Bull. Aust. Math. Soc. **80** (2009), 177–201
doi:10.1017/S0004972709000112

ON THE LENGTHS OF PAIRS OF COMPLEX MATRICES OF SIZE SIX

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(Received 11 September 2008)

Abstract

The length of every pair $\{A, B\}$ of 6×6 complex matrices is shown to be at most 10, that is, the words in A, B of length at most 10, including the empty word, span the unital algebra generated by A, B. This supports the conjecture that the length of every pair of $n \times n$ complex matrices is at most $2n - 2$, known to be true for $n < 6$.

2000 *Mathematics subject classification*: primary 15A30; secondary 47L05.
Keywords and phrases: length, words.

1. Introduction and preliminaries

It is shown in [5, Proposition 5] that, for $n = 2, 3$ or 4, the length of every pair $\{A, B\}$ of $n \times n$ complex matrices is at most $2n - 2$. This was first proved by Paz [6] who conjectured that it was true for every positive integer n. The conjectured result was shown to hold for $n = 5$ in [5, Theorem 5]. (It is also shown in [5], using [3, Example 2], that there are pairs with length $2n - 2$, for every n.) Here we use the notation and extend the techniques of [5] to establish the result for $n = 6$. For related results on more general fields see [1, 2, 5, 6].

Briefly, let $M_n(\mathbb{C})$ denote the set of all $n \times n$ complex matrices. Let $A, B \in M_n(\mathbb{C})$ and, for every positive integer k, define a word in the alphabet A, B to be of length k if it has k factors, counting multiplicities, so that, for example, the word $A^2BAB^2A^3$ has length 9. The word of length zero, also called the empty word, is taken to be the identity matrix. For every natural number k, let $\mathcal{V}_k$ be the subspace of $M_n(\mathbb{C})$ spanned by the words of length at most k (including the empty word). Clearly

$$\mathbb{C}I = \mathcal{V}_0 \subseteq \mathcal{V}_1 \subseteq \mathcal{V}_2 \subseteq \cdots \subseteq \mathcal{V}_i \subseteq \mathcal{V}_{i+1} \subseteq \cdots \subseteq \mathcal{A},$$

where $\mathcal{A}$ is the unital algebra generated by A, B. Since $\mathcal{A}$ is finite-dimensional, there is an integer l such that $\mathcal{V}_l = \mathcal{V}_{l+1}$. Then $\mathcal{V}_k = \mathcal{V}_l$, for every $k > l$, and since

$\mathcal{A} = \bigcup_{k=0}^{\infty} \mathcal{V}_k$, then $\mathcal{A} = \mathcal{V}_l$. The length $l\{A, B\}$ of the pair $\{A, B\}$ is defined to be the smallest integer l for which $\mathcal{V}_l = \mathcal{A}$. Then

$$\mathbb{C}I = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset \cdots \subset \mathcal{V}_i \subset \mathcal{V}_{i+1} \subset \cdots \subset \mathcal{V}_l = \mathcal{A},$$

where '$\subset$' denotes strict inclusion.

2. Main result

The proof begins with a proposition which is the analogue of [5, Proposition 3] for the case $n = 6$.

PROPOSITION 1 [4]. *Let w be a word of length 11 in the symbols a and b satisfying:*

(i) *w has no factor of the form a^6 or b^6;*
(ii) *w is not of any of the forms*

$$(pqrs)^2pqr, \quad p(qrs)^3q, \quad pq(rs)^4r, \quad (pqr)^3ps, \quad p(qr)^4qs, \quad (pq)^4prs,$$

where $\{p, q, r, s\} \subseteq \{a, b\}$.

Then w has at least 37 subwords, including the empty subword.

We leave it to the reader to complete the proof: using considerations analogous to those used in the proof of [5, Proposition 3], it can be shown that the numbers of different subwords of w satisfy the following table:

length	11	10	9	8	7	6	5	4	3	2	1	0
#subwords	1	2	3	4	5	≥ 5	≥ 5	≥ 3	≥ 3	≥ 3	2	1

The following corollary is the analogue of [5, Corollary 2] for the case $n = 6$.

COROLLARY 2. *Let w be a word of length 11 in the symbols a and b. If w has no factors of a^6 or b^6 and has 36 or fewer subwords (including the empty subword) it must be one of*

$$(ab)^5a;\ (ba)^5b;$$
$$(a^2b)^3a^2;\ (b^2a)^3b^2;\ a(ab)^5;\ b(ba)^5;\ (aba)^3ab;$$
$$(bab)^3ba;\ (ab)^5b;\ (ba)^5a;\ (ab^2)^3ab;\ (ba^2)^3ba;$$
$$(a^3b)^2a^3;\ (b^3a)^2b^3;\ a(a^2b)^3a;\ b(b^2a)^3b;\ a^3(ba)^4;\ b^3(ab)^4;$$
$$a^2b(a^3b)^2;\ b^2a(b^3a)^2;\ (a^2b)^3ab;\ (b^2a)^3ba;\ a^2(ba)^4a;\ b^2(ab)^4b;$$
$$(aba^2)^2aba;\ (bab^2)^2bab;\ (aba)^3a^2;\ (bab)^3b^2;\ (ab)^4a^3;\ (ba)^4b^3;\ (ab)^4a^2b;$$
$$(ba)^4b^2a;\ a(bab)^3b;\ b(aba)^3a;\ ab^2(ab)^4;\ ba^2(ba)^4;\ (ab^3)^2ab^2;\ (ba^3)^2ba^2.$$

The words in the first row above each have 22 subwords; those in the second and third rows have 30; those in the remaining rows have 36.

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PROOF. By Proposition 1, w must be of one of the forms

$$(pqrs)^2pqr, \quad p(qrs)^3q, \quad pq(rs)^4r, \quad (pqr)^3ps, \quad p(qr)^4qs, \quad (pq)^4prs,$$

where $\{p, q, r, s\} \subseteq \{a, b\}$. Bearing in mind the fact that w has no factor of a^6 or b^6, it must be one of the words listed or be one of

$$(a^2b^2)^2a^2b;\ (b^2a^2)^2b^2a;\ (ab^2a)^2ab^2;\ (ba^2b)^2ba^2;$$
$$a(ab^2)^3a;\ b(ba^2)^3b;\ (ab^2)^3a^2;\ (ba^2)^3b^2.$$

But each of the latter has 37 subwords. (By symmetry one needs only to verify this for $(a^2b^2)^2a^2b$ and $a(ab^2)^3a$.) Finally, by symmetry, to verify that the actual number of subwords is as claimed, one need only check that, $(ab)^5a$ has 22 subwords, that each of $(a^2b)^3a^2$, $a(ab)^5$, $(aba)^3ab$ has 30 subwords and that each of

$$(a^3b)^2a^3,\ a(a^2b)^3a,\ a^3(ba)^4,\ a^2b(a^3b)^2,$$
$$(a^2b)^3ab,\ a^2b(ba)^4a,\ (aba^2)^2aba,\ (ab)^4a^2b$$

has 36 subwords. □

PROPOSITION 3 [5, Proposition 4]. *Let $n \geq 2$. For all complex $n \times n$ matrices A and B, the matrix $(AB)^{n-1}A$ belongs to $\mathcal{V}_{2n-3}$.*

PROPOSITION 4. *If $A_0, A_1, \ldots, A_m$ are $n \times n$ matrices, $m, n \geq 1$, and $\mathcal{V}$ is a subspace of $M_n(\mathbb{C})$ and $A_0 + A_1\lambda + A_2\lambda^2 + \cdots + A_m\lambda^m \in \mathcal{V}$, for every $\lambda \in \mathbb{C}\backslash\{0\}$, then $A_i \in \mathcal{V}$ for every $i = 1, 2, \ldots, m$.*

PROOF. Let $X_\lambda = A_0 + A_1\lambda + A_2\lambda^2 + \cdots + A_m\lambda^m$. Then $X_\lambda \longrightarrow A_0$ as $\lambda \longrightarrow 0$. Since $\mathcal{V}$ is closed, it follows that $A_0 \in \mathcal{V}$. Then

$$\frac{X_\lambda - A_0}{\lambda} = A_1 + A_2\lambda + \cdots + A_m\lambda^{m-1} \in \mathcal{V},$$

for every $\lambda \in \mathbb{C}\backslash\{0\}$. Repeating the argument just given leads to a proof of the result. □

We can now prove our main result.

THEOREM 5. *The length of every pair $\{A, B\}$ of 6×6 complex matrices is at most 10, that is, the words in A, B of length at most 10, including the empty word, span the unital algebra generated by A, B.*

PROOF. Let $\mathcal{W}$ be the set of all words, including the empty word, in A and B. If $U, V \in \mathcal{W}$ and U and V are the same word we write $U \equiv V$. (So $U \equiv V$ is strictly stronger than $U = V$ where the latter means equality *as matrices*.)

For each integer $k \geq 1$, totally order the words in A and B of length k using dictionary order. (So, if W_1 and W_2 are words of equal nonzero length, we say that $W_1 \preceq W_2$ if $W_1 \equiv XAV_1$ and $W_2 \equiv XBV_2$, where each of X, V_1, V_2 is a word in A, B, possibly empty.) Extend these orders to a total order on $\mathcal{W}$ by additionally defining $W_1 \preceq W_2$ if the length of W_1 is strictly less than the length of W_2. In the totally ordered set $\mathcal{W}$, define $\mathcal{B}$ to be the set of elements which do not belong to the span of their strict predecessors. Then $\mathcal{B}$ is a linearly independent set of matrices, hence finite. Clearly $I \in \mathcal{B}$. Note that, if W is a word in $\mathcal{B}$ of length at least 2, then every proper subword of W belongs to $\mathcal{B}$. For if a word U belongs to the span of words strictly less than it so do the words UV and VU, for any word V.

Now the length of the pair $\{A, B\}$ is at most 10 if and only if $\mathcal{V}_{10}$ is the unital algebra generated by A and B if and only if $\mathcal{B}$ does not contain a word of length 11. Suppose, to the contrary, that $\mathcal{B}$ contains a word of length 11. We derive a contradiction. Let W be the smallest word (in the sense of the total order $\preceq$) of length 11 in $\mathcal{B}$. Then W has length 11, has no factors of the form A^6 or B^6, and has 36 or fewer subwords (including the empty subword). Corollary 2 then shows that W must be one of

$$(ab)^5a;\ (ba)^5b;$$
$$(a^2b)^3a^2;\ (b^2a)^3b^2;\ a(ab)^5;\ b(ba)^5;\ (aba)^3ab;$$
$$(bab)^3ba;\ (ab)^5b;\ (ba)^5a;\ (ab^2)^3ab;\ (ba^2)^3ba;$$
$$(a^3b)^2a^3;\ (b^3a)^2b^3;\ a(a^2b)^3a;\ b(b^2a)^3b;\ a^3(ba)^4;\ b^3(ab)^4;$$
$$a^2b(a^3b)^2;\ b^2a(b^3a)^2;\ (a^2b)^3ab;\ (b^2a)^3ba;\ a^2(ba)^4a;\ b^2(ab)^4b;$$
$$(aba^2)^2aba;\ (bab^2)^2bab;\ (aba)^3a^2;\ (bab)^3b^2;\ (ab)^4a^3;\ (ba)^4b^3;\ (ab)^4a^2b;$$
$$(ba)^4b^2a;\ a(bab)^3b;\ b(aba)^3a;\ ab^2(ab)^4;\ ba^2(ba)^4;\ (ab^3)^2ab^2;\ (ba^3)^2ba^2,$$

taking $a = A$ and $b = B$. By Proposition 3 it cannot be either of the first two words. As we shall see, it is not too hard to prove that it cannot be either of the last 26. It is harder to prove that it cannot be either of the remaining 10.

Let us look at the last 26 first. Suppose W is one of the last 26 words (taking $a = A$ and $b = B$). Each of these words has 36 subwords, so $\mathcal{B}$ consists precisely of these subwords. There are 26 cases to consider: W is one of

(1)	$(A^3B)^2A^3$	(2)	$(B^3A)^2B^3$	(3)	$A(A^2B)^3A$
(4)	$B(B^2A)^3B$	(5)	$A^3(BA)^4$	(6)	$B^3(AB)^4$
(7)	$A^2B(A^3B)^2$	(8)	$B^2A(B^3A)^2$	(9)	$(A^2B)^3AB$
(10)	$(B^2A)^3BA$	(11)	$A^2(BA)^4A$	(12)	$B^2(AB)^4B$
(13)	$(ABA^2)^2ABA$	(14)	$(BAB^2)^2BAB$	(15)	$(ABA)^3A^2$
(16)	$(BAB)^3B^2$	(17)	$(AB)^4A^3$	(18)	$(BA)^4B^3$
(19)	$(AB)^4A^2B$	(20)	$(BA)^4B^2A$	(21)	$A(BAB)^3B$
(22)	$B(ABA)^3A$	(23)	$AB^2(AB)^4$	(24)	$BA^2(BA)^4$
(25)	$(AB^3)^2AB^2$	(26)	$(BA^3)^2BA^2$.		

By the Cayley–Hamilton theorem, $(A + \lambda B)^6 \in \mathcal{V}_5$, for every $\lambda \in \mathbb{C}$. By Proposition 4 it follows that:

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(a) the sum X of the words of length 6 in A, B, with precisely three B factors, belongs to $\mathcal{V}_5$, that is,

$$\begin{aligned} X = A^3B^3 &+ A^2BAB^2 + A^2B^2AB + A^2B^3A + ABA^2B^2 + ABABAB \\ &+ ABAB^2A + AB^2A^2B + AB^2ABA + AB^3A^2 + BA^3B^2 + BA^2BAB \\ &+ BA^2B^2A + BABA^2B + BABABA + BAB^2A^2 + B^2A^3B + B^2A^2BA \\ &+ B^2ABA^2 + B^3A^3 \in \mathcal{V}_5; \end{aligned}$$

(b) the sum Y of the words of length 6 in A, B, with precisely two B factors, belongs to $\mathcal{V}_5$, that is,

$$\begin{aligned} Y = A^4B^2 &+ A^3BAB + A^3B^2A + A^2BA^2B + A^2BABA + A^2B^2A^2 + ABA^3B \\ &+ ABA^2BA + ABABA^2 + AB^2A^3 + BA^4B + BA^3BA + BA^2BA^2 \\ &+ BABA^3 + B^2A^4 \in \mathcal{V}_5; \end{aligned}$$

(c) the sum Z of the words of length 6 in A, B, with precisely two A factors, belongs to $\mathcal{V}_5$, that is,

$$\begin{aligned} Z = A^2B^4 &+ ABAB^3 + AB^2AB^2 + AB^3AB + AB^4A + BA^2B^3 + BABAB^2 \\ &+ BAB^2AB + BAB^3A + B^2A^2B^2 + B^2ABAB + B^2AB^2A + B^3A^2B \\ &+ B^3ABA + B^4A^2 \in \mathcal{V}_5. \end{aligned}$$

The following cases involve X: (5), (11), (18), (20), (23).

(**5**) W is $A^3(BA)^4$

$BABABA$ is a subword of W. In fact, $W \equiv A^3(BABABA)BA$. Thus

$$\begin{aligned} B^2AB^2A(X) = V + W + A^3(&BAB^2A^2 + B^2A^3B + B^2A^2BA + B^2ABA^2 \\ &+ B^3A^3)BA \in \mathcal{V}_{10} \end{aligned}$$

where $V \in \mathcal{V}_{10}$ (since W is the smallest word of length 11 in $\mathcal{B}$). Thus

$$\begin{aligned} W &+ A^3BAB^2A^2BA + A^3B^2A^3B^2A + A^3B^2A^2BABA + A^3B^2ABA^2BA \\ &+ A^3B^3A^3BA \in \mathcal{V}_{10}. \end{aligned} \qquad (*)$$

Since each of A^2, AB, BA is a subword of W, each belongs to $\mathcal{B}$. Since B^2 does not belong to $\mathcal{B}$ we have $B^2 = \alpha A^2 + \beta AB + \gamma BA + U$ for some scalars α, β, γ and some element $U \in \mathcal{V}_1$. Thus

$$\begin{aligned} A^3BAB^2A^2BA = A^3BA(B^2)A^2BA &= A^3BA(\alpha A^2 + \beta AB + \gamma BA + U)A^2BA \\ &= (A^3BA^2)(\alpha A^3BA + \beta BA^2BA) + A^3BA(U)A^2BA \\ &\quad + \gamma A^3B(ABA^2)ABA. \end{aligned} \qquad (1)$$

Now the first five words in A, B of length four are A^4, A^3B, A^2BA, A^2B^2, ABA^2. Of these, only A^3B and A^2BA belong to $\mathcal{B}$ (since they are subwords of W). Thus

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$A^4 \in \mathcal{V}_3$ and $ABA^2 = \lambda A^3B + \mu A^2BA + S$, for some scalars λ, μ and some element $S \in \mathcal{V}_3$. Thus

$$\begin{aligned} A^3BA^2 &= A^2(ABA^2) = A^2(\lambda A^3B + \mu A^2BA + S) \\ &= A^4(\lambda AB + \mu BA) + A^2S \in \mathcal{V}_5. \end{aligned}$$

It follows that

$$\begin{aligned} A^3B(ABA^2)ABA &= A^3B(\lambda A^3B + \mu A^2BA + S)ABA \\ &= A^3BA^2(\lambda AB + \mu BA)ABA + A^3B(S)ABA \in \mathcal{V}_{10}. \end{aligned}$$

Using the facts that $A^3BA^2 \in \mathcal{V}_5$, $A^3B(ABA^2)ABA$, $A^3BA(U)A^2BA \in \mathcal{V}_{10}$ in (1) gives that $A^3BAB^2A^2BA \in \mathcal{V}_{10}$. We have

$$\begin{aligned} A^3B^2A^3B^2A &= A^3(B^2)A^3B^2A = A^3(\alpha A^2 + \beta AB + \gamma BA + U)A^3B^2A \\ &= A^4(\alpha A + \beta B)A^3B^2A + \gamma A^3B(A^4)B^2A + A^3(U)A^3B^2A \in \mathcal{V}_{10} \end{aligned}$$

since $A^4 \in \mathcal{V}_3$ and $U \in \mathcal{V}_1$.

Also,

$$\begin{aligned} A^3B^2A^2BABA &= A^3(B^2)A^2BABA = A^3(\alpha A^2 + \beta AB + \gamma BA + U)A^2BABA \\ &= A^4(\alpha A + \beta B)A^2BABA + \gamma A^3BA^2(ABABA) + A^3(U)A^2BABA \in \mathcal{V}_{10} \end{aligned}$$

since $A^4 \in \mathcal{V}_3$, $A^3BA^2 \in \mathcal{V}_5$ and $U \in \mathcal{V}_1$.

Continuing,

$$\begin{aligned} A^3B^2ABA^2BA &= A^3(B^2)ABA^2BA = A^3(\alpha A^2 + \beta AB + \gamma BA + U)ABA^2BA \\ &= A^4(\alpha A + \beta B)ABA^2BA + \gamma A^3BA^2(BA^2BA) + A^3(U)ABA^2BA \in \mathcal{V}_{10} \end{aligned}$$

since $A^4 \in \mathcal{V}_3$, $A^3BA^2 \in \mathcal{V}_5$ and $U \in \mathcal{V}_1$.

Finally,

$$\begin{aligned} A^3B^3A^3BA &= A^3(B^2)BA^3BA = A^3(\alpha A^2 + \beta AB + \gamma BA + U)BA^3BA \\ &= A^4(\alpha A + \beta B)BA^3BA + \gamma A^3BABA^3BA + A^3(U)BA^3BA \in \mathcal{V}_{10} \end{aligned}$$

since $A^4 \in \mathcal{V}_3$, $A^3BABA^3BA \in \mathcal{V}_{10}$ and $U \in \mathcal{V}_1$.
It now follows from $(*)$ that $W \in \mathcal{V}_{10}$. This is a contradiction.

(**11**) W is $A^2(BA)^4A$

$BABABA$ is a subword of W. In fact, $W \equiv A^2(BABABA)BA^2$. Thus

$$\begin{aligned} A^2(X)BA^2 = V + W + A^2(&BAB^2A^2 + B^2A^3B + B^2A^2BA + B^2ABA^2 \\ &+ B^3A^3)BA^2 \in \mathcal{V}_{10} \end{aligned}$$

where $V \in \mathcal{V}_{10}$ (since W is the smallest word of length 11 in $\mathcal{B}$). Thus

$$\begin{aligned} W &+ A^2BAB^2A^2BA^2 + A^2B^2A^3B^2A^2 + A^2B^2A^2BABA^2 + A^2B^2ABA^2BA^2 \\ &+ A^2B^3A^3BA^2 \in \mathcal{V}_{10}. \qquad (*) \end{aligned}$$

Now A^3 is not a subword of W so does not belong to $\mathcal{B}$. Hence $A^3 \in \mathcal{V}_2$. Also, none of A^5, A^4B, A^3BA, A^3B^2, A^2BA^2 is a subword of W, so each, in particular A^2BA^2, belongs to $\mathcal{V}_4$. Of the words A^3, A^2B, ABA, AB^2 only A^2B, ABA belong to $\mathcal{B}$. Thus $AB^2 = \alpha A^2B + \beta ABA + U$ for some scalars α, β and some element $U \in \mathcal{V}_2$. Thus $AB^2A^2 = \alpha A^2BA^2 + \beta ABA^3 + UA^2 \in \mathcal{V}_4$ since $A^2BA^2 \in \mathcal{V}_4$, $A^3 \in \mathcal{V}_2$ and $U \in \mathcal{V}_2$. Using the facts that $A^3 \in \mathcal{V}_2$, $A^2BA^2 \in \mathcal{V}_4$, $AB^2A^2 \in \mathcal{V}_4$ in $(*)$ gives that $W \in \mathcal{V}_{10}$. This is a contradiction.

(**18**) W is $(BA)^4B^3$

$BABABA$ is a subword of W. In fact, $W \equiv (BABABA)BAB^3$. Thus

$$(X)BABA^3 = V + W + (BAB^2A^2 + B^2A^3B + B^2A^2BA + B^2ABA^2 + B^3A^3)BABA^3 \in \mathcal{V}_{10}$$

where $V \in \mathcal{V}_{10}$ (since W is the smallest word of length 11 in $\mathcal{B}$). Thus

$$W + BAB^2A^2BABA^3 + B^2A^3B^2ABA^3 + B^2A^2BABABA^3 + B^2ABA^2BABA^3 + B^3A^3BABA^3 \in \mathcal{V}_{10}. \tag{$*$}$$

Now A^2 is not a subword of W so does not belong to $\mathcal{B}$. Hence $A^2 \in \mathcal{V}_1$. It follows from $(*)$ that $W \in \mathcal{V}_{10}$. This is a contradiction.

(**20**) W is $(BA)^4B^2A$

$BABABA$ is a subword of W. In fact, $W \equiv (BABABA)BAB^2A$. Thus

$$B^2AB^2A(X) = V + W + (BAB^2A^2 + B^2A^3B + B^2A^2BA + B^2ABA^2 + B^3A^3)BAB^2A \in \mathcal{V}_{10}$$

where $V \in \mathcal{V}_{10}$ (since W is the smallest word of length 11 in $\mathcal{B}$). Thus

$$W + BAB^2A^2BAB^2A + B^2A^3B^2AB^2A + B^2A^2BABAB^2A + B^2ABA^2BAB^2A + B^3A^3BAB^2A \in \mathcal{V}_{10}. \tag{$*$}$$

Now A^2 is not a subword of W so does not belong to $\mathcal{B}$. Hence $A^2 \in \mathcal{V}_1$. It now follows from $(*)$ that $W \in \mathcal{V}_{10}$. This is a contradiction.

(**23**) W is $AB^2(AB)^4$

$BABABA$ is a subword of W. In fact, $W \equiv AB^2A(BABABA)B$. Thus

$$B^2AB^2A(X) = V + W + AB^2A(BAB^2A^2 + B^2A^3B + B^2A^2BA + B^2ABA^2 + B^3A^3)B \in \mathcal{V}_{10}$$

where $V \in \mathcal{V}_{10}$ (since W is the smallest word of length 11 in $\mathcal{B}$). Thus

$$W + AB^2ABAB^2A^2B + AB^2AB^2A^3B^2 + AB^2AB^2A^2BAB + AB^2AB^2ABA^2B + AB^2AB^3A^3B \in \mathcal{V}_{10}. \tag{$*$}$$

Now A^2 is not a subword of W so does not belong to $\mathcal{B}$. Hence $A^2 \in \mathcal{V}_1$. It now follows from $(*)$ that $W \in \mathcal{V}_{10}$. This is a contradiction.

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The following cases involve Y: (1), (3), (7), (9), (13), (15), (17), (19), (22), (24), (26).

(**1**) W is $(A^3B)^2A^3$

BA^3BA is a subword of W. In fact, $W \equiv A^3(BA^3BA)A^2$. Thus

$$A^3(Y)A^2 = V + W + A^3(BA^2BA^2 + BABA^3 + B^2A^4)A^2 \in \mathcal{V}_{10}$$

where $V \in \mathcal{V}_{10}$ (since W is the smallest word of length 11 in $\mathcal{B}$). Thus

$$W + A^3BA^2BA^4 + A^3BABA^5 + A^3B^2A^6 \in \mathcal{V}_{10}. \qquad (*)$$

Now A^4 is not a subword of W so does not belong to $\mathcal{B}$. Hence $A^4 \in \mathcal{V}_3$ and it follows from $(*)$ that $W \in \mathcal{V}_{10}$. This is a contradiction.

(**3**) W is $A(A^2B)^3A$

BA^2BA^2 is a subword of W. In fact, $W \equiv A^3(BA^2BA^2)BA$. Thus

$$A^3(Y)BA = V + W + A^3(BABA^3 + B^2A^4)BA \in \mathcal{V}_{10}$$

where $V \in \mathcal{V}_{10}$ (since W is the smallest word of length 11 in $\mathcal{B}$). Thus

$$W + A^3BABA^3BA + A^3B^2A^4BA \in \mathcal{V}_{10}. \qquad (*)$$

Now A^4 is not a subword of W so does not belong to $\mathcal{B}$. Hence $A^4 \in \mathcal{V}_3$ and it follows that $A^3B^2A^4BA \in \mathcal{V}_{10}$. Also $A^3BABA^3BA = A^2(ABAB)A^3BA \in \mathcal{V}_{10}$, for the following reason. The first six words of length 4 in A, B are, in increasing order, A^4, A^3B, A^2BA, A^2B^2, ABA^2, $ABAB$. Of these, the elements of $\mathcal{B}$ (that is, the subwords of W), are A^3B, A^2BA, ABA^2. By the definition of $\mathcal{B}$ this means that there exist scalars α, β, γ such that $ABAB - \alpha A^3B - \beta A^2BA - \gamma ABA^2 \in \mathcal{V}_3$. Then $A^2(ABAB - \alpha A^3B - \beta A^2BA - \gamma ABA^2)A^3BA = A^3BABA^3BA - \alpha A^5BA^3BA - \beta A^4BA^4BA - \gamma A^3BA^5BA \in \mathcal{V}_{10}$. However, $A^4 \in \mathcal{V}_3$, and so we have $A^3BABA^3BA \in \mathcal{V}_{10}$. From $(*)$ it now follows that $W \in \mathcal{V}_{10}$. This is a contradiction.

(**7**) W is $A^2B(A^3B)^2$

BA^3BA is a subword of W. In fact, $W \equiv A^2(BA^3BA)A^2B$. Thus

$$A^2(Y)A^2B = V + W + A^2(BA^2BA^2 + BABA^3 + B^2A^4)A^2B \in \mathcal{V}_{10}$$

where $V \in \mathcal{V}_{10}$ (since W is the smallest word of length 11 in $\mathcal{B}$). Thus

$$W + A^2BA^2BA^4B + A^2BABA^5B + A^2B^2A^6B \in \mathcal{V}_{10}. \qquad (*)$$

Now A^4 is not a subword of W so does not belong to $\mathcal{B}$. Hence $A^4 \in \mathcal{V}_3$ and it follows from $(*)$ that $W \in \mathcal{V}_{10}$. This is a contradiction.

(**9**) W is $(A^2B)^3AB$

BA^2BA^2 is a subword of W. In fact, $W \equiv A^2(BA^2BA^2)BAB$. Thus

$$A^2(Y)BAB = V + W + A^2(BABA^3 + B^2A^4)BAB \in \mathcal{V}_{10}$$

where $V \in \mathcal{V}_{10}$ (since W is the smallest word of length 11 in $\mathcal{B}$). Thus

$$W + A^2BABA^3BAB + A^2B^2A^4BAB \in \mathcal{V}_{10}. \qquad (*)$$

Now A^3 is not a subword of W so does not belong to $\mathcal{B}$. Hence $A^3 \in \mathcal{V}_2$ and it follows from $(*)$ that $W \in \mathcal{V}_{10}$. This is a contradiction.

(**13**) W is $(ABA^2)^2ABA$

BA^3BA is a subword of W. In fact, $W \equiv A(BA^3BA)A^2BA$. Thus

$$A^2(Y)A^2B = V + W + A(BA^2BA^2 + BABA^3 + B^2A^4)A^2BA \in \mathcal{V}_{10}$$

where $V \in \mathcal{V}_{10}$ (since W is the smallest word of length 11 in $\mathcal{B}$). Thus

$$W + ABA^2BA^4BA + ABABA^5BA + AB^2A^6BA \in \mathcal{V}_{10}. \qquad (*)$$

Now A^4 is not a subword of W so does not belong to $\mathcal{B}$. Hence $A^4 \in \mathcal{V}_3$ and it follows from $(*)$ that $W \in \mathcal{V}_{10}$. This is a contradiction.

(**15**) W is $(ABA)^3A^2$

BA^2BA^2 is a subword of W. In fact, $W \equiv ABA^2(BA^2BA^2)A$. Thus

$$ABA^2(Y)A = V + W + ABA^2(BABA^3 + B^2A^4)A \in \mathcal{V}_{10}$$

where $V \in \mathcal{V}_{10}$ (since W is the smallest word of length 11 in $\mathcal{B}$). Thus

$$W + ABA^2BABA^4 + ABA^2B^2A^5 \in \mathcal{V}_{10}. \qquad (*)$$

Now A^4 is not a subword of W so does not belong to $\mathcal{B}$. Hence $A^4 \in \mathcal{V}_3$ and it follows from $(*)$ that $W \in \mathcal{V}_{10}$. This is a contradiction.

(**17**) W is $(AB)^4A^3$

$BABA^3$ is a subword of W. In fact, $W \equiv ABABA(BABA^3)$. Thus

$$ABABA(Y) = V + W + ABABA(B^2A^4) \in \mathcal{V}_{10}$$

where $V \in \mathcal{V}_{10}$ (since W is the smallest word of length 11 in $\mathcal{B}$). Thus

$$W + ABABAB^2A^4 \in \mathcal{V}_{10}. \qquad (*)$$

Now A^4 is not a subword of W so does not belong to $\mathcal{B}$. Hence $A^4 \in \mathcal{V}_3$ and it follows from $(*)$ that $W \in \mathcal{V}_{10}$. This is a contradiction.

(**19**) W is $(AB)^4A^2B$

$ABABA^2$ is a subword of W. In fact, $W \equiv ABAB(ABABA^2)B$. Thus

$$\begin{aligned} ABAB(Y)B = V + W + ABAB(&AB^2A^3 + BA^4B + BA^3BA + BA^2BA^2 \\ &+ BABA^3 + B^2A^4)B \in \mathcal{V}_{10} \end{aligned}$$

where $V \in \mathcal{V}_{10}$ (since W is the smallest word of length 11 in $\mathcal{B}$). Thus

$$\begin{aligned} W &+ ABABAB^2A^3B + ABAB^2A^4B^2 + ABAB^2A^3BAB + ABAB^2A^2BA^2B \\ &+ ABAB^2ABA^3B + ABAB^3A^4B \in \mathcal{V}_{10}. \qquad (*) \end{aligned}$$

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Now A^4, A^3B, A^2BA belong to $\mathcal{V}_3$ since neither is a subword of W. It follows from $(*)$ that $W \in \mathcal{V}_{10}$. This is a contradiction.

(**22**) W is $B(ABA)^3A$

BA^2BA^2 is a subword of W. In fact, $W \equiv BABA^2(BA^2BA^2)$. Thus

$$BABA^2(Y) = V + W + BABA^2(BABA^3 + B^2A^4) \in \mathcal{V}_{10}$$

where $V \in \mathcal{V}_{10}$ (since W is the smallest word of length 11 in $\mathcal{B}$). Thus

$$W + BABA^2BABA^3 + BABA^2B^2A^4 \in \mathcal{V}_{10}. \qquad (*)$$

Now A^3 is not a subword of W so does not belong to $\mathcal{B}$. Hence $A^3 \in \mathcal{V}_2$ and it follows from $(*)$ that $W \in \mathcal{V}_{10}$. This is a contradiction.

(**24**) W is $BA^2(BA)^4$

A^2BABA is a subword of W. In fact, $W \equiv B(A^2BABA)BABA$. Thus

$$\begin{aligned} B(Y)BABA = V + W + B(&A^2B^2A^2 + ABA^3B + ABA^2BA \\ &+ ABABA^2 + AB^2A^3 + BA^4B + BA^3BA + BA^2BA^2 \\ &+ BABA^3 + B^2A^4)BABA \in \mathcal{V}_{10} \end{aligned}$$

where $V \in \mathcal{V}_{10}$ (since W is the smallest word of length 11 in $\mathcal{B}$). Thus

$$\begin{aligned} &W + BA^2B^2A^2BABA + BABA^3B^2ABA + BABA^2BABABA \\ &\quad + BABABA^2BABA + BAB^2A^3BABA + B^2A^4B^2ABA + B^2A^3BABABA \\ &\quad + B^2A^2BA^2BABA + B^2ABA^3BABA + B^3A^4BABA \in \mathcal{V}_{10}. \qquad (*) \end{aligned}$$

Now none of A^4, A^3B, A^2B^2, ABA^2, AB^2A, AB^3, BA^3, $BABA$ is a subword of W, so none belongs to $\mathcal{B}$. On the other hand, A^2BA, $ABAB$, BA^2B are subwords of W so each belongs to $\mathcal{B}$. It follows that $ABA^2 - \alpha A^2BA \in \mathcal{V}_3$ and $BABA - \beta A^2BA - \gamma ABAB - \delta BA^2B \in \mathcal{V}_3$ for some scalars α, β, γ, δ. Using this, and the fact that $A^3 \in \mathcal{V}_2$ since it is not a subword of W and so does not belong to $\mathcal{B}$, in $(*)$ we obtain $W \in \mathcal{V}_{10}$. This is a contradiction.

(**26**) W is $(BA^3)^2BA^2$

BA^3BA is a subword of W. In fact, $W \equiv (BA^3BA)A^2BA^2$. Thus

$$(Y)A^2BA^2 = V + W + (BA^2BA^2 + BABA^3 + B^2A^4)A^2BA^2 \in \mathcal{V}_{10}$$

where $V \in \mathcal{V}_{10}$ (since W is the smallest word of length 11 in $\mathcal{B}$). Thus

$$W + BA^2BA^4BA^2 + BABA^5BA^2 + B^2A^6BA^2 \in \mathcal{V}_{10}. \qquad (*)$$

Now A^4 is not a subword of W so does not belong to $\mathcal{B}$. Hence $A^4 \in \mathcal{V}_3$ and it follows from $(*)$ that $W \in \mathcal{V}_{10}$. This is a contradiction.

The following cases involve Z: **(2), (4), (6), (8), (10), (12), (14), (16), (21), (25).**

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(**2**) W is $(B^3A)^2B^3$
BAB^3A is a subword of W. In fact, $W \equiv B^2(BAB^3A)B^3$. Thus

$$B^2(Z)B^3 = V + W + B^2(B^2A^2B^2 + B^2ABAB + B^2AB^2A + B^3A^2B + B^3ABA + B^4A^2)B^3 \in \mathcal{V}_{10}$$

where $V \in \mathcal{V}_{10}$ (since W is the smallest word of length 11 in $\mathcal{B}$). Thus

$$W + B^4A^2B^5 + B^4ABAB^4 + B^4AB^2AB^3 + B^5A^2B^4 + B^5ABAB^3 + B^6A^2B^3 \in \mathcal{V}_{10}. \quad (*)$$

Now A^2 is not a subword of W so does not belong to $\mathcal{B}$. Hence $A^2 \in \mathcal{V}_1$. Also, none of A^3, A^2B, ABA is a subword of W so each, in particular ABA, belongs to $\mathcal{V}_2$. Also, none of A^4, A^3B, A^2BA, A^2B^2, ABA^2, $ABAB$, AB^2A is a subword of W so each, in particular AB^2A, belongs to $\mathcal{V}_3$. It now follows from $(*)$ that $W \in \mathcal{V}_{10}$. This is a contradiction.

(**4**) W is $B(B^2A)^3B$
B^2AB^2A is a subword of W. In fact, $W \equiv B^3A(B^2AB^2A)B$. Thus

$$B^3A(Z)B = V + W + B^3A(B^3A^2B + B^3ABA + B^4A^2)B \in \mathcal{V}_{10}$$

where $V \in \mathcal{V}_{10}$ (since W is the smallest word of length 11 in $\mathcal{B}$). Thus

$$W + B^3AB^3A^2B^2 + B^3AB^3ABAB + B^3AB^4A^2B \in \mathcal{V}_{10}. \quad (*)$$

Now A^2 is not a subword of W so does not belong to $\mathcal{B}$. Hence $A^2 \in \mathcal{V}_1$. Also, none of A^3, A^2B, ABA is a subword of W so each, in particular ABA, belongs to $\mathcal{V}_2$. It now follows from $(*)$ that $W \in \mathcal{V}_{10}$. This is a contradiction.

(**6**) W is $B^3(AB)^4$
B^3ABA is a subword of W. In fact, $W \equiv (B^3ABA)BABAB$. Thus

$$(Z)BABAB = V + W + (B^4A^2)BABAB \in \mathcal{V}_{10}$$

where $V \in \mathcal{V}_{10}$ (since W is the smallest word of length 11 in $\mathcal{B}$). Thus

$$W + B^4A^2BABAB \in \mathcal{V}_{10}. \quad (*)$$

Now A^2 is not a subword of W so does not belong to $\mathcal{B}$. Hence $A^2 \in \mathcal{V}_1$. It now follows from $(*)$ that $W \in \mathcal{V}_{10}$. This is a contradiction.

(**8**) W is $B^2A(B^3A)^2$
BAB^3A is a subword of W. In fact, $W \equiv B^2AB^2(BAB^3A)$. Thus

$$B^2AB^2(Z) = V + W + B^2AB^2(B^2A^2B^2 + B^2ABAB + B^2AB^2A + B^3A^2B + B^3ABA + B^4A^2) \in \mathcal{V}_{10}$$

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where $V \in \mathcal{V}_{10}$ (since W is the smallest word of length 11 in $\mathcal{B}$). Thus

$$W + B^2AB^4A^2B^2 + B^2AB^4ABAB + B^2AB^4AB^2A + B^2AB^5A^2B + B^2AB^5ABA + B^2AB^6A^2 \in \mathcal{V}_{10}. \qquad (*)$$

Now A^2 is not a subword of W so does not belong to $\mathcal{B}$. Hence $A^2 \in \mathcal{V}_1$. None of A^3, A^2B, ABA is a subword of W so each, in particular ABA, belongs to ν_2. Also, none of A^4, A^3B, A^2BA, A^2B^2, ABA^2, $ABAB$, AB^2A is a subword of W so each, in particular AB^2A, belongs to $\mathcal{V}_3$. It now follows from $(*)$ that $W \in \mathcal{V}_{10}$. This is a contradiction.

(**10**) W is $(B^2A)^3BA$

B^2AB^2A is a subword of W. In fact, $W \equiv (B^2AB^2A)B^2ABA$. Thus

$$(Z)B^2ABA = V + W + (B^3A^2B + B^3ABA + B^4A^2)B^2ABA \in \mathcal{V}_{10}$$

where $V \in \mathcal{V}_{10}$ (since W is the smallest word of length 11 in $\mathcal{B}$). Thus

$$W + B^3A^2B^3ABA + B^3ABAB^2ABA + B^4A^2B^2ABA \in \mathcal{V}_{10}. \qquad (*)$$

Now A^2 is not a subword of W so does not belong to $\mathcal{B}$. Hence $A^2 \in \mathcal{V}_1$. Also, none of A^4, A^3B, A^2BA, A^2B^2, ABA^2, $ABAB$ is a subword of W so each, in particular $ABAB$, belongs to $\mathcal{V}_3$. It now follows from $(*)$ that $W \in \mathcal{V}_{10}$. This is a contradiction.

(**12**) W is $B^2(AB)^4B$

B^2ABAB is a subword of W. In fact, $W \equiv (B^2ABAB)ABAB^2$. Thus

$$(Z)ABAB^2 = V + W + (B^2AB^2A + B^3A^2B + B^3ABA + B^4A^2)ABAB^2 \in \mathcal{V}_{10}$$

where $V \in \mathcal{V}_{10}$ (since W is the smallest word of length 11 in $\mathcal{B}$). Thus

$$W + B^2AB^2A^2BAB^2 + B^3A^2BABAB^2 + B^3ABA^2BAB^2 + B^4A^3BAB^2 \in \mathcal{V}_{10}. \qquad (*)$$

Now A^2 is not a subword of W so does not belong to $\mathcal{B}$. Hence $A^2 \in \mathcal{V}_1$. It now follows from $(*)$ that $W \in \mathcal{V}_{10}$. This is a contradiction.

(**14**) W is $(BAB^2)^2BAB$

BAB^3A is a subword of W. In fact, $W \equiv BAB^2(BAB^3A)B$. Thus

$$BAB^2(Z)B = V + W + BAB^2(B^2A^2B^2 + B^2ABAB + B^2AB^2A + B^3A^2B + B^3ABA + B^4A^2)B \in \mathcal{V}_{10}$$

where $V \in \mathcal{V}_{10}$ (since W is the smallest word of length 11 in $\mathcal{B}$). Thus

$$W + BAB^4A^2B^3 + BAB^4ABAB^2 + BAB^4AB^2AB + BAB^5A^2B^2 + BAB^5ABAB + BAB^6A^2B \in \mathcal{V}_{10}. \qquad (*)$$

Now A^2 is not a subword of W so does not belong to $\mathcal{B}$. Hence $A^2 \in \mathcal{V}_1$. Also, none of A^3, A^2B, ABA is a subword of W so each, in particular ABA, belongs to $\mathcal{V}_2$. Also, none of A^4, A^3B, A^2BA, A^2B^2, ABA^2, $ABAB$, AB^2A is a subword of W so each, in particular AB^2A, belongs to $\mathcal{V}_3$. It now follows from $(*)$ that $W \in \mathcal{V}_{10}$. This is a contradiction.

(**16**) W is $(BAB)^3B^2$

B^2AB^2A is a subword of W. In fact, $W \equiv BA(B^2AB^2A)B^3$. Thus

$$BA(Z)B^3 = V + W + BA(B^3A^2B + B^3ABA + B^4A^2)B^3 \in \mathcal{V}_{10}$$

where $V \in \mathcal{V}_{10}$ (since W is the smallest word of length 11 in $\mathcal{B}$). Thus

$$W + BAB^3A^2B^4 + BAB^3ABAB^3 + BAB^4A^2B^3 \in \mathcal{V}_{10}. \tag{$*$}$$

Now A^2 is not a subword of W so does not belong to $\mathcal{B}$. Hence $A^2 \in \mathcal{V}_1$. Also, none of A^3, A^2B, ABA is a subword of W so each, in particular ABA, belongs to $\mathcal{V}_2$. It now follows from $(*)$ that $W \in \mathcal{V}_{10}$. This is a contradiction.

(**21**) W is $A(BAB)^3B$

B^2AB^2A is a subword of W. In fact, $W \equiv ABA(B^2AB^2A)B^2$. Thus

$$ABA(Z)B^2 = V + W + ABA(B^3A^2B + B^3ABA + B^4A^2)B^2 \in \mathcal{V}_{10}$$

where $V \in \mathcal{V}_{10}$ (since W is the smallest word of length 11 in $\mathcal{B}$). Thus

$$W + ABAB^3A^2B^3 + ABAB^3ABAB^2 + ABAB^4A^2B^2 \in \mathcal{V}_{10}. \tag{$*$}$$

There is only one subword of W of length 6 strictly preceding $ABAB^3$ in the lexicographic ordering, namely $ABAB^2A$. Thus, $ABAB^3 = \alpha ABAB^2A + U$, for some scalar α and some $U \in \mathcal{V}_5$. Then $(ABAB^3)ABAB^2 = \alpha ABAB^2A^2BAB^2 + UABAB^2$. Now A^2 is not a subword of W so does not belong to $\mathcal{B}$. Hence $A^2 \in \mathcal{V}_1$. It follows that $ABAB^3ABAB^2 \in \mathcal{V}_{10}$, and, from $(*)$, that $W \in \mathcal{V}_{10}$. This is a contradiction.

(**25**) W is $(AB^3)^2AB^2$

BAB^3A is a subword of W. In fact, $W \equiv AB^2(BAB^3A)B^2$. Thus

$$\begin{aligned}AB^2(Z)B^2 = V + W + AB^2(&B^2A^2B^2 + B^2ABAB + B^2AB^2A + B^3A^2B \\ &+ B^3ABA + B^4A^2)B^2 \in \mathcal{V}_{10}\end{aligned}$$

where $V \in \mathcal{V}_{10}$ (since W is the smallest word of length 11 in $\mathcal{B}$). Thus

$$\begin{aligned}W &+ AB^4A^2B^4 + AB^4ABAB^3 + AB^4AB^2AB^2 + AB^5A^2B^3 + AB^5ABAB^2 \\ &+ AB^6A^2B^2 \in \mathcal{V}_{10}.\end{aligned} \tag{$*$}$$

Now A^2 is not a subword of W so does not belong to $\mathcal{B}$. Hence $A^2 \in \mathcal{V}_1$. Also, none of A^3, A^2B, ABA is a subword of W, so each, in particular ABA, belongs to $\mathcal{V}_2$. None of the words A^4, A^3B, A^2BA, A^2B^2, ABA^2, $ABAB$, AB^2A is a subword of W so each one, in particular AB^2A belongs to $\mathcal{V}_3$. Using the facts that $A^2 \in \mathcal{V}_1$, $ABA \in \mathcal{V}_2$, $AB^2A \in \mathcal{V}_3$ in $(*)$ gives that $W \in \mathcal{V}_{10}$. This is a contradiction.

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This completes the consideration of the cases when W is one of the last 26 words. Finally, we have to consider the cases where W is one of

$$(A^2B)^3A^2,\ (B^2A)^3B^2,\ A(AB)^5,\ B(BA)^5,\ (ABA)^3AB,$$
$$(BAB)^3BA,\ (AB)^5B,\ (BA)^5A,\ (AB^2)^3AB,\ (BA^2)^3BA.$$

(**I**) W is $(BA)^5A$

Since $(A+\lambda BA)^6 \in \mathcal{V}_{10}$ by the Cayley–Hamilton theorem, it follows from Proposition 4 that

$$A(BA)^5 + BA^2(BA)^4 + (BA)^2A(BA)^3 + (BA)^3A(BA)^2 + (BA)^4ABA + (BA)^5A \in \mathcal{V}_{10}.$$

In the above $(BA)^5A$ is the largest word, so if it was W then each of the other five words would belong to $\mathcal{V}_{10}$, since W is the smallest word of length 11 in $\mathcal{B}$. Then W would belong to $\mathcal{V}_{10}$, and this is a contradiction.

(**II**) W is $B(BA)^5$

Since $(B+\lambda BA)^6 \in \mathcal{V}_{10}$ by the Cayley–Hamilton theorem, it follows from Proposition 4 that

$$(BA)^5B + (BA)^4B^2A + (BA)^3B(BA)^2 + (BA)^2B(BA)^3 + BAB(BA)^4 + B(BA)^5 \in \mathcal{V}_{10}.$$

In the above $B(BA)^5$ is the largest word, so if it was W then each of the other five words would belong to $\mathcal{V}_{10}$, since W is the smallest word of length 11 in $\mathcal{B}$. Then W would belong to $\mathcal{V}_{10}$, and this is a contradiction.

For the remaining 8 words,

$$(A^2B)^3A^2,\ (B^2A)^3B^2,\ A(AB)^5,\ (ABA)^3AB,$$
$$(BAB)^3BA,\ (AB)^5B,\ (AB^2)^3AB,\ (BA^2)^3BA,$$

we first consider the case where $\{A, B\}$ is not an irreducible pair of matrices (that is, they have a common nontrivial invariant subspace). Then the unital algebra generated by the pair has dimension at most 31. Since each of these words has 30 subwords, including the empty word, the dimension of the unital algebra generated by A and B, that is, the number of elements in $\mathcal{B}$, is at least 30. Hence there is at most one other element of $\mathcal{B}$ apart from the subwords of W.

(**III$_\mathbf{r}$**) W is $(A^2B)^3A^2$

BA^2BA^2 is a subword of W. In fact, $W \equiv A^2BA^2(BA^2BA^2)$. Thus

$$A^2BA^2(Y) = V + W + A^2BA^2(BABA^3 + B^2A^4) \in \mathcal{V}_{10}$$

where $V \in \mathcal{V}_{10}$ (since W is the smallest word of length 11 in $\mathcal{B}$). Thus

$$W + A^2BA^2BABA^3 + A^2BA^2B^2A^4 \in \mathcal{V}_{10}. \tag{$*$}$$

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Now A^4 cannot belong to $\mathcal{B}$, otherwise both A^3 and A^4 would belong to $\mathcal{B}$ and $\mathcal{B}$ would have at least 32 elements. Thus $A^4 \in \mathcal{V}_3$ and $A^2BA^2B^2(A^4) \in \mathcal{V}_{10}$.

If $A^3 \notin \mathcal{B}$ then $A^3 \in \mathcal{V}_2$ and $A^2BA^2BABA^3 = A^2BA^2BAB(A^3)$ belongs to $\mathcal{V}_{10}$. It would then follow from $(*)$ that $W \in \mathcal{V}_{10}$. This is a contradiction.

If $A^3 \in \mathcal{B}$ then $BAB = \alpha A^2 + \beta A^2B + \gamma ABA + \delta BA^2 + U$ for some scalars α, β, γ, δ and some $U \in \mathcal{V}_2$. Then

$$A^2BA^2(BAB)A^3 = \alpha A^2BA^7 + \beta A^2BA^4BA^3 + \gamma A^2BA^3BA^4 + \delta A^2BA^2BA^5 + A^2BA^2UA^3 \in \mathcal{V}_{10}$$

since $A^4 \in \mathcal{V}_3$ and $U \in \mathcal{V}_2$. It follows from $(*)$ that $W \in \mathcal{V}_{10}$. This is a contradiction.

(IV$_\mathbf{r}$) W is $(B^2A)^3B^2$

B^2AB^2A is a subword of W. In fact, $W \equiv B^2A(B^2AB^2A)B^2$. Thus

$$B^2A(Z)B^2 = V + W + B^2A(B^3A^2B + B^3ABA + B^4A^2)B^2 \in \mathcal{V}_{10}$$

where $V \in \mathcal{V}_{10}$ (since W is the smallest word of length 11 in $\mathcal{B}$). Thus

$$W + B^2AB^3A^2B^3 + B^2AB^3ABAB^2 + B^2AB^4A^2B^2 \in \mathcal{V}_{10}. \qquad (*)$$

If $A^2 \notin \mathcal{B}$ then $A^2 \in \mathcal{V}_1$ and $B^2AB^3A^2B^3$, $B^2AB^4A^2B^2$ belong to $\mathcal{V}_{10}$. Also, the first six words of length 4 are A^4, A^3B, A^2BA, A^2B^2, ABA^2, $ABAB$. The first five of them contain a factor of A^2 so none of them belongs to $\mathcal{B}$. Hence each belongs to $\mathcal{V}_3$. Now $ABAB$ cannot belong to $\mathcal{B}$ since ABA would too, making $\mathcal{B}$ have at least 32 elements. Thus $ABAB \in \mathcal{V}_3$ and so $B^2AB^3ABAB^2 = B^2AB^3(ABAB)B \in \mathcal{V}_{10}$. It would then follow from $(*)$ that $W \in \mathcal{V}_{10}$. This is a contradiction.

If $A^2 \in \mathcal{B}$ then none of A^3, A^2B, ABA belongs to $\mathcal{B}$ so each belongs to $\mathcal{V}_2$. Then

$$B^2AB^3A^2B^3 + B^2AB^3ABAB^2 + B^2AB^4A^2B^2 = B^2AB^3(A^2B)B^2 + B^2AB^3(ABA)B^2 + B^2AB^4(A^2B)B \in \mathcal{V}_{10}.$$

It follows from $(*)$ that $W \in \mathcal{V}_{10}$. This is a contradiction.

(V$_\mathbf{r}$) W is $A(AB)^5$

$BABABA$ is a subword of W. In fact, $W \equiv A^2(BABABA)BAB$. Thus

$$A^2(X)BAB = V + W + A^2(BAB^2A^2 + B^2A^3B + B^2A^2BA + B^2ABA^2 + B^3A^3)BAB \in \mathcal{V}_{10}$$

where $V \in \mathcal{V}_{10}$ (since W is the smallest word of length 11 in $\mathcal{B}$). Thus

$$W + A^2BAB^2A^2BAB + A^2B^2A^3B^2AB + A^2B^2A^2BABAB + A^2B^2ABA^2BAB + A^2B^3A^3BAB \in \mathcal{V}_{10}. \qquad (*)$$

Neither of A^4, A^3B can belong to $\mathcal{B}$ since otherwise A^3 would too, making $\mathcal{B}$ have more than 31 elements. Thus $A^3B \in \mathcal{V}_3$ and $A^2B^2A^3B^2AB + A^2B^3A^3BAB \in$

$\mathcal{V}_{10}$. Also, A^2B^2 cannot belong to $\mathcal{B}$ since otherwise B^2 would too. Thus $A^2B^2 = \alpha A^2BA + U$ for some scalar α and some $U \in \mathcal{V}_3$. Then $A^2B^2A^2BABAB = (A^2B^2)A^2BABAB = \alpha A^2BA^3BABAB + UA^2BABAB \in \mathcal{V}_{10}$ since $A^3B \in \mathcal{V}_3$ and $U \in \mathcal{V}_3$. All of the words of length 7 preceding A^2BA^2BA contain a factor of A^3 and so belong to $\mathcal{V}_6$, since none can belong to $\mathcal{B}$. Since A^2BA^2BA cannot belong to $\mathcal{B}$, since its subword BA^2 is not a subword of W, it follows that $A^2BA^2BA \in \mathcal{V}_6$. Thus

$$\begin{aligned} A^2B^2ABA^2BAB &= (A^2B^2)ABA^2BAB \\ &= \alpha(A^2BA^2BA)ABAB + UABA^2BAB \in \mathcal{V}_{10}. \end{aligned}$$

Finally,

$$BAB^2 = \beta A^2BA + \gamma ABAB + \delta BABA + T$$

for some scalars β, γ, δ and some element $T \in \mathcal{V}_3$. This is because every word of length 4 which precedes BAB^2, and BAB^2 itself, contains a proper subword which is not a subword of W, unless it is A^2BA, $ABAB$ or $BABA$. Hence

$$\begin{aligned} A^2BAB^2A^2BAB = A^2(BAB^2)A^2BAB &= \beta A^4BA^3BAB + \gamma A^3BABA^2BAB \\ &\quad + \delta A^2BABA^3BAB + A^2TA^2BAB \in \mathcal{V}_{10}, \end{aligned}$$

since $A^3B \in \mathcal{V}_3$ and $T \in \mathcal{V}_3$. It follows from $(*)$ that $W \in \mathcal{V}_{10}$. This is a contradiction.

(**VI$_\mathbf{r}$**) W is $(ABA)^3AB$

BA^2BA^2 is a subword of W. In fact, $W \equiv ABA^2(BA^2BA^2)B$. Thus

$$ABA^2(Y)B = V + W + ABA^2(BABA^3 + B^2A^4)B \in \mathcal{V}_{10}$$

where $V \in \mathcal{V}_{10}$ (since W is the smallest word of length 11 in $\mathcal{B}$). Thus

$$W + ABA^2BABA^3B + ABA^2B^2A^4B \in \mathcal{V}_{10}. \qquad (*)$$

Neither of A^4, A^3B can belong to $\mathcal{B}$ since otherwise A^3 would too, making $\mathcal{B}$ have more than 31 elements. Thus $A^3B \in \mathcal{V}_3$ and $ABA^2BABA^3B + ABA^2B^2A^4B \in \mathcal{V}_{10}$. It follows from $(*)$ that $W \in \mathcal{V}_{10}$. This is a contradiction.

(**VII$_\mathbf{r}$**) W is $(BAB)^3BA$

B^2AB^2A is a subword of W. In fact, $W \equiv BA(B^2AB^2A)B^2A$. Thus

$$BA(Z)B^2A = V + W + BA(B^3A^2B + B^3ABA + B^4A^2)B^2A \in \mathcal{V}_{10}$$

where $V \in \mathcal{V}_{10}$ (since W is the smallest word of length 11 in $\mathcal{B}$). Thus

$$W + BAB^3A^2B^3A + BAB^3ABAB^2A + BAB^4A^2B^2A \in \mathcal{V}_{10}. \qquad (*)$$

Neither of A^3, A^2B can belong to $\mathcal{B}$ since otherwise A^2 would too, making $\mathcal{B}$ have more than 31 elements. Thus $A^2B \in \mathcal{V}_2$ and $BAB^3A^2B^3A + BAB^4A^2B^2A \in \mathcal{V}_{10}$. The first six words of length 4 are A^4, A^3B, A^2BA, A^2B^2, ABA^2, $ABAB$. Each

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of these has a proper subword which is not a subword of W. It follows that each, in particular $ABAB$, belongs to $\mathcal{V}_3$. Hence $BAB^3ABAB^2A = BAB^3(ABAB)BA \in \mathcal{V}_{10}$. It follows from $(*)$ that $W \in \mathcal{V}_{10}$. This is a contradiction.

(VIII$_\mathbf{r}$) W is $(AB)^5B$

$BABABA$ is a subword of W. In fact, $W \equiv ABA(BABABA)B^2$. Thus

$$ABA(X)B^2 = V + W + ABA(BAB^2A^2 + B^2A^3B + B^2A^2BA + B^2ABA^2 + B^3A^3)B^2 \in \mathcal{V}_{10}$$

where $V \in \mathcal{V}_{10}$ (since W is the smallest word of length 11 in $\mathcal{B}$). Thus

$$W + ABABAB^2A^2B^2 + ABAB^2A^3B^3 + ABAB^2A^2BAB^2 + ABAB^2ABA^2B^2 + ABAB^3A^3B^2 \in \mathcal{V}_{10}. \qquad (*)$$

Neither of A^3, A^2B can belong to $\mathcal{B}$ since otherwise A^2 would too, making $\mathcal{B}$ have more than 31 elements. Thus $A^2B \in \mathcal{V}_2$. It follows that

$$ABABAB^2A^2B^2 + ABAB^2A^3B^3 + ABAB^2A^2BAB^2 + ABAB^2ABA^2B^2 + ABAB^3A^3B^2 \in \mathcal{V}_{10}$$

and hence, from $(*)$, that $W \in \mathcal{V}_{10}$. This is a contradiction.

(IX$_\mathbf{r}$) W is $(AB^2)^3AB$

B^2AB^2A is a subword of W. In fact, $W \equiv A(B^2AB^2A)B^2AB$. Thus

$$A(Z)B^2AB = V + W + A(B^3A^2B + B^3ABA + B^4A^2)B^2AB \in \mathcal{V}_{10}$$

where $V \in \mathcal{V}_{10}$ (since W is the smallest word of length 11 in $\mathcal{B}$). Thus

$$W + AB^3A^2B^3AB + AB^3ABAB^2AB + AB^4A^2B^2AB \in \mathcal{V}_{10}. \qquad (*)$$

Neither of A^3, A^2B can belong to $\mathcal{B}$ since otherwise A^2 would too, making $\mathcal{B}$ have more than 31 elements. Thus $A^2B \in \mathcal{V}_2$. It follows that $AB^3A^2B^3AB + AB^4A^2B^2AB \in \mathcal{V}_{10}$. The first six words of length 4 are A^4, A^3B, A^2BA, A^2B^2, ABA^2, $ABAB$. Each of these has a proper subword which is not a subword of W. It follows that each, in particular $ABAB$, belongs to $\mathcal{V}_3$. Hence, $AB^3ABAB^2AB = AB^3(ABAB)BAB \in \mathcal{V}_{10}$. It follows from $(*)$, that $W \in \mathcal{V}_{10}$. This is a contradiction.

(X$_\mathbf{r}$) W is $(BA^2)^3BA$

BA^2BA^2 is a subword of W. In fact, $W \equiv BA^2(BA^2BA^2)BA$. Thus

$$BA^2(Y)BA = V + W + BA^2(BABA^3 + B^2A^4)BA \in \mathcal{V}_{10}$$

where $V \in \mathcal{V}_{10}$ (since W is the smallest word of length 11 in $\mathcal{B}$). Thus

$$W + BA^2BABA^3BA + BA^2B^2A^4BA \in \mathcal{V}_{10}. \qquad (*)$$

Neither of A^4, A^3B can belong to $\mathcal{B}$ since otherwise A^3 would too, making $\mathcal{B}$ have more than 31 elements. Thus $A^3B \in \mathcal{V}_3$. It follows that $BA^2BABA^3BA + BA^2B^2A^4BA \in \mathcal{V}_{10}$, and then from $(*)$, that $W \in \mathcal{V}_{10}$. This is a contradiction.

It only remains to consider the cases where W is one of

$$(A^2B)^3A^2,\ (B^2A)^3B^2,\ A(AB)^5,\ (ABA)^3AB,\ (BAB)^3BA,\ (AB)^5B,\ (AB^2)^3AB,\ (BA^2)^3BA$$

and $\{A, B\}$ is an irreducible pair of matrices. Then of course, by Burnside's theorem, $\mathcal{B}$ has 36 elements, 30 of which are the subwords of W.

(III$_\mathbf{i}$) W is $(A^2B)^3A^2$

BA^2BA^2 is a subword of W. In fact, $W \equiv A^2BA^2(BA^2BA^2)$. Thus

$$A^2BA^2(Y) = V + W + A^2BA^2(BABA^3 + B^2A^4) \in \mathcal{V}_{10}$$

where $V \in \mathcal{V}_{10}$ (since W is the smallest word of length 11 in $\mathcal{B}$). Thus

$$W + A^2BA^2BABA^3 + A^2BA^2B^2A^4 \in \mathcal{V}_{10}. \tag{$*$}$$

Now the word A^2BA^2BAB has precisely six subwords which are not subwords of W. So if $A^2BA^2BAB \in \mathcal{B}$ then $\mathcal{B}$ consists of the 30 subwords of W together with the six aforementioned subwords of A^2BA^2BAB. In particular, since A^3 is neither a subword of W nor of A^2BA^2BAB it follows that $A^3 \in \mathcal{V}_2$. Then $A^2BA^2BABA^3 + A^2BA^2B^2A^4 \in \mathcal{V}_{10}$ and so, from $(*)$, $W \in \mathcal{V}_{10}$. This is a contradiction. Thus $A^2BA^2BAB \notin \mathcal{B}$. Hence

$$A^2BA^2BAB = \alpha A^2BA^2BA^2 + S$$

for some scalar α and some matrix S which is a linear combination those words which strictly precede $A^2BA^2BA^2$. Then

$$(A^2BA^2BAB)A^3 = \alpha A^2BA^2BA^5 + SA^3.$$

Now, $A^2BA^2BA^5$ strictly precedes W and has length 11, so $A^2BA^2BA^5 \in \mathcal{V}_{10}$. Also, $SA^3 \in \mathcal{V}_{10}$ since, if K is any word strictly preceding $A^2BA^2BA^2$, then KA^3 strictly precedes $A^2BA^2BA^5$ and so strictly precedes W. Thus $A^2BA^2BABA^3 \in \mathcal{V}_{10}$. Now the word $A^2BA^2B^2A$ cannot belong to $\mathcal{B}$ since it has more than six subwords which are not subwords of W. Thus

$$A^2BA^2B^2A = \beta A^2BA^2BA^2 + T$$

for some scalar α and some matrix T which is a linear combination of those words which strictly precede $A^2BA^2BA^2$. Then

$$(A^2BA^2B^2A)A^3 = A^2BA^2B^2A^4 = \beta A^2BA^2BA^5 + TA^3.$$

As above, this implies that $A^2BA^2B^2A^4 \in \mathcal{V}_{10}$. It follows from $(*)$ that $W \in \mathcal{V}_{10}$. This is a contradiction.

(IV$_\mathbf{i}$) W is $(B^2A)^3B^2$

B^2AB^2A is a subword of W. In fact, $W \equiv B^2A(B^2AB^2A)B^2$. Thus

$$B^2A(Z)B^2 = V + W + B^2A(B^3A^2B + B^3ABA + B^4A^2)B^2 \in \mathcal{V}_{10}$$

where $V \in \mathcal{V}_{10}$ (since W is the smallest word of length 11 in $\mathcal{B}$). Thus

$$W + B^2AB^3A^2B^3 + B^2AB^3ABAB^2 + B^2AB^4A^2B^2 \in \mathcal{V}_{10}. \qquad (*)$$

The word B^2AB^3A has eight subwords which are not subwords of W so it cannot belong to $\mathcal{B}$. Then

$$B^2AB^3A = \alpha B^2AB^2AB + S$$

for some scalar α and some matrix S which is a linear combination of those words which strictly precede B^2AB^2AB. Thus

$$B^2AB^3A^2B^3 = (B^2AB^3A)AB^3 = \alpha B^2AB^2ABAB^3 + SAB^3$$

where $B^2AB^2ABAB^3$ strictly precedes W so belongs to $\mathcal{V}_{10}$ and where $SAB^3 \in \mathcal{V}_{10}$ since it is a linear combination of words strictly preceding $B^2AB^2ABAB^3$ and so W. It follows that $B^2AB^3A^2B^3 \in \mathcal{V}_{10}$.

Suppose that $B^2AB^3 \notin \mathcal{B}$. Then $B^2AB^3 = \beta B^2AB^2A + T$ for some scalar β and some matrix T which is a linear combination of those words which strictly precede B^2AB^2A. Thus

$$B^2AB^3ABAB^2 = (B^2AB^3)ABAB^2 = \beta B^2AB^2A^2BAB^2 + TABAB^2$$

where $B^2AB^2A^2BAB^2$ strictly precedes W so belongs to $\mathcal{V}_{10}$ and where $TABAB^2 \in \mathcal{V}_{10}$ since it is a linear combination of words strictly preceding $B^2AB^2A^2BAB^2$ and so W. It follows that $B^2AB^3ABAB^2 \in \mathcal{V}_{10}$. Also,

$$B^2AB^4A^2B^2 = (B^2AB^3)BA^2B^2 = \beta B^2AB^2ABA^2B^2 + TBA^2B^2$$

where $B^2AB^2ABA^2B^2$ strictly precedes W so belongs to $\mathcal{V}_{10}$ and where $TBA^2B^2 \in \mathcal{V}_{10}$ since it is a linear combination of words strictly preceding $B^2AB^2ABA^2B^2$ and so W. It follows that $B^2AB^4A^2B^2 \in \mathcal{V}_{10}$. It now follows from $(*)$ that $W \in \mathcal{V}_{10}$, and this is a contradiction. Hence $B^2AB^3 \in \mathcal{B}$, and so therefore are BAB^3, AB^3, B^3. There remain two words in $\mathcal{B}$ which are neither subwords of W nor belong to the set $\{B^2AB^3, BAB^3, AB^3, B^3\}$.

The word $ABAB^2 \in \mathcal{V}_4$ since every word with five letters which is less than or equal to it has at least three subwords not belonging to $\{B^2AB^3, BAB^3, AB^3, B^3\}$ and not a subword of W. It follows that $B^2AB^3ABAB^2 = B^2AB^3(ABAB^2) \in \mathcal{V}_{10}$. The word $A^2B^2 \in \mathcal{V}_3$ since every word with four letters which is less than or equal to it has at least three subwords not belonging to $\{B^2AB^3, BAB^3, AB^3, B^3\}$ and not a

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subword of W. It follows that $B^2AB^4A^2B^2 = B^2AB^4(A^2B^2) \in \mathcal{V}_{10}$. It follows from $(*)$ that $W \in \mathcal{V}_{10}$. This is a contradiction.

(V$_\mathbf{i}$) W is $A(AB)^5$

For any pair of matrices P, Q, and any pair λ, μ of scalars, the pair $\{P, Q\}$ has the same length as the pair $\{P - \lambda I, Q - \mu I\}$. For this reason we may suppose that both A and B are invertible matrices.

By the Cayley–Hamilton theorem, $(AB)^6 = \alpha I + V_1(AB)$, for some scalar α and some matrix $V_1 \in \mathcal{V}_8$, and also $A^{-1} = \beta A^5 + V_2$, $B^{-1} = \gamma B^5 + V_3$, for some scalars β, γ and matrices $V_2, V_3 \in \mathcal{V}_4$. Then $B^{-1}A^{-1} = \beta\gamma B^5A^5 + V_4$ where $V_4 \in \mathcal{V}_9$. Thus

$$\begin{aligned} W \equiv A(AB)^5 &= A(AB)^6(AB)^{-1} = A\{\alpha I + V_1(AB)\}(AB)^{-1} = \alpha A(AB)^{-1} + AV_1 \\ &= \alpha AB^{-1}A^{-1} + AV_1 = \alpha A\{\gamma B^5 + V_3\}\{\beta A^5 + V_2\} + AV_1 \\ &= \alpha\beta\gamma AB^5A^5 + V_5 + AV_1 \end{aligned}$$

where $V_5 + AV_1 \in \mathcal{V}_{10}$. Since W does not belong to $\mathcal{V}_{10}$, neither does AB^5A^5. In particular, $A^5 \notin \mathcal{V}_4$ and since A^5 is the smallest word of length 5 this means that $A^5 \in \mathcal{B}$. Then $A^5, A^4, A^3 \in \mathcal{B}$.

As in the first part of the argument in case (V$_\mathrm{r}$),

$$\begin{aligned} W &+ A^2BAB^2A^2BAB + A^2B^2A^3B^2AB + A^2B^2A^2BABAB + A^2B^2ABA^2BAB \\ &+ A^2B^3A^3BAB \end{aligned} \tag{$*$}$$

belongs to $\mathcal{V}_{10}$. Now $A^2BAB^2 \notin \mathcal{B}$, since otherwise A^2BAB^2, $ABAB^2$, BAB^2, AB^2, $B^2 \in \mathcal{B}$ and $\mathcal{B}$ would have at least 38 elements ($\mathcal{B}$ already contains the 30 subwords of W and A^5, A^4, A^3). Thus $A^2BAB^2 = \sum_{k=1}^{11} \lambda_k U_k + V_6$ where $V_6 \in \mathcal{V}_5$ and U_k is a word in A, B of length 6 less than or equal to A^2BABA. It follows that

$$(A^2BAB^2)A^2BAB = \sum_{k=1}^{11} \lambda_k(U_kA^2BAB) + V_6A^2BAB.$$

But the word U_kA^2BAB has length 11 and is strictly less than W so belongs to $\mathcal{V}_{10}$. This shows that $A^2BAB^2A^2BAB \in \mathcal{V}_{10}$.

Suppose that $A^2B^2 \notin \mathcal{B}$. Then $A^2B^2 = \rho A^4 + \varepsilon A^3B + \tau A^2BA + V_7$ for some scalars ρ, ε, τ and some $V_7 \in \mathcal{V}_3$. Then the equations

$$(A^2B^2)A^3B^2AB = \rho A^7B^2AB + \varepsilon A^3BA^3B^2AB + \tau A^2BA^4B^2AB + V_7A^3B^2AB,$$

$$\begin{aligned} (A^2B^2)A^2BABAB = \rho A^6BABAB &+ \varepsilon A^3BA^2BABAB + \tau A^2BA^3BABAB \\ &+ V_7A^2BABAB, \end{aligned}$$

$$\begin{aligned} (A^2B^2)ABA^2BAB = \rho A^5BA^2BAB &+ \varepsilon A^3BABA^2BAB + \tau A^2BA^2BA^2BAB \\ &+ V_7ABA^2BAB, \end{aligned}$$

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$$\begin{aligned}A^2B^3A^3BAB &= (A^2B^2)BA^3BAB\\ &= \rho A^4BA^3BAB + \varepsilon A^3B^2A^3BAB + \tau A^2BABA^3BAB\\ &\quad + V_7BA^3BAB\end{aligned}$$

show that the matrices $A^2B^2A^3B^2AB$, $A^2B^2A^2BABAB$, $A^2B^2ABA^2BAB$, $A^2B^3A^3BAB$ all belong to $\mathcal{V}_{10}$ since each word of length 11 appearing in the right-hand side of the above four equations is strictly less than W, and $V_7 \in \mathcal{V}_3$. Since $A^2BAB^2A^2BAB$ also belongs to $\mathcal{V}_{10}$ it follows from $(*)$ that $W \in \mathcal{V}_{10}$. This contradiction means that we must have $A^2B^2 \in \mathcal{B}$ and the elements of $\mathcal{B}$ are precisely all the subwords of W together with A^5, A^4, A^3, A^2B^2, AB^2, B^2.

Now that all the elements of $\mathcal{B}$ are known, we must have

$$B^5A^4 = \eta A^2BABABAB + \sigma ABABABABA + \delta BABABABAB + V_8$$

for some scalars η, σ, δ and some $V_8 \in \mathcal{V}_8$. Then

$$\begin{aligned}AB^5A^5 = A(B^5A^4)A &= A(\eta A^2BABABAB + \sigma ABABABABA\\ &\quad + \delta BABABABAB + V_8)A\\ &= \eta A^3BABABABA + \sigma A^2BABABABA^2\\ &\quad + \delta ABABABABABA + AV_8A.\end{aligned}$$

Each of $A^3BABABABA$ and $A^2BABABABA^2$ has length 11 and is strictly less than W, so each belongs to $\mathcal{V}_{10}$. Also $ABABABABABA$ belongs to $\mathcal{V}_9$ by Proposition 3, and AV_8A belongs to $\mathcal{V}_{10}$. This shows that AB^5A^5 belongs to $\mathcal{V}_{10}$, contradicting what we deduced earlier. Hence W cannot be $A(AB)^5$.

($\mathbf{VI_i}$) W is $(ABA)^3AB$.

BA^2BA^2 is a subword of W. In fact, $W \equiv ABA^2(BA^2BA^2)B$. Thus

$$ABA^2(Y)B = V + W + ABA^2(BABA^3 + B^2A^4)B \in \mathcal{V}_{10}$$

where $V \in \mathcal{V}_{10}$ (since W is the smallest word of length 11 in $\mathcal{B}$). Thus

$$W + ABA^2BABA^3B + ABA^2B^2A^4B \in \mathcal{V}_{10}. \tag{$*$}$$

If $A^3 \notin \mathcal{B}$ then $A^3 \in \mathcal{V}_2$. Then both ABA^2BABA^3B, $ABA^2B^2A^4B$ belong to $\mathcal{V}_{10}$ so $W \in \mathcal{V}_{10}$ by $(*)$. Thus $A^3 \in \mathcal{B}$.

Suppose that $ABA^2BAB \in \mathcal{B}$. Then all of the subwords of ABA^2BAB belong to $\mathcal{B}$ so $\mathcal{B}$ consists of all the subwords of W together with ABA^2BAB, BA^2BAB, A^2BAB, $ABAB$, BAB, A^3. Thus $A^4, A^3B \in \mathcal{V}_3$ so ABA^2BABA^3B, $ABA^2B^2A^4B$ belong to $\mathcal{V}_{10}$ which leads to the contradiction that $W \in \mathcal{V}_{10}$. Thus $ABA^2BAB \notin \mathcal{B}$.

Since $ABA^2BAB \notin \mathcal{B}$, there is a scalar α such that $ABA^2BAB = \alpha ABA^2BA^2 + S$, for some matrix S which is a linear combination of words strictly less than ABA^2BA^2. Then

$$ABA^2BABA^3B = (ABA^2BAB)A^3B = \alpha ABA^2BA^5B + SA^3B.$$

Since $ABA^2BA^5B \prec W$ and has length 11, then $ABA^2BA^5B \in \mathcal{V}_{10}$. Also, since SA^3B is a linear combination of words strictly less than ABA^2BA^5B, and so strictly less than W, then $SA^3B \in \mathcal{V}_{10}$. Thus $ABA^2BABA^3B \in \mathcal{V}_{10}$.

Suppose that $ABA^2B^2 \in \mathcal{B}$. Then $\mathcal{B}$ consists of the subwords of W together with ABA^2B^2, BA^2B^2, A^2B^2, AB^2, B^2, A^3. Then $A^4 \in \mathcal{V}_3$ so $ABA^2B^2A^4B \in \mathcal{V}_{10}$ and it follows that $W \in \mathcal{V}_{10}$. Hence $ABA^2B^2 \notin \mathcal{B}$. Then there exists a scalar β such that $ABA^2B^2 = \beta ABA^2BA + T$, for some matrix T which is a linear combination of words strictly less than ABA^2BA. Since $ABA^2BA^5B \prec W$ and has length 11, we have $ABA^2BA^5B \in \mathcal{V}_{10}$. Also, since TA^4B is a linear combination of words strictly less than ABA^2BA^5B, and so strictly less than W, we have $TA^4B \in \mathcal{V}_{10}$. Thus $ABA^2B^2A^4B \in \mathcal{V}_{10}$. This leads to the contradiction that $W \in \mathcal{V}_{10}$.

(**VII$_\mathbf{i}$**) W is $(BAB)^3BA$

B^2AB^2A is a subword of W. In fact, $W \equiv BA(B^2AB^2A)B^2A$. Thus

$$BA(Z)B^2A = V + W + BA(B^3A^2B + B^3ABA + B^4A^2)B^2A \in \mathcal{V}_{10}$$

where $V \in \mathcal{V}_{10}$ (since W is the smallest word of length 11 in $\mathcal{B}$). Thus

$$W + BAB^3A^2B^3A + BAB^3ABAB^2A + BAB^4A^2B^2A \in \mathcal{V}_{10}. \qquad (*)$$

Suppose that $BAB^3 \notin \mathcal{B}$. Then $BAB^3 = \alpha BAB^2A + S$ for some scalar α and some matrix S which is a linear combination of words strictly less than BAB^2A. Thus

$$BAB^3A^2B^3A = (BAB^3)A^2B^3A = \alpha BAB^2A^3B^3A + SA^2B^3A$$

where $BAB^2A^3B^3A$ strictly precedes W so belongs to $\mathcal{V}_{10}$ and where $SA^2B^3A \in \mathcal{V}_{10}$ since it is a linear combination of words strictly preceding $BAB^2A^3B^3A$ and so W. It follows that $BAB^3A^2B^3A \in \mathcal{V}_{10}$. Similarly,

$$BAB^3ABAB^2A = (BAB^3)ABAB^2A = \alpha BAB^2A^2BAB^2A + SABAB^2A$$

and

$$BAB^4A^2B^2A = (BAB^3)BA^2B^2A = \alpha BAB^2ABA^2B^2A + SBA^2B^2A$$

show that BAB^3ABAB^2A, $BAB^4A^2B^2A \in \mathcal{V}_{10}$. It then follows from $(*)$ that $W \in \mathcal{V}_{10}$, and this contradiction shows that $BAB^3 \in \mathcal{B}$. Hence BAB^3, AB^3, $B^3 \in \mathcal{B}$. This accounts for 33 elements of $\mathcal{B}$. Every word with five letters strictly less than $ABAB^2$ has more than three subwords which neither belong to $\{BAB^3, AB^3, B^3\}$ nor are subwords of W. Hence all such words belong to $\mathcal{V}_4$. In particular, A^2B^3 and A^2B^2A belong to $\mathcal{V}_4$ so $BAB^3A^2B^3A = BAB^3(A^2B^3A) \in \mathcal{V}_{10}$ and $BAB^4A^2B^2A = BAB^4(A^2B^2)A \in \mathcal{V}_{10}$. If $ABAB^2 \notin \mathcal{B}$ then $ABAB^2 \in \mathcal{V}_4$ and so $BAB^3ABAB^2A = BAB^3(ABAB^2)A \in \mathcal{V}_{10}$. Then $W \in \mathcal{V}_{10}$ from $(*)$. Thus $ABAB^2 \in \mathcal{B}$ and so $\mathcal{B}$ consists of all the subwords of W together with the words $ABAB^2$, $ABAB$, ABA, BAB^3, AB^3, B^3. Then every word of length 6 less than or equal to $ABAB^2A$ cannot belong to $\mathcal{B}$ and so $ABAB^2A \in \mathcal{V}_5$. It follows that $BAB^3ABAB^2A = BAB^3(ABAB^2A) \in \mathcal{V}_{10}$, and hence, from $(*)$, that $W \in \mathcal{V}_{10}$. This is a contradiction.

(VIII$_\mathbf{i}$) W is $(AB)^5B$

As in case ($\mathrm{V_i}$) we may suppose that both A and B are invertible matrices.

By the Cayley–Hamilton theorem, $(AB)^6 = \alpha I + (AB)S$, for some scalar α and some matrix $S \in \mathcal{V}_8$, and also $A^{-1} = \beta A^5 + T$, $B^{-1} = \gamma B^5 + U$, for some scalars β, γ and matrices T, $U \in \mathcal{V}_4$. Then $B^{-1}A^{-1} = \beta\gamma B^5A^5 + V$ where $V \in \mathcal{V}_9$. Thus

$$\begin{aligned} W &\equiv (AB)^5B = (AB)^{-1}(AB)^6B = (AB)^{-1}\{\alpha I + (AB)S\}B = \alpha(AB)^{-1}B + SB \\ &= \alpha B^{-1}A^{-1}B + SB = \alpha\{\gamma B^5 + U\}\{\beta A^5 + T\}B + SB \\ &= \alpha\beta\gamma B^5A^5B + Y + SB \end{aligned}$$

where $Y + SB \in \mathcal{V}_{10}$. Now A^5B cannot belong to $\mathcal{B}$, since otherwise A^5B, A^5, A^4B, A^4, A^3B, A^3, A^2B, $A^2 \in \mathcal{B}$ and $\mathcal{B}$ would have at least 38 elements which is impossible. Since $A^5B \notin \mathcal{B}$ and $A^6 \in \mathcal{V}_5$ it follows that $A^5B \in \mathcal{V}_5$. But this means that $W \in \mathcal{V}_{10}$ since, from the above, $W - \alpha\beta\gamma B^5A^5B \in \mathcal{V}_{10}$, and this is a contradiction.

(IX$_\mathbf{i}$) W is $(AB^2)^3AB$

B^2AB^2A is a subword of W. In fact, $W \equiv A(B^2AB^2A)B^2AB$. Thus

$$A(Z)B^2AB = V + W + A(B^3A^2B + B^3ABA + B^4A^2)B^2AB \in \mathcal{V}_{10}$$

where $V \in \mathcal{V}_{10}$ (since W is the smallest word of length 11 in $\mathcal{B}$). Thus

$$W + AB^3A^2B^3AB + AB^3ABAB^2AB + AB^4A^2B^2AB \in \mathcal{V}_{10}. \qquad (*)$$

Suppose that $AB^3 \notin \mathcal{B}$. Then

$$AB^3 = \alpha A^4 + \beta A^3B + \gamma A^2BA + \delta A^2B^2 + \rho ABA^2 + \lambda ABAB + \mu AB^2A + S$$

for some scalars α, β, γ, δ, ρ, λ, μ and some matrix $S \in \mathcal{V}_3$. Then

$$\begin{aligned} &AB^3A^2B^3AB \\ &\quad = (AB^3)A^2B^3AB = \alpha A^6B^3AB + \beta A^3BA^2B^3AB + \gamma A^2BA^3B^3AB \\ &\qquad + \delta A^2B^2A^2B^3AB + \rho ABA^4B^3AB + \lambda ABABA^2B^3AB \\ &\qquad + \mu AB^2A^3B^3AB + SA^2B^3AB. \end{aligned}$$

Each of the words A^6B^3AB, $A^3BA^2B^3AB$, $A^2BA^3B^3AB$, $A^2B^2A^2B^3AB$, ABA^4B^3AB, $ABABA^2B^3AB$, $AB^2A^3B^3AB$ is strictly less than W so belongs to $\mathcal{V}_{10}$. The matrix SA^2B^3AB also belongs to $\mathcal{V}_{10}$. Hence $AB^3A^2B^3AB \in \mathcal{V}_{10}$. In a similar way it can be shown that $AB^3ABAB^2AB = (AB^3)ABAB^2AB$ and $AB^4A^2B^2AB = (AB^3)BA^2B^2AB$ belong to $\mathcal{V}_{10}$. Then $W \in \mathcal{V}_{10}$ from $(*)$, and this is a contradiction. Thus $AB^3 \in \mathcal{B}$ and so $B^3 \in \mathcal{B}$.

https://doi.org/10.1017/S0004972709000112 Published online by Cambridge University Press

Suppose that $ABAB^2A \in \mathcal{B}$. Then $\mathcal{B}$ consists of all the subwords of W together with $ABAB^2A$, $ABAB^2$, $ABAB$, ABA, AB^3, B^3. Thus $A^2B \in \mathcal{V}_2$ and $ABAB^2AB \in \mathcal{V}_6$, so each of $AB^3A^2B^3A$, AB^3ABAB^2AB, $AB^4A^2B^2AB$ belongs to $\mathcal{V}_{10}$. Then $W \in \mathcal{V}_{10}$ from $(*)$, and this is a contradiction. Hence $ABAB^2A \notin \mathcal{B}$.

It is not too difficult to check that every word with 6 letters which is strictly less than $ABAB^2A$ has more than four subwords and so cannot belong to $\mathcal{B}$. Thus, by the definition of $\mathcal{B}$, every word with 6 letters which is less than or equal to $ABAB^2A$ belongs to $\mathcal{V}_5$. Since $AB^3A^2B^3AB = AB^3(A^2B^3A)B$, $AB^3ABAB^2AB = AB^3(ABAB^2A)B$ and $AB^4A^2B^2AB = AB^4(A^2B^2AB)$ it follows that each of $AB^3A^2B^3AB$, AB^3ABAB^2AB, $AB^4A^2B^2AB$ belongs to $\mathcal{V}_{10}$ and so $W \in \mathcal{V}_{10}$. This is a contradiction.

($\mathbf{X_i}$) W is $(BA^2)^3BA$

BA^2BA^2 is a subword of W. In fact, $W \equiv BA^2(BA^2BA^2)BA$. Thus

$$BA^2(Y)BA = V + W + BA^2(BABA^3 + B^2A^4)BA \in \mathcal{V}_{10}$$

where $V \in \mathcal{V}_{10}$ (since W is the smallest word of length 11 in $\mathcal{B}$). Thus

$$W + BA^2BABA^3BA + BA^2B^2A^4BA \in \mathcal{V}_{10}. \qquad (*)$$

Clearly we can suppose that $A^3 \in \mathcal{B}$. Suppose that $A^4BA \in \mathcal{B}$. Then $\mathcal{B}$ consists of the 30 subwords of W together with A^4BA, A^4B, A^3BA, A^4, A^3B, A^3. Then $B^2 = \alpha A^2 + \beta AB + \gamma BA + S$ for some scalars α, β, γ and some matrix S which a linear combination of I, A, B. Then

$$\begin{aligned} BA^2B^2A^4BA &= BA^2(B^2)A^4BA \\ &= \alpha BA^8BA + \beta BA^3BA^4BA + \gamma BA^2BA^5BA + BA^2SA^4BA. \end{aligned}$$

Now, each of the words BA^8BA, BA^3BA^4BA, BA^2BA^5BA is strictly less than W in the ordering, so each belongs to $\mathcal{V}_{10}$. The matrix BA^2SA^4BA also belongs to $\mathcal{V}_{10}$, so $BA^2B^2A^4BA \in \mathcal{V}_{10}$. Also, $BAB = \delta A^3 + \lambda A^2B + \mu ABA + \rho BA^2 + T$ for some scalars $\delta, \lambda, \mu, \rho$ and some matrix $T \in \mathcal{V}_2$. Then

$$\begin{aligned} BA^2BABA^3BA = BA^2(BAB)A^3BA &= \delta BA^8BA + \lambda BA^4BA^3BA \\ &\quad + \mu BA^2BA^4BA + \rho BA^2BA^5BA + BA^2TA^3BA \end{aligned}$$

where each of the matrices BA^8BA, BA^4BA^3BA, BA^2BA^4BA, BA^2BA^5BA, $BA^2TA^3BA \in \mathcal{V}_{10}$ since the first three are strictly less than W. It follows from $(*)$ that $W \in \mathcal{V}_{10}$, and this is a contradiction. Thus $A^4BA \notin \mathcal{B}$.

Suppose that $A^5B \in \mathcal{B}$. Then $\mathcal{B}$ consists of the 30 subwords of W together with A^5B, A^5, A^4B, A^4, A^3B, A^3. Again $B^2 = \alpha A^2 + \beta AB + \gamma BA + S$ for some scalars α, β, γ and some matrix $S \in \mathcal{V}_1$, and $BAB = \delta A^3 + \lambda A^2B + \mu ABA + \rho BA^2 + T$ for some scalars $\delta, \lambda, \mu, \rho$ and some matrix $T \in \mathcal{V}_2$. By the same arguments as above, we get BA^2BABA^3BA, $BA^2B^2A^4BA \in \mathcal{V}_{10}$ and so $W \in \mathcal{V}_{10}$. The latter contradiction shows that $A^5B \notin \mathcal{B}$. Hence $A^4BA \in \mathcal{V}_5$ so $BA^2B^2A^4BA \in \mathcal{V}_{10}$.

https://doi.org/10.1017/S0004972709000112 Published online by Cambridge University Press

Consider BA^2BABA^3BA. If $BA^2BAB \notin \mathcal{B}$ then $BA^2BAB = BA^2BA^2 + R$ where R is a linear combination of words strictly less than BA^2BA^2. Thus $BA^2BABA^3BA = (BA^2BAB)A^3BA = BA^2BA^5BA + RA^3BA$. The word BA^2BA^5BA strictly precedes W (and has length 11) so belongs to $\mathcal{V}_{10}$. The matrix RA^3BA is a linear combination of words strictly less than BA^2BA^5BA so belongs to $\mathcal{V}_{10}$. Hence $BA^2BAB \in \mathcal{B}$. It follows from $(*)$ that $W \in \mathcal{V}_{10}$. This is a contradiction. Hence $BA^2BAB \in \mathcal{B}$ and so $\mathcal{B}$ contains, apart from the subwords of W, BA^2BAB, A^2BAB, $ABAB$, BAB, A^3. This amounts to 35 words. It follows that none of A^5, A^4B, A^3BA can belong to $\mathcal{B}$. In particular, $A^3BA \in \mathcal{V}_4$. Hence $BA^2BABA^3BA = BA^2BAB(A^3BA) \in \mathcal{V}_{10}$. Hence, from $(*)$, $W \in \mathcal{V}_{10}$. This is a contradiction.

This completes the proof of the theorem. $\square$

Acknowledgement

The authors wish to thank Oreste Panaia for his comments.

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