = s_0(q)$, $r_0 = s_0(r)$, $\alpha_0 = s_0(\alpha)$, $C_0 = s_0(C)$, which satisfy the relations,

$$\begin{gathered} q_0(\tilde{t}) = k^{-1-2\alpha_0} q_0(t), \quad r_0(\tilde{t}) = k^{2\alpha_0} r_0(t), \\ C_0 = q_0' r_0 - q_0 r_0' - \frac{1}{2} q_0^2 r_0^2 + x q_0 r_0. \end{gathered} \tag{5.11}$$

Inserting (4.5) into the relations above, we can obtain (5.8). The actions of s_1 and π can be proved in a similar way. □

We focus attention on the action of the translation $T \stackrel{\text{def}}{=} s_0\pi \in \tilde{W}$. Using the formulas (4.5) and (4.7), we can write down the action of T on $y(x)$:

$$T(y) = -y - \frac{r'}{r}, \quad T^{-1}(y) = -y + \frac{q'}{q}. \tag{5.12}$$

Using these equations together with (5.6), we obtain

$$T(y) + y + T^{-1}(y) = -2x + \frac{2C}{y}. \tag{5.13}$$

PROPOSITION 4. *If we define $X_n = T^n(y)$, then X_n solves the difference equation* (1.5), *where the parameters $\kappa_1, \kappa_2, \kappa_3$ are given by*

$$\kappa_1 = \frac{1}{2}, \quad \kappa_2 = -\alpha - \frac{1}{4}, \quad \kappa_3 = C - \alpha - \frac{1}{4}. \tag{5.14}$$

Proof. It follows from Lemma 1 that

$$T^{n+1}(\alpha) = T^n(\alpha) - \frac{1}{2}, \quad T^{n+1}(C) = -T^n(C) - 2T^n(\alpha). \tag{5.15}$$

Solving these equations, one obtains

$$T^n(\alpha) = \alpha - \frac{n}{2}, \quad T^n(C) = \frac{n}{2} - \alpha - \frac{1}{4} + (-1)^n\left(C - \alpha - \frac{1}{4}\right). \tag{5.16}$$

Applying T^n to (5.13) and using (5.16), we obtain the equation (1.5) as desired. □

6. Similarity reduction of Wronskian solutions. We shall consider when the special solution in Section 3 have the similarity property (5.2). First we define the Euler operator $\hat{E}$ as

$$\hat{E} \stackrel{\text{def}}{=} t_1\frac{\partial}{\partial t_1} + 2t_2\frac{\partial}{\partial t_2} + 3t_3\frac{\partial}{\partial t_3} + \cdots. \tag{6.1}$$

PROPOSITION 5. *If the data matrix $\Xi(\lambda, t)$ of* (3.4) *satisfies the relation*

$$\left(-\lambda\frac{\partial}{\partial\lambda} + \hat{E} + \alpha\boldsymbol{Q}\right)\Xi(\lambda, t) = \Xi(\lambda, t)\boldsymbol{\Gamma}, \tag{6.2}$$

where $\boldsymbol{\Gamma}$ is a $(2N \times 2N)$-matrix, then the corresponding solution $\widetilde{\boldsymbol{W}}_N(\lambda)$ obeys the condition (5.1).

Proof. Applying $\lambda\partial/\partial\lambda - \hat{E}$ to (3.9) and using (3.13), (6.2), we have

$$\oint \frac{\mathrm{d}\lambda}{2\pi\mathrm{i}}\lambda^{N-1}\left\{\lambda\frac{\partial\widetilde{\boldsymbol{W}}_N(\lambda)}{\partial\lambda} - \hat{E}\widetilde{\boldsymbol{W}}_N(\lambda) + \alpha\widetilde{\boldsymbol{W}}_N(\lambda)\boldsymbol{Q}\right\}\Xi(\lambda) = 0, \tag{6.3}$$

https://doi.org/10.1017/S0017089505002326 Published online by Cambridge University Press

which leads to

$$\lambda \frac{\partial \widetilde{\boldsymbol{W}}_N(\lambda)}{\partial \lambda} - \hat{E}\widetilde{\boldsymbol{W}}_N(\lambda) - \alpha \left[\boldsymbol{Q}, \widetilde{\boldsymbol{W}}_N(\lambda) \right] = 0. \tag{6.4}$$

Integrating (6.4), we obtain the desired result. □

To show an example of the data $\Xi(\lambda, t)$ that satisfies (6.2), we prepare the elementary Schur polynomials $p_n(t)$:

$$\exp(zt_1 + z^2 t_2 + \cdots) = \sum_{n\in\mathbb{Z}} p_n(t) z^n. \tag{6.5}$$

If we choose $f_k^{(j)}$, $g_k^{(j)}$ of (3.5) as

$$f_k^{(j)} = p_{k-j-1}(t), \quad g_k^{(j)} = p_{k-j-1}(-t) \quad (k = 1, \ldots, 2N), \tag{6.6}$$

then $\Xi(\lambda, t)$ satisfies (3.7), (3.8) with $\boldsymbol{\beta}_n = 0$, $\boldsymbol{\gamma} = [\delta_{i+1,j}]_{1\le i,j\le 2N}$, and simultaneously obeys (6.2) with $\alpha = 0$, $\boldsymbol{\Gamma} = \mathrm{diag}[0, 1, \ldots, 2N-1]$. Putting $t_1 = x$, $t_2 = 1/2$, $t_3 = t_4 = \cdots = 0$, we thus obtain a class of rational solutions for the Painlevé IV equation (1.4) and the discrete Painlevé equation (1.5).

In this case, the Schur polynomials $p_n(t)$ degenerate to the Hermite polynomials $H_n(t)$:

$$\exp(zt_1 + z^2 t_2 + \cdots)\big|_{t_1=x,\, t_2=1/2,\, t_3=t_4=\cdots=0} = \exp(xz + z^2/2) = \sum_{n\in\mathbb{Z}} H_n(t) z^n. \tag{6.7}$$

We remark that the rational solutions for the discrete Painlevé I constructed in **[12]** are essentially the same as the above.

7. Concluding remarks. We have formulated the hierarchy of the ∂NLS equation and constructed solutions expressed in terms of determinants. The Weyl group symmetry introduced in this article is isomorphic to $\tilde{W}(A_1^{(1)})$, which does not seems to be a subgroup of the $\tilde{W}(A_2^{(1)})$-symmetry discussed in **[13, 14]**. To understand the relationship of our $\tilde{W}(A_1^{(1)})$-symmetry to the whole symmetry of the Painlevé IV, it seems that we need to consider a larger group that contain both $\tilde{W}(A_1^{(1)})$ and $\tilde{W}(A_2^{(1)})$ as individual subgroups.

Acknowledgments. The authors would like to thank Professors B. Grammaticos, K. Hasegawa, T. Ikeda, G. Kuroki, K. Takasaki, T. Tsuchida, K. Ueno, and R. Willox for their interests and discussions. The first author is partially supported by Rikkyo University Special Fund for Research, and by the Grant-in-Aid for Scientific Research (No. 14740123) from the Ministry of Education, Culture, Sports, Science and Technology. The second author is partially supported by the 21st Century COE program of Tohoku University: Exploring New Science by Bridging Particle-Matter Hierarchy.

REFERENCES

1. M. J. Ablowitz, A. Ramani and H. Segur, A connection between nonlinear evolution equations and ordinary differential equations of P-type. II, *J. Math. Phys.* **21** (1980), 1006–1015.

2. V. S. Gerdjikov and M. I. Ivanov, A quadratic pencil of general type and nonlinear evolution equations. II. Hierarchies of Hamiltonian structures (Russian), *Bulg. J. Phys.* **10** (1983), 130–143.

3. V. G. Drinfel'd and V. V. Sokolov, Lie algebras and equations of Korteweg-de Vries type, *J. Sov. Math.* **30** (1985), 1975–2036.

4. S. Kakei and T. Kikuchi, Lie algebraic approach to a hierarchy of a derivative nonlinear Schrödinger equation, *Int. Math. Res. Not. 2004*: **78** (2004), 4181–4209.

5. B. Grammaticos and A. Ramani, From continuous Painlevé IV to the asymmetric discrete Painlevé I, *J. Phys.* **A31** (1998), 5787–579.

6. M. Noumi and Y. Yamada, Affine Weyl groups, discrete dynamical systems and Painlevé equations, *Commun. Math. Phys.* **199** (1998), 281–295.

7. E. Date, Multisoliton solutions and quasiperiodic solutions of nonlinear equations of sine-Gordon type, *Osaka J. Math.* **19** (1982), 125–158.

8. S. Kakei, T. Ikeda and K. Takasaki, Hierarchy of $(2+1)$-dimensional nonlinear Schrödinger equation, self-dual Yang-Mills equation, and toroidal Lie algebras, *Annales Henri Poincaré* **3** (2002), 817–845.

9. N. C. Freeman and J. J. C. Nimmo, Soliton solutions of the Korteweg-de Vries and Kadomtsev-Petviashvili equations: the wronskian technique, *Phys. Lett.* **95A** (1983), 1–3.

10. N. C. Freeman, Soliton solution of nonlinear evolution equations, *IMA J. Appl. Math.* **32** (1984), 125–145.

11. S. Kakei, N. Sasa and J. Satsuma, Bilinearization of a generalized derivative nonlinear Schrödinger equation, *J. Phys. Soc. Japan* **64** (1995), 1519–1523.

12. Y. Ohta, K. Kajiwara and J. Satsuma, Bilinear structure and exact solutions of the discrete Painlevé I equation, *CRM Proceedings and Lecture Notes* **9** (1996), 265–268.

13. K. Okamoto, Studies on Painlevé equations III, Second and fourth Painlevé equations P_{II} and P_{IV}, *Math. Ann.* **275** (1986), 221–255.

14. M. Noumi and Y. Yamada, Symmetries in the forth Painlevé equations and Okamoto polynomials, *Nagoya Math. J.* **153** (1999), 53–86.