WAVE TRANSPORT IN STRATIFIED, ROTATING FLUIDS

M. E. McIntyre
Department of Applied Mathematics and Theoretical Physics,
University of Cambridge

SUMMARY

Momentum and energy transport by buoyancy-Coriolis waves is illustrated by means of a simple model example. The need for careful consideration of a complete problem for mean-flow evolution is emphasised, especially when moving media are involved. Then a recent generalisation of the wave-action and pseudomomentum concepts is introduced, and used to exhibit in a very general way the roles of wave dissipation, forcing, or transience in the mean flow problem, for a certain class of 'nearly-unidirectional' mean flows. This class includes differentially-rotating stellar interiors, which could well be systematically changed by wave transport of angular momentum. Similar results hold for MHD and self-gravitating fluids. Finally the physical distinction between momentum and pseudomomentum is discussed.

1. INTRODUCTION

Some of the most spectacular natural manifestations of wave transport effects are those believed on the basis of recent evidence to occur in the stratospheres of Earth and Venus[1-4]. Closely analogous effects appear likely to influence the evolution of the rotation of stellar interiors[5], and to be important in other astrophysical contexts[6]. They are often associated with rather complicated kinds of low-frequency fluid-dynamical waves, in which buoyancy and Coriolis forces are essential. The waves set up a 'radiation stress' whereby the mean azimuthal velocity at one height and latitude can undergo systematic acceleration at the expense of a corresponding deceleration at a more or less distant location. Thus transport of angular momentum by the waves is involved. This transport can result in drastic changes to the pattern of differential rotation (which in turn can drastically affect the wave propagation and lead to some interesting feedback effects[7]).

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An idealisation illustrating this kind of 'radiation stress' phenomenon is the model problem suggested in figure 1. Inertio-gravity waves, which are the simplest type of buoyancy-Coriolis wave, are being generated by a slippery, corrugated boundary moving parallel to itself with constant velocity c:

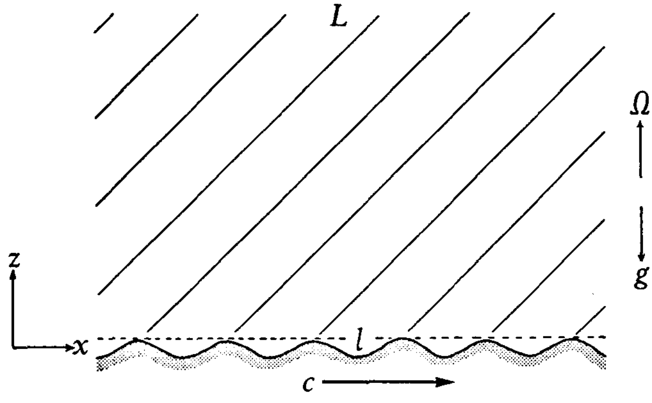

FIGURE 1. Inertio-gravity waves being generated in a stably-stratified fluid (specific entropy increasing upward) by a rigidly-moving boundary. The frame of reference is rotating with constant angular velocity $\underset{\sim}{\Omega} = (0,0,\Omega)$ and the effective gravity $\underset{\sim}{g} = (0,0,g)$ = constant. (It can be shown that the waves have their crests or lines of constant phase sloping forward, unlike the sound waves which might be generated if the boundary moved much faster.)

Here the Cartesian x direction plays the role of the azimuthal direction, and the mean state is assumed independent of x. The mean pressure gradient has no x-component; thus the fluid is free to accelerate in the x direction in response to the radiation stress.

If the waves are being dissipated in some layer L at the top of the picture, there is a systematic tendency for the mean flow to accelerate there. So the wave-drag force which the boundary exerts on the fluid is not felt at the boundary, as far as the mean flow is concerned; it is felt at L. This is a typical radiation-stress effect.

If the waves were generated not at a boundary but by a moving system of heat sources and sinks in some layer in the interior of the fluid, then total momentum would be constant, and the mean acceleration at levels where the waves are dissipated will be accompanied by a corresponding deceleration where they are generated[3,4]. The close connection between mean flow changes and wave dissipation or forcing can be verified by detailed solution of the appropriate sets of equations, but is not usually obvious from the equations themselves.

Where the waves are dissipated will depend not only on the physics of

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the dissipative processes but also upon the solution of the wave propagation problem for the particular mean-flow profile involved. Other things being equal, we usually get enhanced dissipation of the waves in places, if any, where their intrinsic frequency (i.e. frequency in a frame of reference moving with the local mean flow $\bar{u}$) is Doppler shifted towards zero - that is, we tend to get enhanced dissipation near an actual or virtual 'critical line'[8] $\bar{u}(y,z) = c$.

In this review I shall pay particular but not exclusive attention to the class of problems exemplified by figure 1. Their characteristic feature is the existence of a coordinate x (cartesian or curvilinear) such that mean quantities are independent of x, and 'mean' can be defined as an average with respect to x. Such problems, which I shall call 'longitudinally symmetric' happen to comprise an area of recent advances, and also serve to illustrate some of the subtleties and pitfalls which can arise in thinking in general terms about the transport of conservable quantities such as energy and momentum by waves in material media, and most particularly waves in moving media. (We are, of course, dealing with moving media <u>par excellence</u> as soon as Coriolis forces are relevant.) In section 2 some of these points are illustrated by describing in more detail what happens in the problem of figure 1. Most of the phenomena encountered can be found in one or other of several related problems which have been discussed in the literature by Eliassen[9], Phillips[10], Matsuno[11], Uryu[12], Grimshaw[13], and others.

In sections 3 and 4 I turn from illustrative example to general theory, and survey some very recent developments which appear to be of quite wide significance, but which have proved to be especially powerful for longitudinally symmetric problems. A simple yet very general version of the 'wave-action' concept is involved, resulting from a synthesis and extension of ideas from 'classical field theory'[14] and the more recent work of Eckart[15], Hayes[16], Dewar[17], and Bretherton[18]. Equally relevant is the pioneering work of Eliassen & Palm[19] and Charney & Drazin[20]; and a related but not identical line of development is contained in the work of Soward[21] on the Braginskii dynamo problem. A remarkable feature of the general results is that they enable useful statements to be made without requiring validity of approximations of the 'slowly-varying wavetrain' type and attendant concepts like 'group velocity'. Also, they can be developed for finite-amplitude waves[22]. Their special value in longitudinally-symmetric problems is that they lead to ways of expressing the problem for the mean-flow changes which do directly exhibit the abovementioned general connection between those changes and wave dissipation or forcing[22-25].

Finally (section 5) I shall make some remarks about that elusive entity, or rather, nonentity, wave 'momentum'.

2. MORE ABOUT THE PROBLEM OF FIGURE 1

2.1 Equations

The simplest relevant set of model equations is the usual set for a Boussinesq, incompressible, stratified fluid in a rotating frame of reference, with constant angular velocity $\boldsymbol{\Omega}$:

$$\mathbf{u}_{,t} + \mathbf{u}.\nabla\mathbf{u} + 2\boldsymbol{\Omega}_{\wedge}\mathbf{u} + \rho_0^{-1}\nabla p - \theta\hat{\mathbf{z}} = -\mathbf{X} \qquad (2.1a)$$

$$\theta_{,t} + \mathbf{u}.\nabla\theta = -Q \qquad (2.1b)$$

$$\nabla.\mathbf{u} = 0 \quad . \qquad (2.1c)$$

Here $\mathbf{u} = (u,v,w)$ is velocity, $(\ )_{,t}$ stands for $\partial(\ )/\partial t$, $\hat{\mathbf{z}}$ is the unit vector (0,0,1), θ is the buoyancy acceleration given by minus the effective gravity-plus-centrifugal acceleration (assumed constant in this model) times the fractional departure of the density from its constant reference value ρ_0. For descriptive purposes we shall think of θ as a measure of temperature or potential temperature. The departure of the pressure from the hydrostatic value associated with ρ_0 is denoted by p . The terms $\mathbf{X}$ = (X,Y,Z) and Q may be thought of as representing arbitrary body forces and heating, which may or may not be functionally related to the fields of motion but which in any case will be zero if the waves are neither dissipated nor generated internally.

2.2 Excess momentum flux, and the mean-flow problem

Let an average with respect to x be denoted by an overbar: for instance the mean velocity in the x-direction in figure 1 is $\bar{u}(y,z)$. If we average (2.1a) and make use of (2.1c) the result may be written in suffix notation (i,j=1,2,3) as

$$\bar{u}_{i,t} + \{\bar{u}_i\bar{u}_j + \frac{\bar{p}\delta_{ij}}{\rho_0}\}_{,j} + (2\boldsymbol{\Omega}_{\wedge}\bar{\mathbf{u}})_i - \bar{\theta}\hat{z}_i = -\{\overline{u_i'u_j'}\}_{,j} - \bar{X}_i \ . \qquad (2.2a)$$

(It will be convenient in what follows to use (x,y,z) and (u,v,w) interchangeably with (x_1,x_2,x_3) and (u_1,u_2,u_3).) Here $(\ )'$ is defined as $(\ ) - \overline{(\ )}$, the departure from the mean, and $(\ )_{,j}$ means $\partial(\ )/\partial x_j$. Eq.(2.2a) contains mean-flow quantities only, except for the term involving the Reynolds stress $\overline{u_i'u_j'}$. The equation tells us that $\overline{u_i'u_j'}$ is the excess mean momentum flux due to the waves. Note that $\overline{u_i'u_j'}$ is a wave property, by which I mean something which can be self-consistently evaluated as soon as you know the linear wave solution, i.e. when you know the fluctuating quantities $(\ )'$ to leading order.

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It might be tempting to conclude that nothing more need be said: Eq. (2.2a) states that the momentum transport by the waves is equal to $\overline{u_i'u_j'}$; so 'obviously' $-\overline{u_i'u_j'}$ is the stress whose divergence will give the mean acceleration $\bar{u}_{,t}$, or at least the contribution to this acceleration attributable to the waves. The average of Eq.(2.1b), namely

$$\bar{\theta}_{,t} + \{\bar{u}_j\bar{\theta}\}_{,j} = -\overline{\{u_j'\theta'\}}_{,j} - \bar{Q} \quad , \tag{2.2b}$$

is irrelevant, one might think, because how, after all, can the excess heat flux $\overline{u_j'\theta'}$ due to the waves affect momentum transport?

This conclusion would, however, be wrong (for reasons to appear shortly), and the fact that it has appeared in the past literature illustrates the dangers of 'incomplete reasoning' about wave transport effects on the basis of superficial consideration of a relevant-looking wave property - in this case the excess momentum flux $\overline{u_i'u_j'}$. Another illustration will be encountered in section 2.6. In fact the only safe general recipe for getting a self-consistent picture is to include a consideration of the complete problem for the mean flow and its solution correct to second order in the wave amplitude a. In the present example, the wave properties $\overline{u_i'u_j'}$ and $\overline{u_j'\theta'}$ appear as forcing terms in the mean-flow problem; and both turn out to play essential roles.

The result of averaging (2.1c) is

$$\nabla.\bar{u} = 0 \quad , \tag{2.2c}$$

and this completes the set of equations, (2.2), for the mean quantities $\bar{\underline{u}}$ $\bar{p}$ and $\bar{\theta}$. To obtain a well-determined model problem it is simplest to suppose that the flow is bounded laterally by a pair of vertical walls $y = 0, b$ on which the normal component of velocity vanishes, implying that

$$\bar{v} = 0 \quad \text{on} \quad y = 0, b \; . \tag{2.3}$$

We must beware, however, of assuming that $\bar{w}$ vanishes at $z = 0$; in fact for a rigidly-translating, corrugated boundary whose shape is described by a given function h ,

$$z = h(x-ct, y) \quad , \tag{2.4}$$

where $h=O(a)$, $\bar{h}=0$, and c is a (real) constant, it can be shown that

$$\bar{w} = \overline{(v'h)}_{,y} + O(a^3) \quad \text{at} \quad z = 0 \; . \tag{2.5}$$

This illustrates the fact that $\bar{w}$, which is an average along a horizontal line such as ℓ in figure 1, can represent a vertical mass flux, into or out of the thin region betweeen ℓ and the actual boundary, which is continuous with a horizontal, $O(a^2)$ mass flux within that region, associated with any tendency for the disturbance velocity to be one way along troughs and the other way along ridges in the boundary.

In fact, such a tendency turns out to be the rule rather than the exception when Coriolis effects matter; for instance if h is of the form $a \sin k(x-ct)$ then v' for conservative, plane inertio-gravity waves on a uniformly stratified basic state of rest turns out to be exactly in quadrature with w' and therefore exactly in phase with h at z=0. This can easily be verified by setting $\bar{\theta}_{,z}$ = constant, $\bar{u} = \bar{\theta}_{,y} = 0$, and $\underset{\sim}{X}=0$, $Q=0$, and calculating the elementary plane-wave solutions $\propto \exp i(kx + mz - \omega t)$ of the linearised disturbance equations derived from (2.1) (namely (3.2) below). Other pertinent features of such plane-wave solutions are that θ', being proportional to the vertical displacement through the basic stable stratification $\bar{\theta}_{,z}$, is (like h at z=0) in quadrature with the vertical velocity w'; also incompressibility dictates that u' is in phase with w', since (2.1c) implies $iku' + imw' = 0$. Thus $\overline{u'w'}$, $\overline{v'\theta'}$ are nonzero, and $\overline{v'w'}$, $\overline{w'\theta'}$ zero, in a plane inertio-gravity wave. The frequency of such a wave, ω (= kc), satisfies the dispersion relation

$$\omega^2 = (\bar{\theta}_{,z}k^2 + 4\Omega^2m^2)/(k^2 + m^2) \tag{2.6}$$

when $\bar{u} = 0$. (It should be noted that this implies that c^2 must lie between $4\Omega^2/k^2$ and $\bar{\theta}_{,z}/k^2$ for the inertio-gravity waves to be generated.)

2.3 Solution

I shall now describe, for the simplest relevant example, the result of solving the $O(a^2)$ mean flow problem; $\bar{\underset{\sim}{X}}$ and $\bar{Q}$ will be set to zero, so that we are talking about the effect on the mean flow of the waves alone. The waves are supposed to have propagated upwards as far as L either because they are being dissipated there or because a finite time has elapsed since the bottom boundary started moving. Well below L we can take the waves to have reached a steady state and the moticn to be conservative - we assume that $\underset{\sim}{X}'$ and Q' are zero there as well as $\bar{\underset{\sim}{X}}$ and $\bar{Q}$. To keep life as simple as possible we shall assume that $\bar{u} = 0$ initially, and follow its evolution as long as it can be considered to be $O(a^2)$. We also take $\bar{\theta}_{,z}$ = constant + $O(a^2)$ for the moment.

The simplest kind of mathematical analysis for the waves (we omit the details, since the results of section 4 will supersede them) makes the usual kind of 'slowly-varying' approximation, in which the plane wave solution is locally valid. This involves <u>inter alia</u> an assumption that the layer L is sufficiently deep compared with a vertical wavelength. We also take h to be of the form $a.f(y).\sin k(x-ct)$, where f(y) is a sufficiently slowly-varying function (which we assume vanishes at y=0,b). Then by the properties of plane inertio-gravity waves previously mentioned, the important term on the right of the x-component of (2.2a) is $-(\overline{u'w'})_{,z}$ and that on the right of (2.2b) is $-(\overline{v'\theta'})_{,y}$. The remaining terms are not of course exactly zero, because plane waves represent only the leading approximation; but in fact it is consistent to neglect them. The response of the mean flow to the forcing $-(\overline{v'\theta'})_{,y}$ together with the forcing represented by the inhomogeneous boundary condition (2.5) involves a mean 'secondary circulation' indicated schematically by the arrows in figure 2.

The picture assumes that the wave amplitude is a maximum near $y=\frac{1}{2}b$ and falls monotonically to zero on either side, so that $(\overline{v'\theta'})_{,y}$ changes sign once, near $y=\frac{1}{2}b$. The mean flow feels an apparent 'heating' on one side of the channel, and 'cooling' on the other (about which more will be said in section 2.5). This gives rise to an $O(a^2)$ mean vertical velocity $\bar{w}$ which beautifully satisfies the boundary condition (2.5) and, by Eq.(2.2c), demands a mean motion across the channel, i.e. a contribution to $\bar{v}$, in the vicinity of the layer L where the wave amplitude goes to zero with height.

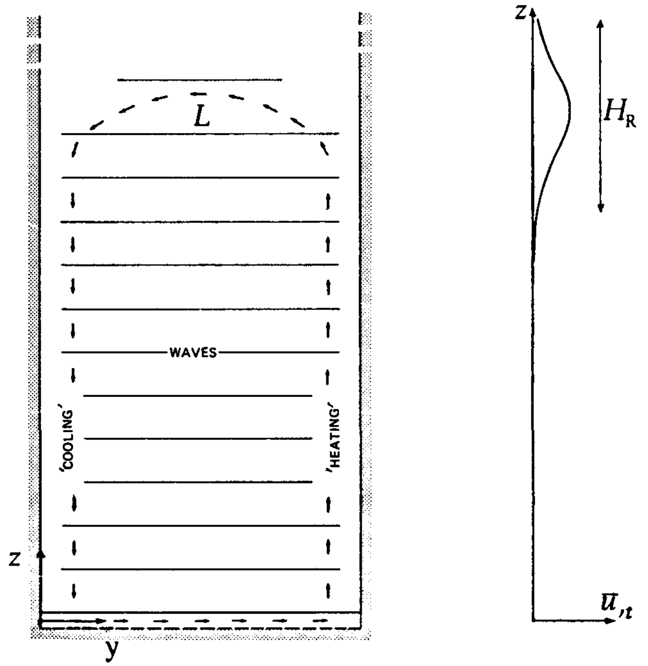

FIGURE 2. Left: end view (looking along the x axis) of the problem of figure 1. Right: typical profile of the mean acceleration in the longitudinal or x direction. The left-hand picture indicates how the secondary circulation $\bar{v}$, $\bar{w}$ is closed by a mass flux 'in the bottom boundary', associated with a positive correlation between the disturbance y-velocity, v', and the depth $-h$ of the corrugations in the boundary.

The Coriolis force associated with this $O(a^2)$ contribution to $\bar{v}$ accounts for a contribution to $\bar{u}_{,t}$ which is generally comparable with that from the Reynolds stress divergence $-(\overline{u'w'})_{,z}$ in the x component of Eq.(2.2a). In fact the two contributions, in the present simple problem, can be shown to stand approximately in the ratio

$$\frac{\text{Reynolds stress divergence}}{\text{Coriolis force associated with wave heat flux}} \doteq \frac{-\omega^2}{4\Omega^2} . \qquad (2.7)$$

The two contributions are comparable in magnitude whenever the Coriolis term is significant in the dispersion relation (2.6); indeed if $k^2 \ll m^2$ in (2.6), $\omega^2 \doteq 4\Omega^2$ and the two contributions are almost equal and opposite. In that case an estimate of effective momentum transport from the Reynolds stress $-\overline{u'w'}$ alone could be too large by an order of magnitude. (It is always wrong in order of magnitude in another example, namely that of quasi-geostrophic, vertically propagating Rossby waves[11,12] – in which case it is too small, by a factor of order the Rossby number.) Only when both contributions are accounted for will the calculated total rate of change of mean momentum $\iint \bar{u}_{,t} dydz$ agree, as it must, with minus the horizontal wave-drag force F exerted by the fluid on the lower boundary. F is defined correct to second order as the integral with respect to y of

$$\mathcal{F} = \overline{p'h_{,x}}\Big|_{z=0} + O(a^3) \quad . \tag{2.8}$$

This agreement (the detailed verification of which is omitted here) provides a useful check on the correctness of the overall picture.

2.4 Lagrangian-mean flow and 'radiation stress'

There are crucial differences between the foregoing picture, which is based on Eulerian averaging, that is averaging at fixed values of (y,z), and the same problem solved using a Lagrangian mean (definable approximately as the mean following a fluid particle). It turns out that the Lagrangian-mean secondary circulation is negligible sufficiently far below L. In a region of steady waves, when Q=0, the fluid particles merely oscillate about a constant mean level, and have no systematic tendency to migrate up or down. This is no more than might be expected for adiabatic motion in stable stratification. So in a Lagrangian-mean description there is no secondary circulation linking the regions of wave generation and dissipation, and thus no 'Coriolis' contribution to the effective transport of momentum by the waves. The analogue, in the Lagrangian-mean momentum equation, of the Reynolds stress $-\overline{u_i'u_j'}$ in the Eulerian-mean momentum equation (2.2a), thus gives a more direct description of the momentum transport, as was recognised by Bretherton[18,26], who suggested that it be identified as the radiation stress. Its xz component, taking the place of $-\overline{u'w'}$ in the present, Eulerian-mean description, is equal to the 'wave-drag' force in the x direction per unit area across a material surface whose undisturbed position is a plane z = constant – which force is evidently the same as (2.8) when z=0. These ideas have been further developed by Grimshaw[13] for the slowly-varying case, and an exact 'generalised Lagrangian-mean' description for arbitrary, finite-amplitude waves has been developed by Andrews & McIntyre[22]. The reader is referred to those papers for more discussion of the differences between the Lagrangian and Eulerian-mean descriptions, and to Bretherton[26] for a sufficient physical explanation, in terms of the average Coriolis force on a thin piece of fluid bounded above by a corrugated material surface and below by a flat, 'Eulerian' control surface, as to why the corresponding momentum fluxes differ in general; see also (4.7) and (4.9) below.

2.5 More details about the Eulerian-mean secondary circulation

Returning to the example of figure 2, we describe in a little more detail how, in the present description, the forcing term $-\overline{(v'\theta')}_{,y}$ 'gives rise' to the Eulerian-mean vertical velocity $\bar{w}$, since this will help motivate the more general analysis of section 4. In the region below L, where we may suppose the waves to be in a steady state and not dissipating ($\underset{\sim}{X}'=0$, $Q'=0$), the forcing term $-\overline{(v'\theta')}_{,y}$ is in fact balanced mainly by $\bar{w}\,\bar{\theta}_{,z}$ on the left of (2.2b), when we rewrite that equation with the aid of (2.2c) as

$$\bar{\theta}_{,t} + \bar{v}\,\bar{\theta}_{,y} + \bar{w}\,\bar{\theta}_{,z} = -\overline{(v'\theta')}_{,y} \quad . \tag{2.9}$$

The term $\bar{v}\,\bar{\theta}_{,y}$ is $O(a^4)$ and therefore negligible, because $\bar{v}$ is $O(a^2)$, and $\bar{\theta}_{,y}$ is also $O(a^2)$ for the following reason. We have taken $\bar{u}=O(a^2)$, which implies that $\bar{\theta}_{,y}=O(a^2)$ because the x component of the curl of (2.2a) gives (when $\underset{\sim}{X}=0$ and $\bar{v},\bar{w}$ are $O(a^2)$)

$$\bar{\theta}_{,y} = -2\Omega\,\bar{u}_{,z} + O(a^2) \quad . \tag{2.10}$$

(This is known in geophysical fluid dynamics as the 'thermal wind equation'.) The other term $\bar{\theta}_{,t}$ on the left of (2.9) turns out to be negligible unless we are within a distance of order the 'Rossby height'

$$H_R = 2\Omega b/(\bar{\theta}_{,z})^{1/2}$$

from the layer L where the waves are unsteady or dissipating[9]. This point will be further explained in section 4.6 below. Thus, sufficiently far below L we have a balance between $\bar{w}\bar{\theta}_{,z}$ and $-\overline{(v'\theta')}_{,y}$, which implies that

$$\bar{w} \doteq -\psi_{,y} \quad , \tag{2.11}$$

where we have defined

$$\psi = \overline{v'\theta'}/(\bar{\theta}_{,z}) \quad , \tag{2.12}$$

again using (2.10) and our temporary assumption that $\bar{u}=O(a^2)$ in order to neglect $\bar{\theta}_{,yz}$.

There is another didactic point to be made here, incidentally: it has often been assumed in the literature, for instance in connection with thermodynamic arguments, that the nonzero value of $\overline{v'\theta'}$ signifies a tendency for the waves to transport heat across the channel. Even more than with $\overline{u'w'}$, this is true but misleading. There is no tendency at all for the mean temperature to rise on one side and fall on the other, if we are sufficiently far below L. The adiabatic heating or cooling associated with $\bar{w}$ closely compensates the divergence of $\overline{v'\theta'}$. This compensation is intrinsic to the nature of the wave motion, as is underlined by the already-mentioned consideration that individual <u>fluid particles</u> are not being heated or cooled below L, because the motion is adiabatic there.

The right-hand half of figure 2 schematically indicates the profile of the mean acceleration $\bar{u}_{,t}$. If the layer L is shallower than the Rossby height H_R, then additional contributions to $\bar{v}$ and $\bar{w}$ arise in a layer of depth H_R centred on L. These adjust the values of $\bar{\theta}_{,t}$ and $\bar{u}_{,t}$ in such a way as to keep the thermal-wind equation (2.10) satisfied; there is 'room' for such a circulation only in a layer of depth H_R (see Eq.(4.12) below).

It turns out that Eq.(2.11) still holds for mean flow profiles $\bar{u}(y,z)$ and $\bar{\theta}(y,z)$ which vary sufficiently slowly and are such that the Richardson number $\bar{\theta}_{,z}/(\bar{u}_{,z})^2$ is large. Then

$$\bar{v} \doteq \psi_{,z} \qquad (2.13)$$

can be significant below L. The associated Coriolis force does not, however, lead to an acceleration of the mean flow well below L, because it turns out that it is always cancelled by an equal and opposite Reynolds stress divergence if the waves are steady and conservative. The fact that this cancellation must take place will be seen as a corollary of the much more general results to be described in section 4.

2.6 Energetics

Our simple example is also quite instructive as regards questions of energy transport. Suppose that the waves are dissipating in the layer L and heating the fluid there. (The amount of heat involved does not change $\bar{\theta}$ significantly within the Boussinesq approximation, but that is beside the point.) What is the source of this energy? Obviously, the work done by the agency moving the bottom boundary. How does the energy get from the boundary up to L ? Answer: there is a vertical energy flux $\overline{p'w'}$ due to the waves. Indeed, the rate of working by the boundary

$$-\mathcal{F}c = -c\overline{p'h_{,x}}\Big|_{z=0} = \overline{p'w'}\Big|_{z=0} \quad , \qquad (2.14)$$

in virtue of (2.8) and the fact that $w' = -ch_{,x}$. All the quantities involved here are wave properties.

What we must not forget, however, is that this simple picture, while correct to $O(a^2)$, depends crucially on the circumstance that $\bar{u}$ is zero, apart from the $O(a^2)$ contribution indicated in figure 2. If we look at the similar problem in which the boundary is brought to rest and the fluid is moving past it with velocity $\bar{u} = -c + O(a^2)$, the picture is quite different. Clearly the boundary can now do no work. The source of energy is now the kinetic energy of the mean flow near L, whose density is changing at a rate ρ_0 times

$$(\tfrac{1}{2}\bar{u}^2)_{,t} = \bar{u}\bar{u}_{,t} = -c\bar{u}_{,t} + O(a^4) \quad . \qquad (2.15)$$

The integral of (2.15) over the yz domain is, indeed, equal to Fc, by the remarks above Eq.(2.8).

In this problem there is no need for the waves to transport any energy into L at all; and indeed it does turn out that there is no net transport, despite the fact that $\overline{p'w'}$ is still the same as before. Note first that in the region of steady, conservative waves below L, the work done

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across a material surface corrugated by the waves is obviously zero, because in the present problem such a surface is immobile. Alternatively, in an Eulerian-mean description of the energy budget correct to $O(a^2)$, based on the model equations (2.1) and (2.2), the total energy flux across a horizontal control surface contains two terms which combine to cancel the contribution $\int \overline{p'w'}dy$. The first comes from the $O(a^2)$ part of the advection by w' of the leading contribution $\frac{1}{2}\rho_0(\bar{u} + u')^2$ to the total kinetic energy:

$$\rho_0 \overline{w'(\tfrac{1}{2}\bar{u}^2 + \bar{u}u' + \tfrac{1}{2}u'^2)} = \rho_0 \bar{u}\,\overline{u'w'} + O(a^2) \quad . \qquad (2.16)$$

(This contribution to the total Eulerian energy flux has been drawn attention to in the literature just about as often as it has been forgotten about!) The second contribution is the mean pressure-working $\int \bar{p}\bar{w}dy$ associated with the y-dependent part of the mean pressure whose gradient balances the Coriolis force associated with $\bar{u}$.

The general conclusion to be drawn is that, whenever moving media are involved, we must expect that a solution to the $O(a^2)$ mean flow problem will be essential to a self-consistent picture of the way in which waves contribute to the energy budget. We must also remember that, as always, use of the energy concept requires us to pay attention to frames of reference! There is no such thing as 'the' net energy transport due to the waves; and the transport can be identified with $\overline{p'\mathbf{u}'}$ only if the medium is everywhere at rest.

None of this affects the quite separate fact that the wave property $\overline{p'\mathbf{u}'}$ is the quantity usually related to the group velocity (when that concept is applicable). It usually turns out that

$$\overline{p'\mathbf{u}'} = E \times (\text{group velocity relative to the local mean flow}) \qquad (2.17)$$

for plane waves, where E is intrinsic wave-energy density, a wave property which in the present problem takes the form

$$E = \tfrac{1}{2}\rho_0(\overline{\mathbf{u}'^2} + \overline{\theta'^2}/\bar{\theta}_{,z}) \quad . \qquad (2.18)$$

(The second term is the 'available potential energy'[27] associated with vertical displacements of particles in the stable stratification; there is no internal energy because our model assumes incompressible flow. Bretherton & Garrett[28] have analysed the idea of 'wave-energy' as a physical concept in some depth, and in particular have established conditions under which E can be uniquely defined in a general manner independent of the mathematical formulation of the wave problem. I shall not go into that here except to say that, roughly speaking, E is the work you would do in setting up the disturbance, in a frame of reference in which the mean flow is at rest - which clearly makes approximate sense in problems of moving media only when the mean state varies sufficiently little over a wavelength.) I said that (2.17) 'usually' holds, by the way, because there are some exotic cases such as Rossby waves where the two sides differ by an identically nondivergent term. However, in the case of our plane inertio-gravity waves the two sides can be verified to be equal.

3. THE GENERALISED WAVE-ACTION PRINCIPLE

I deliberately avoided writing down the full conservation equations for the energy budget, partly because the details get quite complicated for all but the very simplest mean flow structures. Besides, I want to leave space to introduce another kind of conservable quantity, generalised wave-action, which apart from its wider significance will lead to a more powerful approach to problems of the kind just discussed. This approach will depend in no essential way upon any ´slowly-varying´ approximations. Wave-action is an $O(a^2)$ wave property which, in its most general form, satisfies a conservation relation, apart from source terms in $\underset{\sim}{X}'$ and Q', for any mean-flow structure whatever. This remarkable property is to be contrasted with the equation for wave-energy, whose ´right-hand side´ contains a complex of terms representing exchange of energy with the mean flow:

$$\frac{\partial E}{\partial t} + \nabla\cdot(\bar{\underset{\sim}{u}}E + \overline{p'\underset{\sim}{u}'}) =$$
$$= \rho_0[-\bar{u}_{i,j}\overline{u_i'u_j'} - (\delta_{ij}-\hat{z}_i\hat{z}_j)\bar{\theta}_{,i}\overline{u_j'\theta'} - \tfrac{1}{2}\bar{\theta}_{,z}^{-2}\overline{\theta'^2}(\bar{\theta}_{,zt}+\bar{u}_j\bar{\theta}_{,zj})]$$
$$-\rho_0[\overline{\underset{\sim}{u}'\cdot\underset{\sim}{X}'} + \overline{\theta'Q'}/\bar{\theta}_{,z}] \ . \qquad (3.1)$$

Wave-energy fails to be conserved as soon as you have a moving medium. Eq. (3.1) is just for the Boussinesq case, and is derived by dotting the linearised versions of (2.1a) with $\rho_0\underset{\sim}{u}'$ and of (2.1b) with $\rho_0\theta'/\bar{\theta}_{,z}$ and adding. We should always keep in mind that (3.1), as implied by the discussion just given in section 2.6, represents only a part of the whole energy budget. The linearised equations corresponding to (2.1) are

$$D_t\underset{\sim}{u}' + \underset{\sim}{u}'\cdot\nabla\bar{\underset{\sim}{u}} + 2\underset{\sim}{\Omega}\wedge\underset{\sim}{u}' + \rho_0^{-1}\nabla p' - \theta'\hat{\underset{\sim}{z}} = -\underset{\sim}{X}' \qquad (3.2a)$$

$$D_t\theta' + \underset{\sim}{u}'\cdot\nabla\bar{\theta} = -Q' \qquad (3.2b)$$

$$\nabla\cdot\underset{\sim}{u}' = 0 \ , \qquad (3.2c)$$

where the linearised material derivative

$$D_t(\ ) = (\ )_{,t} + \bar{\underset{\sim}{u}}\cdot\nabla(\ ) \ .$$

Here we shall define the generalised wave-action correct to $O(a^2)$ only. (It can be defined exactly, for finite-amplitude disturbances, once one has the generalised Lagrangian-mean description[22], but that is beyond the scope of this review.) Two preliminaries are needed. The first is to introduce the $O(a)$ particle displacement field $\underset{\sim}{\xi}(\underset{\sim}{x},t)$, which is defined to satisfy

$\nabla\cdot\boldsymbol{\xi} = 0$, $\overline{\boldsymbol{\xi}} = 0$, and

$$D_t \boldsymbol{\xi} = \mathbf{u}^{\ell} \ , \qquad (3.3a)$$

where $\mathbf{u}^{\ell}$ is the Lagrangian disturbance velocity

$$\mathbf{u}^{\ell} = \mathbf{u}' + \boldsymbol{\xi}\cdot\nabla\bar{\mathbf{u}} \quad . \qquad (3.3b)$$

For the problem of figure 1, (3.3) simplifies, correct to $O(a^2)$, to

$$D_t\eta = v', \quad D_t\zeta = w', \quad D_t\xi = u' + \eta\bar{u}_{,y} + \zeta\bar{u}_{,z} \ , \qquad (3.4)$$

where ξ, η, and ζ are the components of $\boldsymbol{\xi}$.

The second preliminary is a formal device (see Hayes[16] and Bretherton[25]) which is introduced for the sake of the greatest possible generality: we reinterpret the Eulerian averaging operator $\overline{(\ )}$ as an <u>ensemble average</u>, over an ensemble of wave solutions distinguished by a single, smoothly varying parameter α . In a stochastic problem α would range over a 'sample space'; but random waves are merely one possible case. In the deterministic problem of figure 1, for example, we can generate a suitable ensemble just by translating the boundary and the wave pattern a distance α in the x direction. Then $\overline{(\ )}$ may be trivially re-defined in terms of an integral over α rather than over x. For the axisymmetric mean flows important in astrogeophysical applications the principle is the same but the details less trivial[22,25]. Quite generally, we have the basic property

$$\overline{\{(\ )_{,\alpha}\}} = \{\overline{(\ )}\}_{,\alpha} = 0 \quad , \qquad (3.5)$$

whenever the ensemble of disturbance fields depends differentiably upon the parameter α , which we shall take to be the case.

Instead of dotting the linearised momentum equation (3.2a) with $\rho_0\mathbf{u}'$, we now take its dot product with the derivative $\rho_0\boldsymbol{\xi}_{,\alpha}$ and average. After some manipulation there results[22,23]

$$\boxed{\frac{\partial A}{\partial t} + \nabla\cdot\mathbf{B} = -\rho_0(\overline{\boldsymbol{\xi}_{,\alpha}\cdot\mathbf{X}'} + \overline{\zeta_{,\alpha}\,q'}) + O(a^4)} \qquad (3.6)$$

where

$$A = \rho_0\overline{\boldsymbol{\xi}_{,\alpha}\cdot(\mathbf{u}^{\ell} + \boldsymbol{\Omega}_{\wedge}\boldsymbol{\xi})} \qquad (3.7a)$$

$$\mathbf{B} = \bar{\mathbf{u}}\,A + \overline{\boldsymbol{\xi}_{,\alpha}\,p'} \qquad (3.7b)$$

and

$$q' = -\,\theta' - \boldsymbol{\xi}\cdot\nabla\bar{\theta} + O(a^3), \qquad (3.8a)$$

so that $\overline{q'} = 0$ and

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(3.8b)

Thus $q' = 0$ when $Q' = 0$, i.e. $q' = 0$ for adiabatic motion. So A is conserved, with flux $\underset{\sim}{B}$, whenever $\underset{\sim}{X}'$ and Q' are zero.

In the derivation of (3.6), the property (3.5), and its corollary that if $f'(\underset{\sim}{x},t,\alpha)$ and $g'(\underset{\sim}{x},t,\alpha)$ are any two disturbance fields then

$$\overline{f'_{,\alpha} g'} = -\overline{f'g'_{,\alpha}} \ , \tag{3.9}$$

are needed a number of times. Also (2.2) are used, with $\overline{\underset{\sim}{X}}$ and $\overline{Q} = O(a^2)$.

Eq. (3.6) corresponds when $\underset{\sim}{X}'$ and Q' are zero to one of a class of exact conservation relations pointed out by Hayes[16], which arise from replacing certain space or time derivatives by $(\ )_{,\alpha}$ in the classical 'energy-momentum-tensor' formalism[14]. The relationship between these exact conservation laws and the adiabatic conservation laws discovered by Whitham[29] is discussed in some detail in Hayes' paper. Hayes took the range of α to be $(0,2\pi)$, which makes A unique and is convenient for applications to periodic or almost periodic waves, since α may then be interpreted as phase. However, it is convenient here not to fix the range of α , in order to leave a little more flexibility in applications. Then A is defined only up to a multiplicative constant.

Hayes called his conserved quantity the 'action' irrespective of what variational principle was used as the starting point. It can be shown[25] that A (with $0<\alpha<2\pi$) is equal to Hayes' invariant when the governing variational principle is Hamilton's principle, expressed in its classical sense in terms of the particle displacements $\underset{\sim}{\xi}$. A is to be carefully distinguished from the other conservable quantities to which other variational principles may lead, via Hayes' modification of the energy-momentum-tensor formalism, and which may or may not be wave properties. For example, Hayes' invariant is <u>not</u> a wave property when the Clebsch-Herivel-Lin variational principle[30] is used as the starting point.

A, or more properly its generalisation to finite amplitude[22], represents the fundamental, exactly conservable wave property which, in problems of slowly-varying, conservative waves ($\underset{\sim}{X}$, Q both zero), reduces to the adiabatically-conserved wave-action whose physical meaning and precise relation to Whitham's adiabatic invariants was elucidated by Bretherton & Garrett[28]. The connection between A and Bretherton & Garrett's wave-action can be made via the scalar virial theorem[22], i.e. the result of dotting the momentum equation with $\underset{\sim}{\xi}$ rather than with $\underset{\sim}{\xi}_{,\alpha}$. (This reduces to 'equipartition of energy' in the non-rotating case.) Bretherton & Garrett's wave-action is defined in those 'slowly-varying' circumstances in which the wave-energy E is uniquely defined, and is then equal to E divided by the intrinsic frequency, or frequency in a frame of reference moving with the local mean flow, ω^+.

The result (3.6), or more generally (3.11) below, appears to have two distinct types of application. One is the same as that envisaged by

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Bretherton & Garrett, namely to computing the spatial and temporal dependence of wave amplitude. The second application is to the calculation of mean-flow evolution. Both applications depend on the fact that A and $\underset{\sim}{B}$ are wave properties, and it has been found in both applications that the required information is obtained, in at least some cases[23], from far less computation than would otherwise be needed.

For reference we quote the corresponding result for a compressible fluid of density ρ and general equation of state

$$\rho = S(\theta,p) \quad , \tag{3.10}$$

where p is pressure and θ potential temperature or specific entropy; θ still satisfies Eq. (2.1c) in either case. The result is almost as simple as before (again quite unlike the corresponding wave-energy equation):

$$\boxed{\frac{\partial A}{\partial t} + \nabla.\underset{\sim}{B} = -\bar{\rho}\,\overline{\underset{\sim}{\xi}_{,\alpha}\cdot\underset{\sim}{X}'} - \bar{s}\;\overline{p^{\ell}_{,\alpha}\,q'} + O(a^3) \quad ,} \tag{3.11}$$

where A and $\underset{\sim}{B}$ are still given by (3.7) with ρ_0 replaced by $\bar{\rho}$, the mean density, and[32]

$$p^{\ell} = p' + \underset{\sim}{\xi}.\nabla\bar{p} \tag{3.12}$$

$$\bar{s} = \bar{\rho}^{-1}\partial S(\bar{\theta},\bar{p})/\partial\bar{\theta} \quad . \tag{3.13}$$

The definitions of $\underset{\sim}{\xi}$ and q' are the same as before, except that because of compressibility we have $\nabla.\underset{\sim}{\xi} = -\rho^{\ell}/\bar{\rho} + O(a^2)$ $(\rho^{\ell} = \rho' + \underset{\sim}{\xi}.\nabla\bar{\rho})$.

4. THE GENERALISED ELIASSEN-PALM RELATIONS, AND THE CONNECTION BETWEEN MEAN-FLOW ACCELERATION AND WAVE GENERATION OR DISSIPATION IN LONGITUDINALLY-SYMMETRIC PROBLEMS

4.1 Conservation of pseudomomentum

For simplicity we now revert to the assumption that the mean flow is independent of x_i, as in the example of figure 1 (where i=1). That is, the mean flow is invariant to translations in the x_i direction. Associated with this invariance is a conservable wave property

$$P_i = -\rho_0\,\overline{\underset{\sim}{\xi}_{,i}\cdot(\underset{\sim}{u}^{\ell} + \underset{\sim}{\Omega}\wedge\underset{\sim}{\xi})} \tag{4.1a}$$

because in this case we may replace $(\;)_{,\alpha}$ by $-(\;)_{,i}$ in the generalised wave-action principle (3.6). The associated flux is

$$Q_{ij} = \bar{u}_j P_i - \overline{\xi_{j,i}\,p'} \quad , \tag{4.1b}$$

and (3.6) is replaced by

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$$P_{i,t} + Q_{ij,j} = \rho_o(\overline{\xi_{,i}\cdot X'} + \overline{\zeta_{,i}q'}) + O(a^4) \quad . \tag{4.2}$$

(Of course I could have derived (4.2) in the first place by dotting (3.2a) with $-\xi_{,i}$ rather than $\xi_{,\alpha}$. But I wanted to go via the wave-action principle because of its importance as the starting point for applications in various other contexts. For instance it is not quite so obvious how to find the analogue of (4.2) in the more general kind of longitudinally-symmetric problem where the mean flow is rotationally invariant, until we take (3.6) or (3.11) as starting point and apply it to the ensemble generated by rotating the disturbance pattern[22,25].) For reasons to be explained in section 5 I shall call P_i the density of pseudomomentum.

There is still no obvious connection between (4.2) and the mean-flow equations (2.2) - although there would have been at this stage had we been working with the generalised Lagrangian-mean description[22]. Because we are using the Eulerian-mean description, some more analysis will be needed. First, we use the remarks in section 2.5 to motivate a simple transformation[23,24,32] of the mean-flow equations, which will take them one step closer to the connection with (4.2). The idea is to subtract out the contribution to the $O(a^2)$ Eulerian-mean secondary circulation expressed by the stream function ψ of (2.12). The second and final step[3,4,23,24,26,32-35] will involve manipulation of the linearised equations in a way foreshadowed by the celebrated work of Eliassen & Palm[19] and recently brought to a very general form by Andrews & McIntyre[23,32].

4.2 Preliminary transformation of the mean-flow problem (2.2)

Now take $x=x_1$ as the direction of symmetry. Define $\bar{v}^*$ and $\bar{w}^*$ by

$$\bar{v} = \psi_{,z} + \bar{v}^* , \qquad \bar{w} = -\psi_{,y} + \bar{w}^* , \tag{4.3}$$

so that $\bar{v}^*$, $\bar{w}^*$ represent the 'residual' $O(a^2)$ mean secondary circulation left over after subtracting out the part corresponding to (2.12). Then a small amount of manipulation converts the mean-flow problem (2.2) into the form

$$\bar{u}_{,t} + U_y\bar{v}^* + U_z\bar{w}^* = -S_{xy,y} - S_{xz,z} - \bar{X} \tag{4.4a}$$

$$2\Omega\bar{u}_{,t} + \bar{p}_{,ty} + \bar{v}^*_{,tt} = -\tilde{Y}_{,t} \tag{4.4b}$$

$$-\bar{\theta}_{,t} + \bar{p}_{,tz} + \bar{w}^*_{,tt} = -\tilde{Z}_{,t} \tag{4.4c}$$

$$\bar{\theta}_{,t} + \bar{\theta}_{,y}\bar{v}^* + \bar{\theta}_{,z}\bar{w}^* = -G_{,z} - \bar{Q} \tag{4.4d}$$

$$\bar{v}^*_{,y} + \bar{w}^*_{,z} = 0 \tag{4.4e}$$

where

$$U_y = \bar{u}_{,y} - 2\Omega , \qquad U_z = \bar{u}_{,z} \tag{4.5a}$$

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$$S_{xy} = \overline{u'v'} - U_z\overline{v'\theta'}/\bar{\theta}_{,z} \tag{4.5b}$$

$$S_{xz} = \overline{u'w'} + U_y\overline{v'\theta'}/\bar{\theta}_{,z} \tag{4.5c}$$

$$G = \overline{w'\theta'} + \overline{v'\theta'}\,\bar{\theta}_{,y}/\bar{\theta}_{,z} \tag{4.5d}$$

$$\tilde{Y} = (\overline{v'^2})_{,y} + (\overline{v'w'})_{,z} + (\overline{v'\theta'}/\bar{\theta}_{,z})_{,zt} + \bar{Y} + O(a^4) \tag{4.5e}$$

$$\tilde{Z} = (\overline{v'w'})_{,y} + (\overline{w'^2})_{,z} - (\overline{v'\theta'}/\bar{\theta}_{,z})_{,yt} + \bar{Z} + O(a^4)\ . \tag{4.5f}$$

Correct to $O(a^2)$, (4.4) can be regarded as a set of equations for the five unknowns

$$\{\bar{u}_{,t}\ ,\ \bar{v}^*,\ \bar{w}^*,\ \bar{\theta}_{,t}\ ,\ \bar{p}_{,t}\}\quad , \tag{4.6}$$

since U_y, U_z, $\bar{\theta}_{,y}$ and $\bar{\theta}_{,z}$ can be regarded as known 'coefficients' apart from contributions $O(a^2)$. We are of course thinking of the linearised wave problem as having been solved, so that the wave properties on the right are known forcing terms.

The transformation (4.3) is dependent on our choice of coordinates; a coordinate-independent preliminary transformation may be used instead, at the cost of getting more complicated-looking versions of the results[32].

4.3 Excess momentum fluxes and generalised Eliassen-Palm relations

We now multiply the x component of (3.2a) by η and then by ζ [26], and average. The resulting pair of relations reduces, after a little manipulation in which we use (3.4), (3.9) (with α replaced by x), and our assumption that $\bar{\underset{\sim}{u}} = (\bar{u},0,0) + O(a^2)$, to

$$\overline{u'v'} + \rho_0^{-1}\,\overline{\eta_{,x}p'} - U_z\overline{\eta w'} = \overline{\eta X'} + \tfrac{1}{2}U_y(\overline{\eta^2})_{,t} + (\overline{\eta u'})_{,t} + O(a^4) \tag{4.7a}$$

$$\overline{u'w'} + \rho_0^{-1}\,\overline{\zeta_{,x}p'} - U_y\overline{\zeta v'} = \overline{\zeta X'} + \tfrac{1}{2}U_z(\overline{\zeta^2})_{,t} + (\overline{\zeta u'})_{,t} + O(a^4). \tag{4.7b}$$

The second term in (4.7b) is to be compared with (2.8), the wave-drag force per unit area, and indeed $-\overline{\eta_{,x}p'}$ and $-\overline{\zeta_{,x}p'}$ are the excess momentum fluxes (or minus the 'radiation stress' components) in one of the forms in which they appear in Lagrangian-mean analogues of (2.2a)[13,18,22]. Eqs.(4.7) relate these to the 'Eulerian' excess momentum fluxes $\overline{u'v'}$ and $\overline{u'w'}$. Note that $-\overline{\eta_{,x}p'}$ and $-\overline{\zeta_{,x}p'}$ are also equal to the y and z components of the nonadvective part of the flux of the pseudomomentum component P_1; see (4.1b).

Next we multiply (3.2b) by η and average to get

$$-\overline{v'\theta'} + (\overline{\eta\theta'})_{,t} + \tfrac{1}{2}\bar{\theta}_{,y}(\overline{\eta^2})_{,t} + \bar{\theta}_{,z}\overline{\eta w'} = -\overline{\eta Q'} + O(a^4)\quad , \tag{4.8}$$

and ζ times the first plus η times the second of (3.4) gives

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$$\overline{(\eta\zeta)}_{,t} = \overline{\zeta v'} + \overline{\eta w'} \quad . \tag{4.9}$$

These may be used to eliminate $\overline{\zeta v'}$ and $\overline{\eta w'}$ from (4.7). The result is a pair of relations like (4.7) except that S_{xy} and S_{xz} appear in place of $\overline{u'v'}$ and $\overline{u'w'}$, and the remaining terms, apart from $\overline{\eta_{,x}p'}$ and $\overline{\zeta_{,x}p'}$, are either of the form $\overline{(\;)}_{,t}$ or contain a factor X' or Q'. If the first relation is differentiated with respect to y and the second with respect to z, and (4.2) used to eliminate the two terms in p', there results

$$\begin{aligned} S_{xy,y} + S_{xz,z} &= \overline{(\eta X')}_{,y} + \overline{(\zeta X')}_{,z} + \\ &+ \overline{\underset{\sim}{\xi}_{,x}\cdot\underset{\sim}{X}'} + \overline{\zeta_{,x}q'} - U_z(\overline{\eta Q'}/\bar{\theta}_{,z})_{,y} + U_y(\overline{\eta Q'}/\bar{\theta}_{,z})_{,z} \\ &+ \frac{\partial}{\partial t}\Big[\overline{(\eta u')}_{,y} + \overline{(\zeta u')}_{,z} - \rho_o^{-1}P_1 - \{U_z[(\overline{\eta\theta'} + \tfrac{1}{2}\bar{\theta}_y\overline{\eta^2})/\bar{\theta}_{,z}] - \tfrac{1}{2}U_y\overline{\eta^2}\}_{,y} \\ &+\{U_y[(\overline{\eta\theta'} + \tfrac{1}{2}\bar{\theta}_y\,\overline{\eta^2})/\bar{\theta}_{,z}] + U_y\overline{\eta\zeta} + \tfrac{1}{2}U_z\overline{\zeta^2}\}_{,z}\Big] + O(a^4) \quad . \end{aligned} \tag{4.10a}$$

By multiplying (3.2b) by θ', averaging, and differentiating with respect to z, we get

$$G_{,z} = -(\overline{\theta'Q'}/\bar{\theta}_{,z})_{,z} + \frac{\partial}{\partial t}[-\tfrac{1}{2}\overline{\theta'^2}/\bar{\theta}_{,z}]_{,z} + O(a^4) \quad . \tag{4.10b}$$

4.4 Deductions

The results (4.10) imply that the $O(a^2)$ forcing terms on the right of the mean-flow problem (4.4) can be expressed as a sum of terms each of which falls into one of three categories:

(i) the Eulerian-mean external forcing terms $\bar{\underset{\sim}{X}}$ and $\bar{Q}$ (which we shall choose to regard as unconnected with the waves),

(ii) wave terms of the time-differentiated form $\overline{(\;)}_{,t}$,and

(iii) wave terms all involving $\underset{\sim}{X}'$, Q', or q' , that is, all depending explicitly on the forcing or dissipation of the waves.

This immediately shows that in the problem of figure 1, for instance, the forcing of mean-flow changes vanishes below the layer L, where the waves are steady and conservative, for any initial profiles of mean flow and stratification, no matter how complicated, and no matter whether or not approximate, 'slowly-varying' descriptions of the waves are valid.

It also carries the implication that although strictly conservative waves or instabilities ($\underset{\sim}{X}'$, Q' and q' zero) can change the mean flow if they themselves are growing or decaying in amplitude, such changes are temporary in that no net change to the mean flow persists if the waves propagate out of the region of interest. This is almost obvious[23] from the fact that all the $O(a^2)$ wave terms on the right of (4.4) can then be written in the form $\overline{(\;)}_{,t}$,with the aid of (4.10). However, if a very small amount of dissipation is present, its effects can be greatly enhanced by the occurrence of mean-flow changes that would otherwise be temporary. For a striking example of this, see reference 11. A change in the mean-flow profile due to wave transience can bring about an approach to 'critical-line' conditions somewhere, giving the dissipation terms a chance to take over locally, in turn causing further and more permanent mean-flow changes. Theoretical work on such highly nonlinear feedback effects, which could easily be important in the evolution of stellar differential rotation, for example, is still in its infancy[7,8].

4.5 Extensions

Precisely analogous results, enabling the same qualitative conclusions about mean-flow evolution to be drawn without solving wave problems in detail, have been derived for:

1) rotationally as well as translationally invariant mean flows, with mean velocity predominantly in the longitudinal (azimuthal) direction[22-25, 32]

2) a fluid with a general equation of state $\rho = S(\theta, p)$; the Boussinesq approximation is not essential[22,25,32]

3) a self-gravitating fluid[22]

4) a conducting fluid, with mean magnetic field as well as mean velocity predominantly longitudinal[31].

4.6 Simplifications for the problem of section 2

It is of interest in connection with the previous discussion to exhibit the approximate form taken by the transformed mean-flow problem for the almost-plane inertio-gravity waves of figure 1. A self-consistent set of approximations requires that the Richardson number $\bar{\theta}_{,z}/(\bar{u}_{,z})^2$ be large, and the depth of the layer L, and other scales of mean variation in the vertical, including the Rossby height $H_R = 2\Omega b/(\bar{\theta}_{,z})^{\frac{1}{2}}$, large compared to the vertical radian wavelength m^{-1}. The horizontal scale b is then large also, compared with k^{-1}, assuming that both terms in the dispersion relation (2.6) are not too different in magnitude. The upshot of such approximations is that all the terms of the form $\overline{(\;)}_{,y}$ or $\overline{(\;)}_{,z}$ in (4.10) become negligible, and (4.4) can be shown to simplify, when $\bar{X}$ and $\bar{Q}$ are zero, to

$$\bar{u}_{,t} - 2\Omega\bar{v}^* = -\overline{\xi_{,x}\cdot X'} - \overline{\zeta_{,x}q'} + \rho_0^{-1}P_{1,t} \tag{4.11a}$$

$$2\Omega\bar{u}_{,t} + \bar{p}_{,ty} = 0 \tag{4.11b}$$

$$-\bar{\theta}_{,t} + \bar{p}_{,tz} = 0 \tag{4.11c}$$

$$\bar{\theta}_{,t} + \bar{\theta}_{,z}\bar{w}^* = 0 \tag{4.11d}$$

$$\bar{v}^*_{,y} + \bar{w}^*_{,z} = 0 \quad . \tag{4.11e}$$

According to the second and third equations the mean flow stays in geostrophic and hydrostatic balance as it changes: knowledge of $\bar{p}_{,t}$ implies knowledge of $\bar{u}_{,t}$ and $\bar{\theta}_{,t}$. (The thermal-wind equation (2.10) holds, with its $O(a^2)$ contribution negligible). If we eliminate $\bar{u}_{,t}$ and $\bar{\theta}_{,t}$ in favour of $\bar{p}_{,t}$ in (4.11a) and (4.11d), and cross-differentiate to eliminate $\bar{v}^*$ and $\bar{w}^*$ via (4.11e), there results

$$\left\{\frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial z}\frac{H_R^2}{b^2}\frac{\partial}{\partial z}\right\}\bar{p}_{,t} = 2\Omega\{\overline{\boldsymbol{\xi}_{,x}\cdot\mathbf{X}'} + \overline{\zeta_{,x}q'} - \rho_0^{-1}\dot{P}_{1,t}\}_{,y} \quad . \qquad (4.12)$$

The form of the elliptic operator on the left shows why the Rossby height $H_R = 2\Omega b/(\bar{\theta}_{,z})^{1/2}$ takes over as the vertical scale of the response of the mean flow, whenever the forcing on the right of (4.12) is confined to a layer L of depth smaller than H_R.

5. PSEUDOMOMENTUM IS NOT MOMENTUM

It is surprising how often one seems to encounter the conceptual mistake that since waves can transport momentum, they must possess it. No less than Lord Rayleigh[36] appears to have been under this impression when he wrote that 'if the reflexion of a train of waves exercises a pressure upon the reflector, it can only be because the train of waves itself involves momentum'. Nowadays one often reads about 'the momentum of the waves' and about waves 'exchanging'(their) momentum with the mean flow, and suchlike; and there appears to be a tendency to assume that this momentum which the waves are supposed to have is to be identified with the wave property P_i defined in (4.1a), or rather the wave property

$$P_i = Ek_i/\omega^+ \ , \qquad (5.1)$$

to which P_i can be shown to reduce in those 'slowly-varying' circumstances where Bretherton & Garrett's[28] arguments apply.

On the other hand, as Brillouin[37] pointed out in 1925, Rayleigh's statement is a non sequitur because in a material medium you can perfectly well have a nonzero <u>flux</u> of momentum unaccompanied by any momentum density – try leaning against a brick wall. Two specific counterexamples to Rayleigh's statement which I happen to know are the obvious one of waves in solids (phonons)[37,38], and a simple fluid-dynamical example I published in 1973[40]. In the latter example, a packet of 'inertia' (pure Coriolis) waves propagates along a waveguide comprising an incompressible, homogeneous liquid between rigid, parallel boundaries in a rotating frame of reference. The mean momentum is <u>zero</u>, for reasons of mass continuity. Nevertheless there is a non-zero recoil force when the wave packet is reflected from an immersed obstacle.

That problem has the further interesting feature that the wave packet does have a well-defined 'fluid impulse', $\underset{\sim}{I}$, which gives the recoil force. $\underset{\sim}{I}$ is not equal to the integral of $\underset{\sim}{P}$; indeed it can have the opposite sense! (In fact the momentum flux due to the mean pressure $\bar{p}$ plays a leading role in this particular problem.)

An example of a somewhat different kind is the celebrated problem of a packet of electromagnetic radiation in a refractive medium. This problem has long been controversial, but has been convincingly clarified in recent years by Penfield & Haus[42], Gordon[39], and Peierls[38], to whose papers, together with the review by Robinson[41], the reader is referred for some very interesting history. In this problem, provided we neglect dispersion, there is a definite, non-zero total momentum M_i which travels with the waves. Again, this is not generally equal to the integral of P_i (or rather its electromagnetic counterpart, the 'Minkowski quantity'), nor is it equal to the electromagnetic part of the total momentum (the 'Abraham quantity'). On the other hand recoil forces, are, this time, simply related to P_i (but not to M_i!) in at least some circumstances[39].

Brillouin's point is that waves don't have to possess momentum; the examples show that in fact they sometimes do and sometimes don't - and that when they do, the momentum is not necessarily related to recoil forces. The wave property P_i may or may not be closely related to either; and whether it is depends in fact on global considerations - on the full $O(a^2)$ mean problem and its boundary conditions. So we must either say that P_i may sometimes 'be interpreted' as momentum, and sometimes not, depending on the global problem - surely a most unsatisfactory conceptual structure - or we must decide that P_i is simply an entity in its own right, not necessarily related to any momentum, whereupon the conceptual problems disappear. This is why I like to have a separate name 'pseudomomentum' for P_i, just as one likes to have separate names for other pairs of quantities, like energy and torque, which have the same dimensions but different physical natures. I want to use the term 'momentum' in its ordinary, elementary sense, of course (which we use when thinking intuitively about forces and accelerations). The terminology follows the usage of workers in solid-state physics, to whom all this has naturally been more obvious than to most. Blount and Gordon[39] have carried the terminology, and the distinction between momentum and pseudomomentum, into classical electrodynamics, and shown how it helps clarify the issues of the so-called Abraham-Minkowski 'controversy' over electromagnetic waves referred to above.

'But,' the reader may say, 'you said in section 3 that Hayes' conservation relation results from replacing certain space or time derivatives by $(\ )_{,\alpha}$ in the variational derivation of the conservation relation for the energy-momentum tensor. So the result of going in the reverse direction, which is just (4.2) without the 'dissipative' terms, is nothing but the conservation law satisfied by the 'momentum' part of the

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energy-momentum tensor itself. So surely P_i is a momentum.'

My reply is that that would follow if the same mathematical formalism always represented the same physical entity. Here, however, the blanket term 'energy-momentum tensor' tends to obscure the fact that different variational principles, springing from different basic formulations of the physical problem, can be used as starting point. Different basic formulations carry different implications about the kinds of invariance properties associated with conservation laws. Certainly Eq. (4.2) is associated with invariance under translations in space, just as is conservation of momentum. But the translational invariances referred to are in fact quite different, a point well made by Peierls[38]. Conservation of the i-component of momentum, for instance, depends on invariance of the basic physical problem, including force potentials (e.g.gravitational), under translations in the x_i direction. Conservation of P_i depends on translational invariance of the mean flow, insofar as it enters into the disturbance problem. This condition does not necessarily involve external force potentials. Other necessary conditions for the two kinds of conservation law to hold are also quite different. For instance, whether or not momentum is conserved has nothing to do with whether or not the motion is adiabatic, while conservation of P_i does depend on the motion being adiabatic since it requires, inter alia, that $q' = 0$ in (4.2).

It is the description in terms of particle displacements (and Hamilton's principle in the classical sense) that lies behind the appearance of pseudomomentum rather than momentum in the 'energy-momentum' tensor. By contrast, if we form components of the Eulerian-mean energy-momentum tensor from the usual pure-Eulerian variational principle in fluid dynamics, namely the Clebsch-Herivel-Lin principle[30], then in place of P_i and its flux Q_{ij} we get the Eulerian-mean density and flux of momentum,

$$\overline{\rho u_i} \quad \text{and} \quad -\bar{p}\delta_{ij} - \overline{\rho u_i u_j} \quad , \tag{5.2}$$

to within an identically nondivergent contribution. This should be no surprise in view of the foregoing remarks on translational invariance. Conservation of pseudomomentum, as distinct from momentum, is connected with invariance to a displacement of the disturbance pattern while mean particle positions are kept fixed, as distinct from a displacement of the whole system, particles as well as disturbance pattern[38]. The idea of 'fixed mean particle positions' cannot be directly expressed within a purely field-theoretic or Eulerian description, which does not keep track of where fluid particles are. But it is implicit in a description of the disturbance in terms of particle displacements (from 'mean particle positions'). (What this means for finite-amplitude disturbances is dealt with in reference 22.)

One possible reason why momentum and pseudomomentum have sometimes been mistaken for one another may be that in certain examples, even more idealised than those already cited, not only are both quantities conserved (requiring $\underset{\sim}{X}'$ and Q' to be zero in the case of pseudomomentum), but also their conservation relations reduce to the same form. If in these examples we generate the waves

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starting from an initial state in which momentum and pseudomomentum are both zero, it can happen that they evolve in parallel and remain equal. The simplest example is the trivial one of an electromagnetic wave in vacuo. Here there is no medium present to make the translational symmetry operations different. But there are also examples involving waves in media, all of them longitudinally-symmetric problems, in which the longitudinal components of pseudomomentum and mean momentum evolve in parallel. Perhaps the most celebrated example is that of Stokes' periodic waves on the surface of an infinite, inviscid ocean. The initial conditions of no motion are hidden in the assumption of irrotational motion.* Approximate longitudinal symmetry would be enough for approximate conservation of pseudomomentum; but we need also that there be no mean horizontal pressure gradient and that, concomitantly, the mean mass continuity equation (which constrains the distribution of mean momentum but not that of pseudomomentum) plays no significant role. Exactly the same considerations explain why a further such example is provided by the problem of section 2, in the case when $H_R << H$, the scale of the layer L. Eqs. (4.11) then imply that, for conservative waves,

$$\rho_0 \bar{u}_{,t} = p_{1,t} \qquad (\underset{\sim}{X}=0,\ Q=0,\ H_R<<H). \tag{5.3}$$

These examples are very special, and in any case the question of whether or when there happens to be a momentum density equal to p_i is not the most relevant one in practice. Statements which are more useful and general can be made about the fluxes of momentum and pseudomomentum. Especially when a Lagrangian-mean description is used, the excess momentum flux due to the waves is often simply related, although not usually equal, to the flux of pseudomomentum. It is basically this fact which accounts for examples of the kind just mentioned. It is also why (4.2) could be used to eliminate the terms in p' during the derivation of (4.10a) from (4.7). The reasons for the existence of such relations are hinted at by Eqs. (2.14), (2.17), (5.1), and the well-known argument about the relation between wave-drag, phase speed, and the rate of working across a material surface.

More explicitly, in many 'slowly-varying' situations it turns out that the mean flow can be defined in such a way that

(1) the excess momentum flux is the only wave term in the leading approximation to the $O(a^2)$ mean-flow problem (Eqs. (4.11) provide an example of this), and

(2) the excess momentum flux is then either equal to the pseudomomentum flux, or differs from it by a contribution C_{ij} which in some cases does not cause systematic mean-flow changes because it can be balanced quasi-statically by the reaction of the medium.

When MHD effects are not involved, and the fluid is compressible, C_{ij} is an isotropic, pressure-like contribution which can be thought of as a kind of acoustic 'hard-spring' effect coming from the nonlinearity of the equation of state[18,37,44]. Analytically this results from redefining the mean pressure in such a way as to avoid having a wave term in the equation of state for the mean flow. For electromagnetic waves in refractive media C_{ij} is again isotropic[42,39] and comes from 'electrostrictive' and 'mag-

* The same remark applies in the classical theory of sound waves.

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netostrictive' effects[41]. In MHD problems C_{ij} is not, however, isotropic[17].

Whether or not such statements are helpful or misleading depends on whether or not, in the problem in question, the difference C_{ij} between the pseudomomentum and excess momentum fluxes has time to be balanced quasi-statically by the mean stress in the medium (it usually does have enough time in 'slowly-varying' situations), as well as on whether that mean stress happens to affect the answer to the particular question being posed. (To take a classical example, C_{ij} is relevant to the force exerted by the absorption of acoustic waves into the end wall of a closed container, but not into an absorber immersed within a larger volume of fluid[43,44]. It is presumably the former situation more than the latter, incidentally, to which the problem of solar wind acceleration by Alfvén radiation pressure is analogous.) When (5.1) holds, and 'group velocity' is meaningful, it is often true that the analogue of (2.17) holds also, namely that Q_{ij} equals P_i times the jth component of the group velocity. So there are some slowly-varying situations (those in which the questions being asked permit C_{ij} to be ignored in some sense) where one can say mnemonically that the waves transport momentum as if a local momentum density equal to P_i were being carried along through a vacuum at the group velocity. But the difficulty of saying in general terms when this mnemonic is applicable, and when it isn't, brings us back in the end to the point made earlier: the only safe and completely general recipe for studying wave transport effects is to consider not only the 'wave properties', which can be evaluated from the O(a), linearised problem, but also a self-consistent analysis, correct to $O(a^2)$, of whatever global mean-flow problem is relevant.

REFERENCES

1. Lindzen,R.S. 1973 Boundary-layer Meteorol.4, 327-43
2. Holton,J.R. 1975 The dynamic meteorology of the stratosphere and mesosphere. Boston, Amer.Met. Soc., 218pp.
3. Fels,S.B. & Lindzen,R.S. 1974 Geophys.Fluid Dyn.6, 149-91
4. Plumb,R.A. 1975 Q.J.Roy.Meteorol.Soc.101, 763-76
5. Spiegel,E.A., Gough,D.O., personal communication
6. E.g. Sakurai,T. 1976 Astrophys.& Space Sci.41, 15-25
7. Holton,J.R. & Mass,C. 1976 J.Atmos.Sci.33, 2218-25
8. E.g. Grimshaw,R. 1975 J.Atmos.Sci.32, 1779-93
9. Eliassen,A. 1952 Astrophysica Norvegica 5, 19-60
10. Phillips,N.A. 1954 Tellus 6, 273-86
11. Matsuno,T. 1971 J.Atmos.Sci. 28, 1479-94
12. Uryu,M. 1974 J.Meteorol.Soc.Japan 52,481-90
13. Grimshaw,R. 1975 J.Fluid Mech.71, 497-512
14. Landau,L.D. & Lifshitz,E.M. 1975 The classical theory of fields, 4th English edition. Pergamon, 402 pp. See also Dougherty,J.P. 1970 J. Plasma Phys. 4, 761-85
15. Eckart,C. 1963 Phys.Fluids 6, 1037-41
16. Hayes,W.D. 1970 Proc.Roy.Soc.A 320, 187-208
17. Dewar,R.L. 1970 Phys.Fluids 13, 2710-20
18. Bretherton,F.P. 1971 Lectures in Appl. Math. 13, 61-102 (Amer.Math.Soc.)

19.Eliassen,A. & Palm,E. 1961 Geofys.Publ.,22#3, 1-23
20.Charney,J.G. & Drazin,P.G. 1961 J.Geophys.Res.66, 83-109
21.Moffatt,H.K. 1977 Magnetic field generation. Cambridge Univ.Press (to appear)
22.Andrews,D.G. & McIntyre,M.E. 1977 Submitted to J.Fluid Mech.
23.Andrews,D.G. & McIntyre,M.E. 1976 J.Atmos.Sci.33, 2031-48
24.Boyd,J. 1976 J.Atmos.Sci. 33, 2285-91
25.Bretherton,F.P. 1977 Submitted to J.Fluid Mech.
26.Bretherton,F.P. 1969 Q.J.Roy.Meteorol.Soc.95, 213-43
27.Lorenz,E.N. 1955 Tellus 7, 157-67
28.Bretherton,F.P. & Garrett.C.J.R. 1968 Proc.Roy.Soc.A 302, 529-54
29.Whitham,G.B. 1974 Linear and nonlinear waves. Wiley.
30.Bretherton,F.P. 1970 J.Fluid Mech 44, 19-31
31.Andrews,D.G., personal communication
32.Andrews,D.G. & McIntyre,M.E. 1977 To appear in J.Atmos.Sci.
33.Eliassen,A. 1968 Geofys.Publ.27#6, 1-15
34.Dickinson,R.E. 1969 J.Atmos.Sci.26, 73-81
35.Uryu,M. 1973 J.Meteorol.Soc.Japan 51,86-92
36.Rayleigh,Lord 1905 Phil.Mag.10, 364-74. (Sci.Papers,5, 262-71)
37.Brillouin,L. 1925 Annales de Physique 4, 528-86
38.Peierls,R. 1976 Proc.Roy.Soc.A 347, 475-91
39.Gordon,J.P. 1973 Phys.Rev.A 8, 14-21
40.McIntyre,M.E. 1973 J.Fluid Mech 60, 801-11
41.Robinson,F.N.H. 1975 Phys.Reports (Sec.C.of Physics Letters) 16, 314-54
42.Penfield,P. & Haus,H.A. 1966 Phys,Fluids 9, 1195-1204
43.Brillouin,L. 1936 Revue d'Acoustique 5, 99-111
44.Rooney,J.A. & Nyborg,W.L. 1972 Amer.J.Phys. 40, 1825-30