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BOUNDARY AND INTERIOR CONTROL FOR PARTIAL DIFFERENTIAL EQUATIONS

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1. Introduction. From the time that the basic existence and regularity problems for partial differential equations have been solved many interesting new variational and control problems could be studied. In general a differential equation or boundary value problem is used to define a class of admissible functions, and then the problem is that of finding the extrema of a given functional defined on that class of functions.

Consider for example the Dirichlet problem for a given second order elliptic differential expression L on a domain G with boundary ∂G: $Lu(x) = f(x)$, $x \in G$; $u(x) = g(x)$, $x \in \partial G$. To generate a class of admissible functions one could, for example, vary one of the coefficients of L within a given collection of functions. Another possibility is to vary f or write $f(x) = h(x, p(x))$ and vary p and the same could be done for the function g. A collection of functions in which the variations occur is called a control set. For parabolic, hyperbolic and higher order equations one can proceed analogously.

The variational problem is that of finding elements in the class of admissible functions for which a given functional attains an extremum. The control problem is that of finding an element in the control set to generate such a function.

These problems appear in physics and engineering in the control of processes that are described by partial differential equations. Also from a purely mathematical point of view the problems are of importance. Their study often leads to new boundary value problems and in variational problems it makes good sense to restrict *a priori* the class of functions that is considered to functions that can actually occur or that are otherwise interesting.

An extensive study of this type of control problems is given in the book of J. L. Lions; Contrôle optimal de systèmes gouvernés par des equations aux dérivées partielles, [**7**]. In this book many references to the literature and historical remarks are given.

Elliptic variational problems, the main subject of this paper is closely related to the classical calculus of variations. It is the variational problem for the common functional of the calculus of variations where the class of admissible functions consists of all sufficiently smooth solutions of an elliptic partial differential equation. Of course it can also be considered as a control problem, for example, modelled on the Dirichlet problem where the control is exercised

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through the boundary by freely choosing sufficiently smooth values for the boundary conditions.

A more explicit description for second order equation and first order functional omitting some smoothness requirements is the following. Let G be an open bounded domain in R^ν with boundary ∂G and let Du denote the first order partial derivatives of u. Let a functional J be given by

$$(1.1) \qquad J(u) = \int_G F(x, u(x), Du(x))\, dx$$

and let a uniformly elliptic differential equation be given by

$$(1.2) \qquad L = \sum_{|\alpha| \leq 2} a_\alpha(x) D^\alpha, \quad x \in G.$$

The class of admissible functions U consists of all sufficiently smooth solutions of the equation $Lu(x) = f(x)$, $x \in G$, where f is a given function. For this problem necessary conditions equivalent to the vanishing of the first variation of J at u with respect to all admissible variations are given by

$$(1.3) \qquad \begin{aligned} Lu(x) &= f(x), x \in G, \\ L^*v(x) &= [F]_{,u}, x \in G \\ v(x) &= 0, x \in \partial G, \\ \partial v/\partial \mathbf{n} &= -\Big\{ \sum_{|\alpha|=2} n^\alpha a_\alpha(x) \Big\}^{-1} \sum_{\beta=1} n^\beta (\partial F/\partial(D^\beta u)), \quad x \in \partial G. \end{aligned}$$

(L^* is the formal adjoint of L.) Thus if J has a relative extremum within U at u then corresponding to that u there exists a function v that satisfies the last three conditions of (1, 3). Such a function v is called a *variational adjoint* of u.

The result (1.3) was first published in [**4**]. Here it was assumed that L was uniformly strongly elliptic with uniquely solvable Dirichlet problem. In [**6**] the results were extended to the case where the differential expression is of order m and the functional is of order $m/2$. In addition, the problem with parabolic differential expression was considered. In these papers some examples of how (1.3) can be used are given. In the sections 3 and 4 of the present paper, necessary conditions analogous to (1.4) are derived for the variational problem where the differential expression is uniformly strongly elliptic of order m and the functional is of order m. It is no longer required that the Dirichlet problem is uniquely solvable. The improvements were made possible and the derivations were greatly simplified by Lemma 2.4.

In section 5, the techniques that are developed for the variational problem are used to obtain necessary conditions for a more specific control problem. In the second order elliptic boundary and interior control problem for J and L as given in (1.1) and (1.2) the class of admissible functions W is the set of all sufficiently differentiable solutions of

$$(1.4) \qquad \begin{aligned} Lw(x) &= f(x, p(x)), \quad p \in P, x \in G, \\ w(x) &= g(x, q(x)), \quad q \in Q, x \in \partial G. \end{aligned}$$

It is assumed that the Dirichlet problem for L is uniquely solvable, f and g are given functions, P is the interior control set and Q is the boundary control set. For simplicity in this chapter $P = C^{\infty}(\bar{G})$ and $Q = C^{\infty}(\partial G)$. Now the necessary conditions take the form

$$(1.5) \qquad \begin{aligned} &Lu(x) = f(x, p(x)), p \in P, x \in G \\ &L^*v(x) = [F]_{,u}, x \in G \\ &v(x)\partial f(x, p(x))/\partial p = 0, x \in G. \\ &u(x) = g(x, q(x)), x \in \partial G, q \in Q \\ &v(x) = 0, x \in \partial G \\ &\left\{\frac{\partial v}{\partial \mathbf{n}} \sum_{|\alpha|=2} n^{\alpha} a_{\alpha} + \sum_{|\alpha|=1} \frac{n^{\alpha} \partial F}{\partial (D^{\alpha} u)}\right\} \frac{\partial g}{\partial q} = 0, x \in \partial G. \end{aligned}$$

This set of equations is called the control boundary value problem.

Let $A = \{x \in G : \partial f(x, p(x))/\partial p \neq 0\}$, let ∂A be the boundary of A and $B = \{x \in \partial G : \partial g(x, q(x))/\partial q = 0\}$. Then one observes that u satisfies the Euler equation in A and the transversality condition on $(\partial A \cap \partial G) \sim B$ while the variational boundary value problem is satisfied in $G \sim A$ and on $\partial G \sim B$. Thus both the classical necessary conditions of the calculus of variations and the variational boundary value problem are contained in (1.5).

The methods of this paper are basically integration by parts and classical variation techniques. While the results of the Dirichlet problem are used no Sobolev space methods appear. The results are necessary conditions in the form of boundary value problems for sufficiently differentiable minimizing functions. Except for some isolated and simple results (see e.g. [**4**; **6**]), little yet is known about sufficient conditions. For certain special functionals (see e.g. [**8**]) the problems are within the range of the methods in J. L. Lions [**7**, Chapters I, II]. In this book, using a generalization of the Lax-Milgram lemma, existence conditions in Sobolev spaces are established. The differentiability results however are not sufficient to justify the derivation of the necessary conditions of this paper.

An analysis of the elliptic variational problem in a generalized sense is given in our paper *Critical points on closed elliptic affine subspaces* to appear in Proc. Amer. Math. Soc. In this paper an extension of the Lagrange multiplyer method, based on Banach's closed ranged theorem, is derived and then is used to get necessary conditions which are equivalent to the elliptic boundary value problem if the differentiability conditions of this paper are satisfied. The paper also contains existence results.

Notation. R^ν is the ν-dimensional Euclidian space. G is an open domain in R^ν with boundary ∂G. $\partial G \in C^k$ denotes that G is k-times continuously differentiable. (See, e.g., [**12**].) If $\partial G \in C^1$ then $\mathbf{n}$ is the outward unit normal to ∂G. If $A \subset R^\nu$ then nbh. A is an open set in R^ν that contains A. The usual multi-index notation is used. (See, e.g., [**11**].) Multi-indices are denoted by α, β, γ and δ. $D_i u(x) = \partial u(x)/\partial x_i$. $D_i^k u(x) = \partial^k u(x)/\partial x_i^k$. $D^k u(x)$ denotes the k'th order partial derivatives of u. $(\partial/\partial \mathbf{n})^j u(x)$ denotes the j'th normal derivative of u. $C^k(G)$ is the collection of all k-times continuously differentiable functions defined on G. $C^k(\bar{G})$ consists of all elements of $C^k(G)$ whose derivatives of order less than or equal to k can be extended as continuous functions in $\bar{G}$.

2. Preliminary theorems. In this section some results that are needed in the sequel are stated. Proofs of the new material are given in the appendix. The Lemmas 1, 2, 3 and 4 are technical in nature.

LEMMA 2.1. *Let G be a bounded domain in R^ν with $\partial G \in C^{k+h}$, $k \geqq 0$, $h \geqq 0$, and let Ω be an open domain in R^ν that contains $\bar{G}$. Then to every set of functions $g_j(x) \in C^{k+h-j}(\partial G)$, $0 \leqq j \leqq k$, there exists at least one function $w \in C^{k+h}(\Omega)$ that satisfies $(\partial/\partial \mathbf{n})^j w = g_j(x)$, $x \in \partial G$, $0 \leqq j \leqq k$.*

LEMMA 2.2. *Assume that:*
(1) *G is bounded domain in R^ν with $\partial G \in C^k$;*
(2) *$w \in C^k(\bar{G})$;*
(3) *$(\partial/\partial \mathbf{n})^j w = 0$, $x \in \partial G$, $0 \leqq j \leqq k-1$.*
Then $D^\alpha w = 0$, $x \in \partial G$, $0 \leqq |\alpha| \leqq k-1$, and $D^\alpha w = \mathbf{n}^\alpha (\partial/\partial \mathbf{n})^k w$, $x \in \partial G$, $|\alpha| = k$.

The proofs can be found in [**12**] and in [**6**] respectively.

LEMMA 2.3. *Let G, ∂G, Ω and the functions g_j, $0 \leqq j \leqq k$, be as in Lemma 2.1 and let B_j, $0 \leqq j \leqq k$ be a system of differential expressions defined near ∂G, given by*

$$B_j = \sum_{|\alpha| \leqq j} b_j^\alpha(x) D^\alpha, \quad b_j^\alpha \in C^{k+h-j}(\text{nbh. } \partial G), \textit{ with}$$

$$\sum_{|\alpha|=j} \mathbf{n}^\alpha(x) b_j^\alpha(x) \neq 0, \quad x \in \partial G.$$

Then there exists at least one function $w \in C^{k+h}(\Omega)$ that satisfies $B_j w(x) = g_j(x)$, $x \in \partial G$, $0 \leqq j \leqq k$.

An analogous Lemma, without precise differentiability results, was proved, in [**3**]. This proof depends on a transformation of the system $N_j w(x) = g_j(x)$, $x \in \partial G$, $0 \leqq j \leqq k$, into a system that acts directly on the normal derivatives of w. A simple proof depending only on Lemmas 1 and 2 is given in the appendix.

LEMMA 2.4. *Let an integral I be given by*

$$(2.1)\qquad I = \int_{\partial G} \sum_{|\alpha|=0}^{m} b^{\alpha}(x)\, D^{\alpha} q(x) dS$$

and assume that G *is a bounded domain in* R^{ν} *with* $\partial G \in C^{r+1}$, $r \geqq m$, *that* $b^{\alpha} \in C^{|\alpha|}(nbh.\ \partial G)$ *and that* $q \in C^{m}(\text{nbh. } \partial G)$. *Then there exist linear differential expressions* M_j^{α}

$$(2.2)\qquad M_j^{\alpha} = \sum_{|\beta|=0}^{|\alpha|-j} m_j^{\alpha\beta}(x) D^{\beta}, \quad m_j^{\alpha\beta} \in C^{t+j+|\beta|-|\alpha|}(\text{nbh. } \partial G),$$

such that

$$(2.3)\qquad I = \int_{\partial G} \sum_{j=0}^{m} (\partial/\partial \mathbf{n})^{j} q \sum_{|\alpha|=j}^{m} M_j^{\alpha} b^{\alpha} dS.$$

A proof is given in the appendix.

In J. L. Lions – E. Magenes [**9**] a very general Green's formula for elliptic differential expression is given. The equation (2.4) below corresponds to the case where one of the systems of boundary operators are the normal derivatives. The Green's identity that is given below is more general than the corresponding special form in [**9**] because it is not restricted to elliptic equations and because it gives explicit information about the leading part of the boundary operators, equation (2.5). This is of vital importance for the derivation of the forth equation in (4.12).

It follows immediately that the boundary is non-characteristic in points that have a neighbourhood where L is elliptic.

THEOREM 2.1. GREEN'S IDENTITY. *Assume that:*
(1) G *is a bounded domain in* R^{ν} *with* $\partial G \in C^{m+t}$, $t \geqq 0$;
(2) $u \in C^{m}(\bar{G})$, $v \in C^{m}(\bar{G})_{i}$;

(3) $L = \sum_{|\alpha| \leqq m} a_{\alpha}(x)\, D^{\alpha}$ *with* $a_{\alpha} \in C^{|\alpha|+s}(\bar{G})$, $s \geqq \max.\ (0, t+1)$.

Then Green's Identity can be written as (2, 4) *with* (2, 5)

$$(2.4)\qquad \int_{G} (uLv - vL^{*}u) dV = \int_{\partial G} \sum_{j=0}^{m-1} (\partial^{j} v/\partial \mathbf{n}^{j}) N_j^{m-1-j} u dS,$$

$$N_j^{m-1-j} = \sum_{|\alpha| \leqq m-1-j} p_j^{\alpha(x)} D^{\alpha},\ 0 \leqq j \leqq m-1, \quad p_j^{\alpha} \in C^{t+|\alpha|+j}(\partial G).$$

Moreover,

$$(2.5)\qquad \sum_{|\alpha|=m-1-j} p_j^{\alpha} \mathbf{n}^{\alpha} = (-1)^{m-1-j} \sum_{|\alpha|=m} a_{\alpha} \mathbf{n}^{\alpha}, \quad x \in \partial G.$$

A proof based on Lemma 2.4. is given in the appendix. L^* is the formal adjoint of L.

3. The variational adjoint. Let a functional J on $C^l(\bar{G})$ into R be given by

$$J(u) = \int_G F(x, u(x), Du(x), \ldots, D^l u(x)) dV$$

where F is continuous with respect to all its variables.

The class of *admissible functions* U consists of all solutions of a given differential equation of order $m, m \geqq l, Lu(x) = f(x), x \in G$, where $L = \sum_{|\alpha| \leqq m} a^\alpha(x) D^\alpha$, that are of class $C^l(\bar{G})$. With norm

$$||u||_l = \sum_{|\alpha| \leqq l} \sup_{x \in G} |D^\alpha u(x)|,$$

$C^l(\bar{G})$ is a Banach space. U is considered as an affine subspace of this Banach space. As usual the class of *admissible variations* U_0 is defined by $U_0 = \{\delta u : \delta u = u_1 - u_2, u_1 \in U, u_2 \in U\}$. It is observed that $\delta u \in U_0$ implies that $L\delta u = 0$ and that U_0 is a linear subspace of the Banach space $C^l(\bar{G})$.

A *relative extremum* of J at $u \in U$ within U is defined with respect to the norm $|| \quad ||_l$. If J has a (Frechet) derivative at $u \in U$ denoted by $J'(u)$ then $J'(u)\delta u$, $\delta u \in U_0$, is called the first variation of J at u with respect to δu. This quantity is written as $\delta J(u; \delta u)$.

LEMMA 3.1. *If F is continuously differentiable with respect to all its arguments, $u \in U$ and $\delta u \in U_0$ then*

$$\delta J(u; \delta u) = \int_G \sum_{|\alpha|=0}^{l} (D^\alpha \delta u)(\partial F / \partial (D^\alpha u)) dV. \tag{3.1}$$

Moreover, if J has a relative extremum within U for some $u \in U$ then $\delta J(u; \delta u) = 0$, for all $\delta u \in U_0$.

The proof follows immediately from application of the mean value theorem to F. Details can be found in [**6**].

LEMMA 4.2. *If F is $(l + 1)$-times continuously differentiable, $u \in C^{2l}(\bar{G})$, $\delta u \in C^l(\bar{G})$, and $\delta G \in C^1$, then $\delta J(u; \delta u)$ can be put in the form*

$$\delta J(u; \delta u) = \int_G \delta u [F]_{,u} dV + \int_{\delta G} \sum_{|\alpha|=0}^{l-1} (D^\alpha \delta u) Q^\alpha u dS \tag{3.2}$$

$$[F]_{,u} = \sum_{|\alpha| \leqq l} (-1)^{|\alpha|} D^\alpha (\partial F / \partial (D^\alpha u)) \tag{3.3}$$

$$Q^\alpha u = \sum_{\substack{|\beta| \leqq l-1-|\alpha|, \\ |\gamma|=1}} (-1)^{|\beta|} \frac{(\alpha + \beta + \gamma)!}{|\alpha + \beta + \gamma|!} \mathbf{n}^\gamma D^\beta \frac{\partial F}{\partial (D^{\alpha+\beta+\gamma} u)}. \tag{3.4}$$

This is well-known. $[F]_{,u}$ is the Euler-expression for J. The differential expressions Q^α appear in the transversality conditions of the classical calculus of variations problem. The proof depends on integration by parts. Details can be found in [**6**].

THEOREM 3.1. *Suppose*

(i) F *is* $(l+1)$*-times continuously differentiable,* $u \in C^{2l}(\bar{G})$, $\delta u \in C^m(\bar{G})$, $\partial G \in C^m$.

(ii) $L\delta u(x) = 0, x \in G$.

(iii) $a_\alpha \in C^{|\alpha|}(\bar{G}), 0 \leqq |\alpha| \leqq m$.

(iv) *There exists a function* $v \in C^m(\bar{G})$ *that satisfies* $L^*v = [F]_{,u}$, $x \in G$. *Then* $\delta J(u; \delta u)$ *can be written as*

$$\delta J(u;\delta u) = \int_{\partial G} \left\{ \sum_{j=0}^{l-1} (\partial^j \delta u/\partial \mathbf{n}^j) \left\{ \sum_{|\alpha|=j}^{l-1} M_j^\alpha (Q^\alpha u) - N_j^{m-1-j} v \right\} - \sum_{j=l}^{m-1} (\partial^j \delta u/\partial \mathbf{n}^j) N_j^{m-1-j} v \right\} dS. \tag{3.5}$$

(For the definitions of M_j^α and N_j^{m-1-j} see (2.2) and Theorem 2.1. respectively.)

Proof. Under the above conditions one obtains from Green's Identity (2.4)

$$\int_G \delta u [F]_{,u} dV = \int_G (\delta u L^* v - vL\delta u) dV$$
$$= \int_{\delta G} \sum_{j=0}^{m-1} -(\partial^j \delta u/\partial \mathbf{n}^j) N_j^{m-1-j} v dS.$$

Substitution of the result into (3.2) and application of Lemma 2.4 to the boundary integral in (3.2) completes the proof.

Definition. Suppose (i) F is $(l+1)$-times continuously differentiable, $a_\alpha \in C^{|\alpha|}(\bar{G})$ for $0 \leqq |\alpha| \leqq m$, and $\partial G \in C^m$.

(ii) $Lu(x) = f(x), x \in G; u \in C^{2l}(\bar{G})$.

(iii) There exists a function $v \in C^m(\bar{G})$ that satisfies

$$L^* v(x) = [F]_{,u}, x \in G, \text{ and} \tag{3.6}$$

$$N_j^{m-1-j} v = \sum_{|\alpha|=j}^{l-1} M_j^\alpha (Q^\alpha u), \quad 0 \leqq j \leqq l-1 \tag{3.7}$$
$$N_j^{m-1-j} v = 0, \quad l \leqq j \leqq m-1, x \in \partial G.$$

Then that function v is called a *variational adjoint* of u with respect to J and L. (If $m = l$, the last set of equations is vacuous.)

If the conditions (i), (ii) of the above definition are satisfied and if u has a variational adjoint with respect to J and L then $\delta J(u; \delta u) = 0$ for all $\delta u \in U_0 \cap C^l(\bar{G})$. In general, with some additional conditions, the converse is also true. For strongly elliptic equations with uniquely solvable Dirichlet problem, $l \leqq \frac{1}{2} m$, and for uniformly parabolic equations, $l \leqq \frac{1}{2}m$, this was

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shown in [6]. In section 4 of this paper the converse is proved for general strongly elliptic equations, $l \leqq m$.

THEOREM 3.2. *Let v be a variational adjoint of u and let $\partial G'$ denote the non-characteristic part of ∂G. Then the variational adjoint boundary conditions* (3.7), *for $x \in \partial G'$, are equivalent with*

$$(3.8) \quad N_j^{m-1-j} v = \sum_{|\alpha|=j}^{l-1} M_j^{\alpha}(Q^{\alpha}u), 0 \leqq j \leqq l-2.$$

$$(3.9) \quad (\partial/\partial \mathbf{n})^{m-l} v = \sum_{|\alpha|=l, |\beta|=m} (-1)^{m-l}(\mathbf{n}^{\beta}a_{\beta})^{-1}\mathbf{n}^{\alpha}(\partial F/\partial D^{\alpha}u).$$

$$(3.10) \quad (\partial/\partial \mathbf{n})^{m-1-j} v = 0, l \leqq j \leqq m-1 \text{ (vacuous if } l = m).$$

Proof. Let $l < m$. From Theorem 2.1 one obtains $N_{m-1}{}^0 v = p_{m-1}v = \sum_{|\alpha|=m} a_{\alpha}\mathbf{n}^{\alpha}v$ so that $N_{m-1}{}^0 v = 0$, $x \in \partial G'$, implies that $v = 0$, $x \in \partial G'$, and conversely.

If $l < m - 1$, then $v = 0$, $x \in \partial G'$, hence by Theorem 2.1 and Lemma 2.2

$$N_{m-2}{}^1 v = \sum_{|\alpha| \leqq 1} p_{m-2}{}^{\alpha} D^{\alpha} v = - \sum_{|\alpha|=m} a_{\alpha}\mathbf{n}^{\alpha}(\partial v/\partial \mathbf{n}).$$

Hence, under the present conditions, $N_{m-2}{}^1 v = 0, x \in \partial G'$ implies $(\partial v/\partial \mathbf{n}) = 0$, $x \in \partial G'$.

Repeating this process one obtains the equivalence on $\partial G'$ of the last set of equations of (3.7) to (3.10). (If $l = m$, the above part of the proof should be omitted.)

It remains to be shown that the equation

$$N_{l-1}^{m-l} v = \sum_{|\alpha|=l-1} M_{l-1}{}^{\alpha}(Q^{\alpha}u), \quad x \in \partial G',$$

can be written as (3.9) if (3.10) is satisfied. If the last is true then using Theorem 2.1 and Lemma 2.2 one obtains as before

$$(3.11) \quad N_{l-1}^{m-l} v = (-1)^{m-l} \sum_{|\alpha|=m} a_{\alpha}\mathbf{n}^{\alpha}(\partial/\partial \mathbf{n})^{m-l} v.$$

As in the proof of Theorem 1 of this section one obtains from Lemma 2.4 that

$$\int_{\partial G} \sum_{|\alpha|=0}^{l-1} (D^{\alpha}q)(Q^{\alpha}u)dS = \int_{\partial G} \sum_{j=0}^{l-1} (\partial/\partial \mathbf{n})^j q \sum_{|\alpha|=j}^{l-1} M_j{}^{\alpha}(Q^{\alpha}u)dS$$

where q can be chosen freely in C^{l-1} (nbh. ∂G). Letting $(\partial/\partial \mathbf{n})^j q = 0, 0 \leqq j \leqq l-2$, $x \in \partial G$, one obtains from varying $(\partial/\partial \mathbf{n})^{l-1} q$ that

$$(3.12) \quad \sum_{|\alpha|=l-1} n^{\alpha}(Q^{\alpha}u) = \sum_{|\alpha|=l-1} M_{j1}^{\alpha}(Q^{\alpha}u)$$

and from substitution of (3.4),

$$(3.13) \quad \sum_{|\alpha|=l-1} \mathbf{n}^{\alpha}(Q^{\alpha}u) = \sum_{|\alpha|=l} \mathbf{n}^{\alpha}(\partial F/\partial(D^{\alpha}u)).$$

From (3.11), (3.12) and (3.13) one gets the desired equation.

4. Necessary conditions for the elliptic variational problem. In this section L is a *uniformly strongly elliptic* differential expression of order m defined on an open bounded domain G in R^ν. So

$$(4.1) \qquad L = \sum_{|\alpha| \leqq m} a_\alpha(x) D^\alpha, \quad x \in G,$$

with uniformly bounded principal coefficients and

$$(-1)^{m/2} \sum_{|\alpha| \leqq m} a_\alpha(x) \xi^\alpha \geqq c|\xi|,$$

for some positive constant c and arbitrary ν-vector ξ.

First a well-known existence and regularity theorem for the classical Dirichlet problem will be stated (see e.g. [**11**] and Lemma 2.1). It will be used in the derivation of Theorem 2, the main result of this paper.

THEOREM 4.1. *Let the following conditions be satisfied for integers p and t, $p \geqq 0$, $t = p + [\nu/2] + 1$.*

(i) *G is a bounded domain in R^ν with $\partial G \in C^{m+t}$.*

(ii) *L is uniformly strongly elliptic in G, with $a_\alpha \in C^t(\bar{G})$, $0 \leqq |\alpha| \leqq m/2$; $a_\alpha \in C^{|\alpha|+t-m/2}(\bar{G})$ $m/2 \leqq |\alpha| \leqq m$.*

(iii) *$f \in C^t(\bar{G})$.*

(iv) *$g_j \in C^{m+t-j}(\partial G)$, $0 \leqq j \leqq m/2 - 1$.*

Then the Fredholm alternative holds for the classical Dirichlet problem

$$(4.2) \qquad \begin{aligned} &Lu(x) = f(x), \quad x \in G, \\ &(\partial/\partial \mathbf{n})^j u(x) = g_j(x), \quad x \in \partial G, 0 \leqq j \leqq m/2 - 1, \end{aligned}$$

while any solution is of class $C^{m+p}(\bar{G})$.

FREDHOLM ALTERNATIVE. Let $N(L)$ and $N(L^*)$ denote the null space of L and L^*, respectively. Then $N(L)$ and $N(L^*)$ are subspaces of $L_2(G)$ of the same finite dimension. If this dimension is zero, then the Dirichlet problem is uniquely solvable, independent of the particular choice of the functions f and g_j. If the dimension of the null spaces is positive then the Dirichlet problem

$$(4.3) \qquad \begin{aligned} &Lu(x) = f(x), \quad x \in G \\ &\partial^j u/\partial \mathbf{n}^j = 0, \quad 0 \leqq j \leqq m/2 - 1, x \in \partial G, \end{aligned}$$

is solvable if and only if f is orthogonal in $L_2(G)$ to $N(L^*)$.

THEOREM 4.2. *Let k be a positive integer and suppose that*

(i) *G is bounded in R^ν with $\partial G \in C^{m+[\nu/2]+1+k}$;*

(ii) *L is uniformly strongly elliptic with $a_\alpha \in C^{k+[\nu/2]+1+|\alpha|}(\bar{G})$, $0 \leqq |\alpha| \leqq m/2$, and, $a_\alpha \in C^{k+[\nu/2]+1+2|\alpha|-m/2}(\bar{G})$, $m/2 + 1 \leqq |\alpha| \leqq m$;*

(iii) *$f \in C^{k+[\nu/2]+1}(\bar{G})$;*

(iv) *F is $l + [\nu/2] + 1$ times continuously differentiable with respect to all its arguments;*

(v) *U is the collection of all solutions of $Lu = f$. Then $U \subset C^{m+k}(\bar{G})$ and if there exists a $u \in U$ such that $\delta J(u; \delta u) = 0$ for all $\delta u \in U_0$* (U_0 is the class of admissible variations corresponding to U), *then to that function u there exists a variational adjoint with respect to L and J. Conversely, if there exists a variational adjoint to some $u \in U$ then $\delta J(u; \delta u) = 0$ for all $\delta u \in U_0$.*

Proof. In consequence of the Fredholm alternative two cases are to be considered. They are treated separately.

I. $N(L)$ and $N(L^*)$ both consist of the trivial solution only.

Let $v \in C^m(\bar{G})$ be a solution of

$$\begin{aligned} &L^*v(x) = [F]_{,u}, x \in G \\ (4.4)\qquad &N_j^{m-1-j}v = 0, m-1 \geqq j \geqq \max.(l, m/2), x \in \partial G, \\ &N_j^{m-1-j}v = \sum_{|\alpha|=j}^{l-1} M_j^{\alpha}(Q^{\alpha}u), l-1 \geqq j \geqq m/2, x \in \partial G. \end{aligned}$$

(If $l \leqq m/2$, the last condition is vacuous). That such a function v exists follows directly from Lemma (2.3) and Theorem 1 of this section. The N_j^{m-1-j}'s, given in (2.4) satisfy the conditions of the Lemma because of property (2.5) together with the uniform strong ellipticity condition on L (∂G is non-characteristic for all N_j^{m-1-j}'s).

With this function v one obtains from Theorem 3.1

$$(4.5)\qquad \delta J(u; \delta u) = \int_{\partial G} \sum_{j=0}^{m/2-1} (\partial/\partial\mathbf{n})^j \delta u \left\{ \sum_{|\alpha|=j}^{l-1} M_j^{\alpha}(Q^{\alpha}u) - N_j^{m-1-j}v \right\} dS.$$

(If $j \geqq l$ then $M_j^{\alpha}(Q^{\alpha}u)$ must be omitted.) For any choice of sufficiently differentiable functions g_j the system

$$(4.6)\qquad \begin{aligned} &L\delta u(x) = 0, \quad x \in G, \\ &(\partial/\partial\mathbf{n})^j \delta u(x) = g_j(x), \quad x \in \partial G \end{aligned}$$

has a solution which is an element of U_0. As $\delta J(u; \delta u) = 0$, $\delta u \in U_0$, one obtains from varying the normal derivatives in (4.5)

$$(4.7)\qquad \begin{aligned} &N_j^{m-1-j}v = 0, \quad m/2 - 1 \geqq j \geqq l, x \in \partial G, \\ &N_j^{m-1-j}v = \sum_{|\alpha|=j}^{l-1} M_j^{\alpha}(Q^{\alpha}v), \quad l-1 \geqq j \geqq 0, x \in \partial G. \end{aligned}$$

(If $l \geqq m/2$, the first set of equations is vacuous.) As v satisfies both (4.4) and (4.7) it is a variational adjoint.

II. Dim. $N(L)$ and Dim. $N(L^*)$ are positive.

Again, first it will be shown that there exists a function $v \in C^m(\bar{G})$ that satisfies (4.4). In view of Theorem 1 this is true if and only if

$$(4.8)\qquad (\delta u, [F]_{,u} - L^*w) = 0, \quad \delta u \in N(L)$$

Here $w \in C^{m+[\nu/2]+1}(\bar{G})$ is any function that satisfies the boundary conditions of (4.4), $w \equiv 0$ if $l \leqq m/2$. Now it will be shown that

$$(4.9) \qquad (\delta u, [F]_{,u} - L^*w) = \delta J(u; \delta u), \quad \delta u \in N(L).$$

If $l \leqq m/2$, this follows immediately from (3.2). If $l > m/2$ and $\delta u \in N(L)$ one has

$$\begin{aligned}(\delta u, [F]_{,u} - L^*w) &= (\delta u, [F]_{,u}) + (wL\delta u - \delta u L^*w) = \\ &(\delta u, [F]_{,u}) + \int_{\partial G} \sum_{j=m/2}^{l-1} ((\partial/\partial \mathbf{n})^j \delta u) N_j^{m-1-j} w \, dS = \\ &(\delta u, [F]_{,u}) + \int_{\partial G} \sum_{j=m/2}^{l-1} ((\partial/\partial \mathbf{n})^j \delta u) \sum_{|\alpha|=j} M_j^{\alpha}(Q^{\alpha} u) \, dS = \\ &\qquad (\delta u, [F]_{,u}) + \int_{\partial G} \sum_{|\alpha|=m/2}^{l-1} (D^{\alpha}\delta u) Q^{\alpha} u \, dS\end{aligned}$$

and again (4.9) follows from (3.2). As $N(L) \subset U_0$ and $\delta J(u; \delta u) = 0, \delta u \in U_0$, one obtains (4.8) from (4.9).

Let v' be one of the functions that satisfy (4.4). With this function $\delta J(u; \delta u)$ can be written again as in (4.5). This expression must vanish for all $\delta u \in U_0$. The normal derivatives of δu can not be chosen freely. Using Green's Identity one gets that

$$(4.10) \qquad \int_{\delta G} \sum_{j=0}^{m/2-1} ((\partial/\partial \mathbf{n})^j \delta u) N_j^{m-1-j} z \, dS, \quad \text{for all } z \in N(L^*).$$

must be satisfied. Therefore

$$(4.11) \qquad \sum_{|\alpha|=j}^{l-1} M_j^{\alpha}(Q^{\alpha} u) - N_j^{m-1-j} v' = N_j^{m-1-j} z', \quad m/2 - 1 \geqq j \geqq 0,$$

where $z' \in N(L^*)$. (If $j \geqq l$ then $\sum M_j^{\alpha}(Q^{\alpha}u)$ must be omitted.) From (4.4) and (4.11) it follows that $v = v' + z'$ is a variational adjoint of u with respect to L and J.

In both cases the converse follows directly from Theorem 1 of the previous section. This completes the proof.

COROLLARY. *If the conditions* (i)–(v) *of Theorem* 2 *are satisfied and if J has a relative extremum within U for some $u \in U$ then there exists a function $v \in C^m(\bar{G})$ such that*

$$(4.12) \qquad \begin{aligned}&Lu(x) = f(x), \quad x \in G, \\ &L^*v(x) = [F]_{,u}, \quad x \in G, \\ &(\partial/\partial \mathbf{n})^j v(x) = 0, \quad x \in \partial G, 0 \leqq j \leqq m - l - 1, \\ &(\partial/\partial \mathbf{n})^{m-l} v(x) = \sum_{|\alpha|=l, |\beta|=m} (-1)^{m-l} \frac{\mathbf{n}^{\alpha}}{(\mathbf{n}^{\beta} a_{\beta})} \frac{\partial F}{\partial (D^{\alpha} u)}, \quad x \in \partial G, \\ &N_j^{m-1-j} v(x) = \sum_{|\alpha|=j}^{l-1} M_j^{\alpha}(Q^{\alpha} u), \quad l - 2 \geqq j \geqq 0.\end{aligned}$$

The above set of equations is called the *variational boundary value problem.* The first set of boundary conditions is vacuous if $l = m$. If $l = 1$ the last equation is vacuous. In this case the boundary value problem takes a much simpler form as no complicated computations for the functions M_j^α and Q^α need to be carried out.

5. The elliptic boundary and interior control problem. As in the previous sections a functional J is given by

$$J(w) = \int_G F(x, w(x), Dw(x), \ldots, D^l w(x) dV,$$

where F is continuous with respect to all its arguments and a differential expression L is given by

$$L = \sum_{|\alpha| \leqq m} a^\alpha(x) D^\alpha, \quad x \in G, m \geqq l.$$

Throughout this section it is assumed that L is uniformly strongly elliptic and that the Dirichlet problem is uniquely solvable. ($N(L) = \{0\}$.)

The *interior control set* P and the *boundary control sets* Q_j, $0 \leqq j \leqq m/2 - 1$, are given linear function spaces defined on G and ∂G respectively. The class of *admissible functions* W is the set of all solutions of

$$\begin{aligned} &Lw(x) = f(x, p(x)), \quad p \in P, x \in G, \\ &(\partial/\partial \mathbf{n})^j w(x) = g_j(x, q_j(x)), \quad q_j \in Q_j, 0 \leqq j \leqq m/2 - 1, x \in \partial G, \end{aligned} \tag{5.1}$$

which are of class $C^l(\bar{G})$. The functions f and g_j, $0 \leqq j \leqq m/2 - 1$, are given fixed functions defined on $(\bar{G} \times R)$ and $(\partial G \times R)$, respectively. The functions p and q_j can be chosen freely in P and Q_j, $0 \leqq j \leqq m/2 - 1$, respectively. The class of *admissable* variations δW and a relative extremum are defined as in section 3. In this section $\delta J(u; \delta u)$ will not simply be $J'(u)\delta u$ but $J'(u)\delta_1 u$ where $\delta_1 u$ is the principal part of δu which will be given a precise meaning in the sequel ($\delta u = \delta_1 u + \delta_2 u$ and see (5.3), (5.4)).

THEOREM 5.1. *Let the following conditions be satisfied.*

(i) *F is continuously differentiable.*

(ii) $a_\alpha \in C^{[\nu/2]+2}(\bar{G})$, $\quad 0 \leqq |\alpha| \leqq m/2$,
$a_\alpha \in C^{|\alpha|+[\nu/2]+2-m/2}(\bar{G})$, $\quad m/2 < |\alpha| \leqq m$.

(iii) *The functions f, g_j and their first and second derivatives with respect to their second variable are of class $C^{[\nu/2]+1}(\bar{G} \times R)$ and $C^{[\nu/2]+2+m-j}(\partial G \times R)$, $0 \leqq j \leqq m/2 - 1$, respectively.*

(iv) $P \subset C^{[\nu/2]+2}(\bar{G})$; $Q_j \subset C^{[\nu/2]+2+m-j}(\partial G)$, $0 \leqq j \leqq m/2 - 1$.

Then

$$\delta J(w, \delta w) = \int_G \sum_{|\alpha|=0}^{l} (D^\alpha \delta_1 w)(\partial F/\partial D^\alpha w) dV, \tag{5.2}$$

where $\delta_1 w$ is a solution of

$$\text{(5.3)}\qquad \begin{cases} L\delta_1 w = \delta p(\partial f/\partial p) \\ (\partial/\partial \mathbf{n})^j \delta_1 w = \delta g_j(\partial g_j/\partial q_j), \quad 0 \leqq j \leqq m/2 - 1, \end{cases}$$

equals zero for all $\delta w \in \delta W$ if J has a relative extremum within W at $w \in W$.

Proof. The existence and regularity requirements are based on Theorem 4.1. They will not be mentioned explicitely.

The function δw is the solution of

$$L\delta w = f(x, p + \delta p) - f(x, p)$$
$$(\partial/\partial \mathbf{n})^j \delta w = g_j(x, g_j + \delta q_j) - g_j(x, q_j), 0 \leqq j \leqq m/2 - 1,$$

so that $\delta w = \delta_1 w + \delta_2 w$, where $\delta_1 w$ is given above and $\delta_2 w$ is the solution of

$$L\delta_2 w = (\delta p)^2 \partial^2 \bar{f}/\partial p^2$$
$$(\partial/\partial \mathbf{n})^j \delta_2 w = (\delta g_j)^2 \partial^2 \bar{g}_j/\partial q_j^2, \quad 0 \leqq j \leqq m/2 - 1.$$

(An overbar indicates that the function must be evaluated for intermediate values; e.g., $\partial^2 \bar{f}/\partial p^2 = \partial^2 f(x, p(x) + \theta(x)\delta p(x))/\partial p^2$, $0 < \theta(x) < 1$, $x \in G$.) As unique solvability of the Dirichlet problem is assumed the following Schauder estimate for $\delta_2 w$ is valid. (See e.g. [**2**, Theorem 7.3. and Remark 2].)

$$||\delta_2 w||_{m+\rho} \leqq K\left\{||\delta p^2 (\partial^2 \bar{f}/\partial p^2)|| + \sum_{j=0}^{m/2-1} ||\delta g_j^2 (\partial^2 \bar{g}_j/\partial g_j^2||_{m+\rho-j}\right\}$$

($\rho \in (0, 1)$ is the exponent of Hölder continuity), so that

$$\text{(5.4)}\qquad \lim_{a\to 0} \frac{1}{a} ||\delta_2 w(x; a\delta p, a\delta g)||_{m+\rho} = 0.$$

Applying the mean value theorem to the integrand of $\Delta J(w; \delta w)$, $\Delta J = J(w + \delta w) - J(w)$, one obtains

$$\text{(5.5)}\qquad \Delta J(w, \delta w) = \delta J(w, \delta w) + R(w, \delta w), \textit{with},$$

$$\text{(5.6)}\qquad R(w, \delta w) = \int_G \sum_{|\alpha| \leqq l} \{(D^\alpha \delta_2 w)(\partial \bar{F}/\partial D^\alpha w) + (D^\alpha \delta_1 w)(\partial \bar{F}/\partial D^\alpha w - \partial F/\partial D^\alpha w)\} dV.$$

From (5.3), (5.4) and $\partial \bar{F}/\partial D^\alpha w$, evaluated for $\partial w(x, a\delta p, a\delta q) \to \partial F/\partial D^\alpha w$, uniformly on $\bar{G}$, as $a \to 0$ it follows that

$$\text{(5.7)}\qquad \lim_{a\to 0} \frac{1}{a} R(w; \delta w(x, a\delta p, a\delta q)) = 0.$$

From (6.3) and (6.4) it follows that

$$\text{(5.8)}\qquad \lim_{a\to 0} ||\delta w(x, a\delta p, a\delta q)||_l = 0.$$

If J has a relative extremum at w then $\Delta J(w, \delta w)$ must be either non-positive or else non-negative for all $\delta w \in \delta w$ with $||\delta w||_l \leqq \delta$, for some positive δ. As $\Delta J(w, \delta w(x, a\delta p, a\delta g)) = a\{\delta J(w, \delta w(x; \delta p, \delta g)) + (1/a)R(w, \delta w(x; a\delta p, a\delta g))$ and (5.7), (5.8) this can only be true if $\delta J(w; \delta w) = 0$, $\delta w \in \delta W$.

THEOREM 5.2. *Let k be a non-negative integer,* $k = [\nu/2] + 1 + \max.\ (1, k)$, $\lambda_j = [\nu/2] + 1 + m + \max.\ (1, k) - j$ *and let the following conditions be satisfied.*

(i) *G is bounded in R^ν with* $\partial G \in C^{m+[\nu/2]+1}$.

(ii) *L is uniformly strongly elliptic,* $N(L^*) = \{0\}$,
$a_\alpha \in C^{[\nu/2]+1+\max(1,|\alpha|)}(\bar{G})$, $0 \leqq |\alpha| \leqq m/2$, *and*
$a_\alpha \in C^{[\nu/2]+1+2|\alpha|-m/2}(\bar{G})$, $m/2 < |\alpha| \leqq m$.

(iii) *The functions f, g_j, and their first two derivatives with respect to their second variable are of class* $C^k(\bar{G} \times R)$ *and* $C^{\lambda_j}(\partial G \times R)$, $0 \leqq j \leqq m/2 - 1$, *respectively.*

(iv) $P \subset C^k(\bar{G})$ *and* $Q_j \subset C^{\lambda_j}(\partial G)$, $0 \leqq j \leqq m/2 - 1$.

(v) *F is $(l + [\nu/2] + 1)$-times continuously differentiable.*

(vi) *The function* $v \in C^m(\bar{G})$ *is the solution of*

$$L^*v(x) = [F]_{,w},\ x \in G, \tag{5.9}$$
$$N_j^{m-1-j}v = \sum_{|\alpha|=j}^{l-1} M_j^\alpha(Q^\alpha w), \quad m/2 \leqq j \leqq l - 1$$
$$N_j^{m-1-j}v = 0, \quad \max.\ (l, m/2) \leqq j \leqq m - 1, x \in \partial G.$$

Then $\delta J(w, \delta w)$, *given in* (5.2), (5.3), *can be written as*

$$\begin{aligned}\delta J(w, \delta w) = {} & \int_G \delta p(v(\partial f/\partial p))dV \\ & + \int_{\partial G} \sum_{j=0}^{\min(l,m/2)-1} \delta g_j(\partial g/\partial q_j) \left\{ \sum_{|\alpha|=j}^{l-1} M_j^\alpha(Q^\alpha w) - N_j^{m-1-j}v \right\} dS \\ & + \int_{\partial G} \sum_{j=l}^{m/2-1} \delta g_j(\partial g/\partial q_j) N_j^{m-1-j}v\, dS.\end{aligned} \tag{5.10}$$

(If $l \leqq m/2$ then the second set of equations in (5.9) is vacuous. If $l \geqq m/2$ then the last integral of (5.10) must be omitted. For the definitions of M_j^α and N_j^{m-1-j} see (2.2) and (2.4) respectively.)

The proof of the above theorem is analogous to that of Lemma 3.2 and Theorem 4.1. The existence and regularity properties are derived from Theorem 4.1.

Necessary conditions for the existence of an optimal solution of the boundary and interior control problem follow directly from the Theorems 5.1 and 5.2. The most interesting conditions are given in the next theorem.

THEOREM 5.3. *If*

(i) *the conditions of Theorem* 5.2 *are satisfied,*

(ii) $C^\infty(\bar{G}) \subset P$ *and* $C^\infty(\partial G) \subset Qj$, $0 \leqq j \leqq m/2 - 1$,

(iii) *J has a relative extremum within W at* $w \in W$, *then the solution of* (5.9) *also satisfies*

$$(5.11)\quad \begin{aligned} &(\partial f/\partial p)v = 0, \quad x \in G, \quad (\partial g_j/\partial q_j)\left\{\sum_{|\alpha|=j}^{l-1} M_j^{\alpha}(Q^{\alpha}w) - N_j^{m-1-j}v\right\} = 0, \\ &x \in \partial G, 0 \leqq j \leqq \min\,(l, m/2) - 1, \ (\partial g_j/\partial q_j)\,\{N_j^{m-1-j}v\} = 0, \\ &\qquad\qquad x \in \partial G, l \leqq j \leqq m/2 - 1. \end{aligned}$$

(*If* $l \geqq m/2$ *then the last set of equations is vacuous.*)

Definition. Let the conditions of Theorem 5.3 be satisfied. Then (5.1), (5.9) and (5.11) together constitute the *control boundary value problem.*

Remark 1. If $\partial f(x, p(x))/\partial p \neq 0, x \in G$, and $\partial g_j(x, q_j(x))/\partial q_j \neq 0, 0 \leqq j \leqq m/2 - 1$, $x \in \partial G$, then the control boundary value problem reduces to the Euler equation with transversality conditions for J.

Remark 2. If $f = f(x)$ and $\partial g_j(x, q_j(x))/\partial q_j \neq 0, 0 \leqq j \leqq m/2 - 1, x \in \partial G$, then the control boundary value problem reduces to the variational boundary value problem.

6. Appendix. The appendix contains the proofs of the results stated in section 2. The following additional notation is used. If α is a multi-index then perm α is an $|\alpha|$-vector the components of which take values in $\{1, 2, \ldots \nu\}$ such that $i \in \{1, 2, \ldots \nu\}$ occurs α_i times (e.g., let $\alpha = (2, 1)$ then perm $\alpha \in \{1, 1, 2), (1, 2, 1), (2, 1, 1)\}$). In the proof of Lemma 2.4 summation indices α_i and β_i are used. If repeated in the same term they run independently from 1 to ν, primed indices run from 1 to $\nu - 1$.

Proof of Lemma 2.3. Using Lemma 2.1, functions w_j, $0 \leqq j \leqq k$, each of class $C^{k+h}(\Omega)$, can be chosen, that satisfy

$$w_0(x) = g_0(x)/b_0(x), \quad x \in \partial G,$$

and for $l = 1, l = 2, \ldots, l = k$, respectively

$$(\partial/\partial \mathbf{n})^j w_l = 0, 0 \leqq j \leqq l - 1, x \in \partial G,$$

$$(\partial/\partial \mathbf{n})^l w_l = \left\{\sum_{|\alpha|=l} n^{\alpha} b_l^{\alpha}\right\}^{-1}\left\{g_l - B_l\left(\sum_{i=0}^{l-1} w_i\right)\right\}, \quad x \in \partial G.$$

Using Lemma 2 one obtains

$$D^{\alpha} w_l(x) = 0, \quad 0 \leqq |\alpha| \leqq l - 1, x \in \partial G$$
$$B_l w_l(x) = g_l(x) - B_l\left(\sum_{i=0}^{l-1} w_i(x)\right), \quad x \in \partial G$$

so that $w = \sum_{l \leqq k} w_k$ satisfies $B_j w(x) = g_j(x), x \in \partial G, 0 \leqq j \leqq k$.

Proof of Lemma 2.4. Let $m \leqq 1$. Since ∂G is compact in R^{ν} and at least of class C^2 there exists a small enough open strip about ∂G in which new coordinates can be introduced as follows. Let $\{O_h\}$ be an open covering of ∂G

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such that corresponding to each h a mapping T_h of an open set $P_h \subset R^\nu$ onto O_h, which is one to one and together with its inverse of class C^r can be defined by:

$$(6.1)\quad \begin{aligned} x_i(t) &= t_i + t_\nu n_i(t'), \quad i = 1, -, j-1 \\ x_i(t) &= t_{i-1} + t_\nu n_i(t'), \quad i = j+1, -, \nu \\ x_j(t) &= f_h(t') + t_\nu n_j(t'), \end{aligned}$$

where $f_h(x_{1'} - {}'x_{j-1'}x_{j+1'} - {}'x_\nu) = x_j$, $x \in O_h \cap \partial G$, $t_\nu = \pm$dist. $(x, \partial G)$, $+$ or $-$ as $x \notin G$ or $x \in G$; $t' = (t_{1'} - {}'t_{\nu-1})$.

Let $\{\Phi_h\}$ be a partition of unity subordinate to $\{O_h\}$ and put

$$(6.2)\quad I_h = \int_{O_h \cap \partial G} \Phi_h(x) \sum_{i=0}^{m} b^{\alpha_1,-,\alpha_i}(x)\, D_{\alpha_1} \ldots D_{\alpha_i} q(x)\, dS$$

with $b^{\alpha_1,-,\alpha_i} = (\alpha!/|\alpha|!)b^\alpha$, $\alpha_{1'} - {}'\alpha_i =$ perm. α and $D_{\alpha_i} = \partial/\partial x_{\alpha_i}$, so that $I = \sum I_h$.

Going over to t-coordinates one gets

$$(6.3)\quad I_h = \int_{Q_h} \sum_{j=0}^{m} B_h^{\beta_1,-,\beta_j}(t)\, D_{\beta_1} \ldots D_{\beta_j} q_h(t) dt_1, \ldots, dt_{\nu-1},$$

with $Q_h = \{t \in R^\nu : t \in P_h \wedge t_\nu = 0\}$; $D_{\beta_1} = \partial/\partial t_{\beta_1}$; $q_h(t) = q(T_h(t))$, $t \in P_h$, and

$$B_h^{\beta_1,-,\beta_j}(t) = \sum_{k=j}^{m} J_h(t)\Phi_h(T_h(t)) b^{\alpha_1,-,\alpha k\alpha_1,-,\alpha k'\beta_1,-,\beta_j}(T_h(t))R(t).$$

In the last formula J_h is an extension into $C^r(P_h)$ of $(1 + (\partial f_h/\partial t_1)^2 + \ldots + (\partial f_h/\partial t_{\nu-1})^2)^{\frac{1}{2}}$ and the functions $R^{\alpha_1,-,\alpha_k,\beta_1,-,\beta_j} \in C^{r+j-k}(P_h)$ are the transformation coefficients of the derivatives of q. One observes that the $B_h^{\beta_1,-,\beta_j}$ have compact support in P_h and are symmetric in $\beta_{1'} - {}'\beta_j$. Thus after rearrangement of the summation the integrand of (6.3) takes the form

$$\sum_{j=0}^{m} \sum_{k=0}^{m-j} \binom{j+k}{k} B_h^{\nu,-,\nu'\beta_{k+1'},-,\beta_{j+k'}} D_\nu^{\,j} D_{\beta_{j+1'}} \ldots D_{\beta_{j+k'}} q_{h'}$$

and after integration by parts it is

$$\sum_{j=0}^{m} (\partial^i q_h/\partial t_\nu^{\,j}) \sum_{k=0}^{m-j} (-1)^k \binom{k+j}{k} D_{\beta_{j+1'}} \ldots D_{\beta_{j+k'}} B_h^{\nu,-,\nu\beta_{j+1'},-,\beta_{j+k'}}.$$

From $(\partial q_h/\partial t_\nu) = (\partial x_{\alpha_1}/\partial t_\nu)D_{\alpha_1} q = n_{\alpha_1} D_{\alpha_1} q(x)$, $x \in O_h$, it follows that $(\partial^j q_h(t)/\partial t_\nu^{\,j}) = (\partial/\partial \mathbf{n})^j q(x)$, $x \in O_h$, so that in x-coordinates

$$(6.4)\quad I_h = \int_{O_h \cap \partial G} \sum_{j=0}^{m} ((\partial/\partial \mathbf{n})^j q) E_{jh}\, dS, \textit{ where}$$

$$(6.5)\quad \begin{aligned} E_{jh}(x) = \sum_{k=0}^{m-j} \sum_{l=0}^{k} (-1)^k \binom{k+j}{k} (J_h(T_h^{-1}(x))^{-1} \\ \times S_h^{\beta_{j+1'},-,\beta_{j+k''}\gamma_1,-,\gamma_l}(x)\, D_{\gamma_1} \ldots D_{\gamma_l} B_h^{\nu,-,\nu'\beta_{j+1'},-,\beta_{j+k'}}. \end{aligned}$$

The functions $S_h^{\beta_{j+1'},\ldots,\beta_{j+k'},\gamma_1,\ldots,\gamma_l} \in C^{r+l-k}(O_h)$ are transformation coefficients, $D_{\gamma_i} = \partial/\partial x_{\gamma_i}$.

Summing again over h one obtains

$$(6.6) \qquad I = \int_{\partial G} \sum_{j=0}^{m} (\partial/\partial \mathbf{n})^j q \sum_h E_{jh}\, dS.$$

Finally, from (6, 5) and the information below (6, 3) it follows that

$$(6.7) \qquad \sum_h E_{jh} = \sum_{|\alpha|=j}^{m} \sum_{|\beta|=0}^{|\alpha|-j} m_j^{\alpha\beta}(x) D^\beta b(x), \text{ with } m_j^{\alpha\beta} \in C^{t+j+|\beta|-|\alpha|}(\text{nbh. } \partial G).$$

Remark. The expression M_j are independent of m and of the coordinate transformations used in the sense that if

$$I' = \int_{\partial G} \sum_{|\alpha|=0}^{m'} b'^\alpha(x) D^\alpha q'(x)\, dS = \int_{\partial G} \sum_{j=0}^{m} (\partial/\partial \mathbf{n})^j q \sum_{\alpha=j}^{m} M'^{\alpha}_j b'^\alpha\, dS,$$

and correspondingly for I'', then $M'^{\alpha}_j = M''^{\alpha}_j$, $|\alpha| \leqq \min(m', m'')$ in some neighbourhood of ∂G. For choose $(\partial/\partial \mathbf{n})^i q' = (\partial/\partial \mathbf{n})^i q'' = 0$, $i \neq j$, $(\partial/\partial \mathbf{n})^j q' = (\partial/\partial \mathbf{n})^j q'' = r$, $b'^\beta = b''^\beta = 0$, $\beta \neq \alpha$ and $b^\alpha = s$. Then from varying r one obtains $M'^{\alpha}_j s = M''^{\alpha}_j s$ independent of the particular choice of s so that $M'^{\alpha}_j = M''^{\alpha}_j$.

Proof of Theorem 2.1. For two functions u and v both of class $C^m(\Omega)$ and a domain $G \subset R^\nu$ with $\partial G \in C^1$, Green's identity is

$$(6.8) \qquad \int_G (uLv - vL^*u)\, dV = \int_{\partial G} \sum_{|\alpha|=0}^{m-1} \sum_{|\beta|=0}^{m-|\alpha|-1} D^\alpha u D^\beta v P^{\alpha\beta} dS,$$

where the functions $P^{\alpha\beta}$ are given by

$$\sum_{|\delta|=1}^{m-|\alpha|-|\beta|-1} \sum_{|\gamma|=0} (-1)^{|\alpha|+|\gamma|} \binom{|\alpha| + |\gamma|}{|\alpha|} \frac{(\alpha+\beta+\gamma+\delta)!}{|\alpha+\beta+\gamma+\delta|!} D^\gamma a_{\alpha+\beta+\gamma+\delta} \mathbf{n}^\delta.$$

This result depends only on integration by parts. The computations can be found in [**6**]. The first part of Theorem 2.1 follows immediately from (6.8) and Lemma 2.4. To prove (2.5) choose any integer j in $[0, m-1]$ and let $\partial^k v/\partial \mathbf{n}^k = 0$, $0 \leqq k \leqq j-1$, $x \in \partial G$, and $\partial^k u/\partial \mathbf{n}^k = 0$, $0 \leqq k \leqq m-2-j$, $x \in \partial G$, so that for $x \in \partial G$; $D^\alpha v = 0$, $0 \leqq |\alpha| \leqq j-1$; $D^\alpha v = \mathbf{n}^\alpha(\partial^j v/\partial \mathbf{n}^j)$, $|\alpha| = j$; $D^\alpha u = 0$, $0 \leqq |\alpha| \leqq m-j-2$; $D^\alpha u = \mathbf{n}^\alpha(\partial^{m-j-1} v/\partial \mathbf{n}^{m-j-1})$, $|\alpha| = m-j-1$ (Lemma 2.2). Substitution of these results into (6.8) and (2.4), respectively yields

$$\int_{\partial G} (\partial/\partial \mathbf{n})^j v (\partial/\partial \mathbf{n})^{m-j-1} u \sum_{|\alpha|=m-j-1} \sum_{|\beta|=j} \sum_{|\delta|=1} (-1)^{m-j-1} \frac{(\alpha+\beta+\delta)!}{|\alpha+\beta+\delta|!}$$
$$\times\, a_{\alpha+\beta+\delta} \mathbf{n}^{\alpha+\beta+\delta} dS = \int_{\partial G} (\partial/\partial \mathbf{n})^j v (\partial/\partial \mathbf{n})^{m-j-1} u \sum_{|\alpha|=m-j-1} \mathbf{n}^\alpha p_j^\alpha\, dS.$$

As the above normal derivatives of u and v can be chosen freely one obtains an identity that after rearrangement of the summation takes the form (2.5). This completes the proof.

REFERENCES

1. S. Agmon, *Lectures on elliptic boundary value problems* (D. Van Nostrand Company, Inc., New York, 1965).
2. A. Agmon, A. Douglis, and L. Nirenberg, *Estimates near the boundary for the solutions of elliptic differential satisfying general boundary values. I*, Comm. Pure Appl. Math. *12* (1959), 623–727.
3. N. Aronszajn and A. N. Milgram, *Differential operators on Riemannian manifolds*, Rend. Circ. Mat. Palermo *2* (1953), 266–325.
4. R. Delver, *The Optimization of a functional within the class of solutions of a partial differential equation*, J. Engrg. Math. *4* (1969), 253–264.
5. ——— *Variational problems within the class of solutions of a partial differential equation*, Bull. Amer. Math. Soc. *77* (1971), 1100–1105.
6. ——— *Variational problems within the class of solutions of a partial differential equation*, Trans. Amer. Math. Soc. *180* (1973), 265–289.
7. J. L. Lions, *Contrôle optimal de systèmes gouvernés par des équations aux dérivées partielles* (Dunod et Gauthier-Villars, Paris, 1968).
8. ——— *Sur le contrôle optimal de systèmes decrits par des equations aux dérivées partielles linéaires*, C. R. Acad. Sci. Paris Sér. A-B, *263* (II) (1966), 713–715.
9. J. L. Lions and E. Magenes, *Equations differentielles opérationelles et problèmes aux limites*, Vol. I (Dunod., Paris, 1968).
10. A. Friedman, *Optimal control for parabolic equations*, J. Math. Anal. Appl. *18* (1967), 479–491.
11. ——— *Partial differential equation* (Holt, Rinehart and Winston, New York, 1969).
12. C. Miranda, *Equazioni alle derivate parziali di tipo ellittico*, (Ergebnisse der Mathematik und ihrer Grenzgebiete, Heft 2, Springer-Verlag, Berlin, 1955); English transl., Springer-Verlag, Berlin, 1970.
13. C. B. Morrey, Jr., *Multiple integrals in the calculus of variations* (Springer-Verlag, New York Inc., 1966).

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