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# ON REPRESENTATIONS OF VARIANTS OF SEMIGROUPS

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We construct a family of representations of an arbitrary variant $S_a$ of a semigroup $S$, induced by a given representation of $S$, and investigate properties of such representations and their kernels.

## 1. INTRODUCTION

Let $S$ be a semigroup and $a \in S$. Set $s * t = sat$, $s, t \in S$. The semigroup $(S, *)$ is called a *variant* of $S$ and will be denoted by $S_a$. One of the motivations for the study of variants is their importance to understanding the structure of an original semigroup. In particular, the notion of a regularity-preserving element, which generalises the notion of an invertible element, is defined using variants, see [**2, 3, 5**]. Other situations where variants naturally appear and work are discussed in [**5**].

Let $\cdot : M \times S \to M$ or just $(S, M)$ be a representation of a semigroup $S$ by transformations of a set $M$. It defines a homomorphism from $S$ to the full transformation semigroup $\mathcal{T}(M)$; we denote the image of $(m, s)$ under the function $\cdot$ by $m \cdot s$. For $a \in S$ and a decomposition $a = \beta\alpha$, $\alpha, \beta \in S^1$, we introduce a map $* : M \times S_a \to M$ or just $(S_a, M; \alpha, \beta)$, defined by

$$m * s = m \cdot (\alpha s \beta) \text{ for all } s \in S,\ m \in M.$$

Since

$$m * (s * t) = m * (sat) = m \cdot (\alpha s \beta \alpha t \beta)$$
$$= \big(m \cdot (\alpha s \beta)\big) \cdot (\alpha t \beta) = (m * s) * t \text{ for all } s, t \in S,\ m \in M,$$

it follows that $(S_a, M; \alpha, \beta)$ is a representation of $S_a$ on $M$.

The kernel $\rho$ of a representation of $S$ naturally leads to the consideration of the family of congruences $\{\rho_{b,c} : b, c \in S^1, cb = a\}$ on $S_a$, which are the kernels of the corresponding representations of $S_a$. In the case when $\iota$ is the identity relation on $S$, $\iota_{a,1}$ and $\iota_{1,a}$ coincide with the congruences $\mathfrak{l}$ and $\mathfrak{r}$ on $S_a$ respectively, which were first introduced by Symons (see [**8**]) for the generalised transformation semigroups and many

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properties of which resemble the corresponding properties of the Green's relations $\mathcal{L}$ and $\mathcal{R}$.

The present paper is devoted to the study of the connections between a given representation $(S, M)$ of $S$ on $M$ and the representations $(S_a, M; \alpha, \beta)$; and also to the study of the properties of congruences $\{\rho_{b,c} : b, c \in S^1, cb = a\}$. In Section 2 we prove that for a regular semigroup $S$, which does not contain the bicyclic semigroup, a faithful representation of $S_a$ coincides with certain $(S_a, M; \alpha, \beta)$ if and only if $\alpha$ and $\beta$ are left and right cancellable in $S$ respectively (Theorems 1, 2 and Proposition 1). We also show that for the bicyclic semigroup these conditions are not equivalent. The next result is Theorem 4, claiming that for a regular semigroup $S$ each representation $* : M \times S_a \to M$ of any variant of $S$ is induced by some $\cdot : M \times S \to M$ such that either $m * s = m \cdot (as)$ for all $m \in M$ or $m * s = m \cdot (sa)$ for all $m \in M$, if and only if it is either a left or a right group, giving a new characterisation of the class of left (right) groups. In Section 3 we investigate the properties of the family of congruences $\{\rho_{b,c} : b, c \in S^1, cb = a\}$. In particular, in Theorem 6 we prove that in the case when $c$, $b$ are regular elements of $S$ the map sending $\rho$ to $\rho_{b,c}$ is a homomorphism from the lattice of congruences of $S$ to the lattice of congruences of $S_{cb}$. Without the requirement of regularity of $b, c$ this map always preserves the meet, while may not preserve the join, the corresponding example is provided. Finally, the aim of Section 4 is, given $a \in S^1$ and a congruence $\rho$ on $S$, to define a congruence $\rho_a$ which naturally generalises the congruence d on $S_a$, introduced in [**8**] and studied afterwards in [**3, 6**].

## 2. Connections between representations

Let $S$ be a semigroup. An element $u \in S^1$ will be called *left* [*right*] *cancellable* if $us = ut$ [$su = tu$] implies $s = t$ for all $s, t \in S$.

Fix $a \in S$ and a decomposition $a = \beta\alpha$, $\alpha, \beta \in S^1$.

**Lemma 1.** *The implication $\alpha s\beta = \alpha t\beta \Rightarrow s = t$ for all $s, t \in S$ holds if and only if $\alpha$ is left cancellable and $\beta$ is right cancellable simultaneously.*

Proof: The proof is straightforward. □

A representation $\cdot : M \times S \to M$ is said to be *faithful* if the corresponding homomorphism from $S$ to $\mathcal{T}(M)$ is injective. That is, $\cdot : M \times S \to M$ is faithful if and only if

$$(\forall m \in M)\ [m \cdot s = m \cdot t] \Rightarrow\ [s = t] \text{ for all } s, t \in S.$$

**Lemma 2.** *Suppose that $(S_a, M; \alpha, \beta)$ is faithful. Then so is $(S, M)$.*

Proof: Indeed, by the definition we have

$$(\forall\ m \in M)\big[m \cdot (\alpha s\beta) = m \cdot (\alpha t\beta)\big] \Rightarrow\ [s = t] \text{ for all } s, t \in S.$$

Take $s, t \in S$. If now $m \cdot s = m \cdot t$ for all $m \in M$, then

$$(m \cdot \alpha) \cdot (s\beta) = (m \cdot \alpha) \cdot (t\beta) \text{ for all } m \in M,$$

and so $s = t$. □

**Theorem 1.** *Let $(S, M)$ be a representation of a semigroup $S$ on a set $M$ and $a \in S$. Take $\alpha, \beta \in S^1$ such that $\beta\alpha = a$. Then the following conditions are equivalent:*

1. *$(S_a, M; \alpha, \beta)$ is faithful;*
2. *$(S, M)$ is faithful and $\alpha, \beta$ are left and right cancellable respectively.*

Proof: That 1 implies 2 is due to Lemmas 1 and 2. The opposite implication is a consequence of Lemma 1. □

Recall that an element $u$ of a semigroup $S$ is said to be a *mididentity* in $S$ if $sut = st$ for all $s, t \in S$. For the case when $S$ is regular, we obtain the following theorem.

**Theorem 2.** *Let $S$ be a regular semigroup and $a = \beta\alpha \in S$, $\alpha, \beta \in S^1$. Suppose that there exist $\alpha^*$ and $\beta^*$ of $S^1$, inverses of $\alpha$ and $\beta$ in $S^1$ respectively, such that $\beta^*\beta\alpha\alpha^*$ is a mididentity in $S$. Let also $* : M \times S_a \to M$ be a faithful representation of $S_a$ on $M$. Then the following conditions are equivalent:*

1. *there is a representation $(S, M)$ such that $(S_a, M)$ coincides with $(S_a, M; \alpha, \beta)$;*
2. *$\alpha$ and $\beta$ are left and right cancellable respectively.*

Proof: Due to Theorem 1 we have that 1 implies 2.

Now let $\alpha$ and $\beta$ be left and right cancellable respectively. Let us prove that a function $\cdot : M \times S \to M$, given by

$$m \cdot s = m * (\alpha^* s \beta^*) \text{ for all } s \in S,\ m \in M,$$

defines a representation $(S, M)$ of $S$ such that $(S_a, M)$ coincides with $(S_a, M; \alpha, \beta)$.

Since $\beta^*\beta\alpha\alpha^*$ is a mididentity in $S$, we have

$$\begin{aligned}(m \cdot s) \cdot t = \big(m * (\alpha^* s \beta^*)\big) * (\alpha^* t \beta^*) &= m * (\alpha^* s \beta^* \cdot \beta\alpha \cdot \alpha^* t \beta^*) \\ &= m * (\alpha^* st \beta^*) = m \cdot (st) \text{ for all } s, t \in S,\ m \in M.\end{aligned}$$

Thus $(S, M)$ is indeed a representation.

Note that the equality $\alpha\alpha^*\alpha s = \alpha s$ implies $\alpha^*\alpha s = s$ for all $s \in S$. Analogously, $s\beta\beta^* = s$ for all $s \in S$. Then we have

$$m \cdot (\alpha s \beta) = m * (\alpha^* \alpha s \beta \beta^*) = m * s \text{ for all } s \in S,\ m \in M.$$

The latter means that $(S_a, M)$ coincides with $(S_a, M; \alpha, \beta)$. That is, the function $\cdot$ defines the required representation $(S, M)$. □

The following example shows that the condition of Theorem 2 that there are $\alpha^*$ and $\beta^*$ in $S^1$, such that $\beta^*\beta\alpha\alpha^*$ is a mididentity in $S$, is essential.

EXAMPLE 1. Let $\mathcal{B} = \langle a, b \mid ba = 1\rangle$ be the bicyclic semigroup. Consider the Cayley representation of $\mathcal{B}$:

$$m * s = ms \text{ for all } m, s \in \mathcal{B}.$$

Then there is no faithful representation $\circ : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ such that

$$m \circ (asb) = m * s = ms \text{ for all } m, s \in \mathcal{B}.$$

Since $\mathcal{B}$ is an inverse semigroup with the identity element, 1, its Cayley representation is faithful. That $ba = 1$ implies $\mathcal{B}_{ba} = \mathcal{B}$. Note that $b^{-1}baa^{-1} = a \cdot ba \cdot b = ab$ is not a mididentity in $\mathcal{B}$, as $1 \cdot ab \cdot 1 = ab \neq 1$.

Assume that there is a faithful representation $\circ : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ such that

$$m \circ (asb) = m * s = ms \text{ for all } m, s \in \mathcal{B}.$$

Then $m \circ (ab) = m * 1 = m$ for all $m \in \mathcal{B}$. So we have

$$m \circ a = (m \circ a) \circ (ab) = m \circ (a^2 b) = m * a = ma \text{ and}$$
$$m \circ b = \big(m \circ (ab)\big) \circ b = m \circ (ab^2) = m * b = mb \text{ for all } m \in \mathcal{B}.$$

But this leads to a contradiction:

$$1 = 1 \circ (ab) = (1 \circ a) \circ b = a \circ b = ab.$$

Let us recall (see [**1**, Corollary 1.32]) that every homomorphic image of $\mathcal{B}$ is either a cyclic group or isomorphic to $\mathcal{B}$. Using this fact, we are going to prove that in the case when $S$ is a regular semigroup, which does not contain a subsemigroup, isomorphic to $\mathcal{B}$, one obtains that the conditions 1 and 2 of Theorem 2 are equivalent.

PROPOSITION 1. *Let $S$ be a regular semigroup, which does not contain a subsemigroup isomorphic to $\mathcal{B}$. Let $a = \beta\alpha \in S$, $\alpha, \beta \in S^1$. Let also $* : M \times S_a \to M$ be a faithful representation of $S_a$ on $M$. Then the conditions 1 and 2 of Theorem 2 are equivalent.*

PROOF: We have only to prove that 2 implies 1.

Let now $\alpha$ and $\beta$ be left and right cancellable respectively. Take $\alpha^*$ and $\beta^*$, inverses of $\alpha$ and $\beta$ in $S^1$ respectively. Then the equality $\alpha\alpha^*\alpha s = \alpha s$ implies $\alpha^*\alpha s = s$ for all $s \in S$. Also $s\beta\beta^*\beta = s\beta$ implies $s\beta\beta^* = s$ for all $s \in S$. It follows that $\alpha^*\alpha$ and $\beta\beta^*$ are left and right identity elements of $S^1$ respectively. Consider three possible cases.

CASE 1. Let $\alpha, \beta \in S$. Then $\alpha^*\alpha$ and $\beta\beta^*$ are both in $S$. It follows that $S$ has the identity $1 = \alpha^*\alpha = \beta\beta^*$. Assume that $\alpha\alpha^* \neq 1$. Then $\langle \alpha, \alpha^*\rangle$ is a homomorphic image of $\mathcal{B} = \langle a, b \mid ba = 1\rangle$ under a map, which sends $a$ to $\alpha$ and $b$ to $\alpha^*$. Since $\langle \alpha, \alpha^*\rangle$ is not isomorphic to a cyclic group, then $\langle \alpha, \alpha^*\rangle$ is a subsemigroup, isomorphic to $\mathcal{B}$. We get a contradiction. Thus $\alpha\alpha^* = 1$. Analogously, one shows that $\beta^*\beta = 1$. Then $\beta^*\beta\alpha\alpha^*$ is a mididentity in $S$. Now due to Theorem 2, we have that 1 holds.

CASE 2. Let $\beta \notin S$ and $\alpha = a \in S$. Arguments are similar to those of Case 1. Then $\alpha\alpha^* = \alpha^*\alpha$ is again a mididentity in $S$ and to prove that 1 holds, it remains to use Theorem 2.

CASE 3. Let $\alpha \notin S$ and $\beta = a \in S$. Analogously to Case 2, $\beta^*\beta = \beta\beta^*$ is a mididentity in $S$. Thus, according to Theorem 2, 1 holds. □

Recall (see [**1**]) that a *right* [*left*] *group* is a semigroup which is right [left] simple and left [right] cancellative. We shall need the following known fact.

**THEOREM 3.** ([**1**, Theorem 1.27].) *The following conditions for a semigroup $S$ are equivalent:*

1. *$S$ is a left [right] group;*
2. *$S$ is right [left] simple and contains an idempotent;*
3. *$S$ is a direct product $G \times U$ of a group $G$ and a left [right] zero semigroup $U$.*

Among all the decompositions $a = \beta\alpha$, $\alpha, \beta \in S^1$, two are rather special, namely $a = a \cdot 1$ and $a = 1 \cdot a$. The main result of the following theorem is the characterisation of semigroups $S$ such that for all $a \in S$ and for each representation $* : M \times S_a \to M$ there exists a representation $\cdot : M \times S \to M$ such that either $m * s = m \cdot (as)$ for all $m \in M$ or $m * s = m \cdot (sa)$ for all $m \in M$.

**THEOREM 4.** *Let $S$ be a regular semigroup with the set of idempotents $E$. Then the following conditions are equivalent:*

1. *for all $a \in S$ and for each representation $* : M \times S_a \to M$ there exists a representation $\cdot : M \times S \to M$ such that either $m * s = m \cdot (as)$ for all $m \in M$ or $m * s = m \cdot (sa)$ for all $m \in M$;*
2. *for all $e \in E$ there are a faithful representation $* : M \times S_e \to M$ and a representation $\cdot : M \times S \to M$ such that either $m * s = m \cdot (es)$ for all $m \in M$ or $m * s = m \cdot (se)$ for all $m \in M$;*
3. *every element of $E$ is either a left or a right identity in $S$;*
4. *$S$ is either a left or a right group.*

PROOF: Obviously, 1 implies 2.

Assume that 2 holds. Take an idempotent $e \in E$. The first case is that there are a set $M$, a faithful representation $* : M \times S_e \to M$ and a representation $\cdot : M \times S \to M$ such that $m * s = m \cdot (es)$ for all $m \in M$. Then $m * s = m * (es)$ for all $m \in M$ and $s \in S$ and so $s = es$ for all $s \in S$. In the second case we obtain $s = se$ for all $s \in S$. Thus, 2 implies 3.

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Let us prove now that 3 implies 4 . Note that $E$ is nonempty as $S$ is regular. If $S$ contains both left and right identities then it contains the identity 1, thus each idempotent coincides with 1. Hence, $S$ is a regular semigroup with the unique idempotent 1. It follows now from [**7**, Lemma II.2.10] that $S$ is a group and so also a left group. If every idempotent $e \in E$ is a left [right] unit in $S$, then

$$sS^1 = sS = ss'S = S[S^1 s = Ss = Ss's = S]$$

for all $s \in S$ and for each $s'$, inverse of $s$, and so $S$ is right [left] simple. But due to Theorem 3 we obtain now that $S$ is a right [left] group.

Finally, assume that $S$ is, for instance, a left group. Then by Theorem 3 there are a group $G$ and a left zero semigroup $U$ such that $S = G \times U$. Take $(g,u) \in S$ and a representation $* : M \times S_{(g,u)} \to M$. Put $m \cdot (h,v) = m * (hg^{-1}, v)$ for all $(h,v) \in S$. Then $m \cdot \big((h,v)(g,u)\big) = m * (h,v)$ for all $(h,v) \in S$. We are left to prove that $\cdot$ is a representation of $S$. Indeed, we have

$$\begin{aligned} m \cdot (h_1h_2, v_1v_2) &= m * (h_1h_2g^{-1}, v_1v_2) = m * \big((h_1g^{-1}, v_1) * (h_2g^{-1}, v_2)\big) \\ &= \big(m \cdot (h_1, v_1)\big) \cdot (h_2, v_2) \text{ for all } (h_1,v_1),(h_2,v_2) \in S,\ m \in M. \end{aligned}$$

This completes the proof. ∎

Let $S$ be a semigroup and $\cdot : M \times S \to M$ a representation of $S$. Denote by $M \cdot \alpha$ the set $\{m \cdot \alpha : m \in M\}$ and by $m \cdot S$ the set $\{m \cdot s : s \in S\}$. In the case when $S \neq S^1$ set $M \cdot 1 = M$. A representation $(S, M)$ is said to be *cyclic* (see [**1**]) if there is a *generating* element $m \in M$ for $(S, M)$, that is, $m \cdot S = M$.

**Proposition 2.** *Let $S$ be an arbitrary semigroup and $a \in S$, $a = \beta\alpha$, $\alpha, \beta \in S^1$. Let $(S, M)$ be a representation of $S$ on $M$. The following conditions are equivalent:*

1. *$(S_a, M; \alpha, \beta)$ is cyclic;*
2. *$(S, M)$ is cyclic, $M \cdot \beta = M$ and $M \cdot \alpha$ contains a generating element for $(S, M)$.*

Proof: Suppose that 1 holds. It follows immediately from the definition of cyclic representations that $(S, M)$ is cyclic. If $m \in M$ is a generating element for $(S_a, M; \alpha, \beta)$ then $m \cdot (\alpha S \beta) = M$ which implies $M \cdot \beta = M$. Since

$$(m \cdot \alpha) \cdot S \supseteq m \cdot (\alpha S \beta) = M,$$

we have that $M \cdot \alpha$ contains a generating element for $(S, M)$.

Suppose now that 2 holds. Let $m \in M \cdot \alpha$ be a generating element for $(S, M)$ such that $m = m_0\alpha$ for some $m_0 \in M$. Let us prove that $m_0$ is a generating element for $(S_a, M; \alpha, \beta)$. Indeed, take $m_1 \in M$. Then there exists $\overline{m} \in M$ such that $m_1 = \overline{m} \cdot \beta$. Also there is $s \in S$ such that $\overline{m} = (m_0\alpha) \cdot s$. Then we obtain $m_0 * s = m_0 \cdot (\alpha s \beta) = \big((m_0\alpha) \cdot s\big) \cdot \beta = m_1$. Thus $m_0 * S = M$. ∎

## 3. Congruences $\rho_{b,c}$

Let $\rho$ be a congruence on a semigroup $S$ and $a \in S$, $a = cb$, $b, c \in S^1$. Define a relation $\rho_{b,c}$ on $S_a$ as follows:

$$(s,t) \in \rho_{b,c} \text{ if and only if } (bsc, btc) \in \rho \text{ for all } s,t \in S.$$

It is straightforward that $\rho_{b,c}$ is a congruence on $S_a$. If now we have a representation $\cdot : M \times S \to M$ of $S$ on $M$, then the congruence $\nu$, related to it, is given by

$$(s,t) \in \nu \text{ if and only if } m \cdot s = m \cdot t \text{ for all } m \in M.$$

Then one can easily see that the congruence on $S_a$, related to the representation $(S_a, M; b, c)$, coincides with $\nu_{b,c}$.

Denote by $\mathrm{Cong}(S)$ the set of all congruences on a semigroup $S$. Set $\rho_{1,1} = \rho$ for all $\rho \in \mathrm{Cong}(S)$. In the case when $S \neq S^1$, set also $S_1 = S$.

**Proposition 3.** *Let $S$ be a semigroup, $b, c \in S^1$, $\rho \in \mathrm{Cong}(S)$. Then*

$$S_{cb}/\rho_{b,c} \cong bSc/\rho \cap (bSc \times bSc).$$

Proof: Define a map $\varphi : S_{cb} \to bSc/\rho \cap (bSc \times bSc)$ as follows:

$$\varphi(x) = (bxc)\rho \text{ for all } x \in S.$$

One can easily show that $\varphi$ is an onto homomorphism and $\varphi \circ \varphi^{-1} = \rho_{b,c}$. These two facts complete the proof. ∎

Let $S \neq S^1$ and $\rho \in \mathrm{Cong}(S)$. Then we identify $(1)\rho$ with the identity element of $(S/\rho)^1$.

**Theorem 5.** *Let $S$ be a semigroup, $b, c, b_1, c_1 \in S^1$, $\rho, \sigma \in \mathrm{Cong}(S)$. Then*

1. *if $b\rho$ and $b_1\rho$ are $\mathcal{L}$-related, $c\rho$ and $c_1\rho$ are $\mathcal{R}$-related in $(S/\rho)^1$, then $\rho_{b,c} = \rho_{b_1,c_1}$;*
2. *$\rho \subseteq \rho_{b,c}$;*
3. *$\rho_{b,c} = \rho$ if and only if $b\rho$ and $c\rho$ are left and right cancellable in $(S/\rho)^1$ respectively;*
4. *if $\rho \subseteq \sigma$ then $\rho_{b,c} \subseteq \sigma_{b,c}$. If $b\sigma$ and $c\sigma$ are left and right cancellable in $(S/\sigma)^1$ then $\rho \subseteq \sigma$ if and only if $\rho_{b,c} \subseteq \sigma_{b,c}$.*

Proof: Statements 1 and 2 follow immediately from the definition of $\rho_{b,c}$. Statement 3 follows from Lemma 1 and the fact that $\rho_{b,c} = \rho$ is equivalent to the implication

$$(b\rho)x(c\rho) = (b\rho)y(c\rho) \Rightarrow [x = y] \text{ for all } x, y \in S/\rho.$$

Finally, let us prove 4. If $\rho \subseteq \sigma$ then $(x,y) \in \rho_{b,c}$ implies $(bxc, byc) \in \rho \subseteq \sigma$ which, in turn, implies $(x,y) \in \sigma_{b,c}$. Thus, if $\rho \subseteq \sigma$, then $\rho_{b,c} \subseteq \sigma_{b,c}$.

Suppose now that $b\sigma$ and $c\sigma$ are left and right cancellable in $(S/\sigma)^1$ and $\rho_{b,c} \subseteq \sigma_{b,c}$. Take $(x,y) \in \rho$. Then $(bxc, byc) \in \rho$ which implies $(x,y) \in \rho_{b,c} \subseteq \sigma_{b,c}$ or just $(b\sigma)(x\sigma)(c\sigma) = (b\sigma)(y\sigma)(c\sigma)$, we have $x\sigma = y\sigma$. Thus, we obtain that $\rho \subseteq \sigma$. This completes the proof. ∎

The converse statement of 1 of Theorem 5 is not true in general as the following easy example shows.

EXAMPLE 2. Consider a semilattice $E = \{a, b\}$, where $a \leqslant b$. Let $\rho$ be the identity relation on $E$. Then $\rho_{a,a} = \rho_{a,b}$ but $(a,b) \notin \mathcal{R}$.

Set $\rho_a^r = \rho_{1,a}$ and $\rho_a^l = \rho_{a,1}$ for all congruences $\rho$ on a semigroup $S$ and $a \in S^1$. The following proposition shows that the converse statement of 1 of Theorem 5 is true in the case when $b = b_1 = 1$ $[c = c_1 = 1]$ and $S/\rho$ is inverse.

**PROPOSITION 4.** *Let $S$ be a semigroup, $b, c \in S^1$, $\rho \in \mathrm{Cong}(S)$. Suppose that $S/\rho$ is inverse. Then $\rho_b^r = \rho_c^r$ $[\rho_b^l = \rho_c^l]$ if and only if $b\rho$ and $c\rho$ are $\mathcal{R}$-related $[\mathcal{L}$-related$]$ in $(S/\rho)^1$.*

PROOF: The sufficiency follows from 1 of Theorem 5. Let now assume that $\rho_b^r = \rho_c^r$. Set $b\rho = u$ and $c\rho = v$. Then we have $xu = yu$ if and only if $xv = yv$ for all $x, y \in S/\rho$. In particular, $xu = xuu^{-1}u$ gives us $xv = xuu^{-1}v$ for all $x \in S/\rho$. Hence, $v = vv^{-1}v = vv^{-1}uu^{-1}v = uu^{-1}v$, thus $vv^{-1} = uu^{-1}vv^{-1}$. Analogously, $uu^{-1} = vv^{-1}uu^{-1}$. Thus, $vv^{-1} = uu^{-1}$ which is well-known to be equivalent to $u\mathcal{R}v$ (see [4]). ∎

Take $b, c \in S^1$. Set a map $\varphi_{b,c} : \mathrm{Cong}(S) \to \mathrm{Cong}(S_{cb})$ as follows:

$$\varphi_{b,c}(\rho) = \rho_{b,c} \text{ for all } \rho \in \mathrm{Cong}(S).$$

It is well-known that if one has $\rho, \sigma \in \mathrm{Cong}(S)$ then $\rho \vee \sigma$ coincides with the transitive closure $(\rho \cup \sigma)^t$. Denote by $\mathcal{SL}(S)$ the lower semilattice $(\mathrm{Cong}(S); \subseteq, \cap)$ of congruences on $S$ and by $\mathcal{L}(S)$ the lattice $(\mathrm{Cong}(S); \subseteq, \cap, \vee)$ of congruences on $S$.

**THEOREM 6.** *Let $S$ be a semigroup and $b, c \in S^1$. Then*

1. *$\varphi_{b,c}$ is a homomorphism from $\mathcal{SL}(S)$ to $\mathcal{SL}(S_{cb})$;*
2. *$\rho_{b,c} \vee \sigma_{b,c} \subseteq (\rho \vee \sigma)_{b,c}$ for all $\rho, \sigma \in \mathrm{Cong}(S)$;*
3. *if $b$ and $c$ are regular in $S^1$ then $\varphi_{b,c}$ is a homomorphism from $\mathcal{L}(S)$ to $\mathcal{L}(S_{cb})$;*
4. *if $bSc = S$ then $\varphi_{b,c}$ is injective homomorphism from $\mathcal{L}(S)$ to $\mathcal{L}(S_{cb})$.*

PROOF: Take $\rho, \sigma \in \mathrm{Cong}(S)$. Then $(x,y) \in \rho_{b,c} \cap \sigma_{b,c}$ if and only if $(bxc, byc) \in \rho \cap \sigma$, which is, in turn, equivalent to $(x,y) \in (\rho \cap \sigma)_{b,c}$ for all $x, y \in S$. Thus, $\rho_{b,c} \cap \sigma_{b,c} = (\rho \cap \sigma)_{b,c}$, which completes the proof of 1.

To prove 2, we note that due to $\rho_{b,c} \subseteq (\rho \vee \sigma)_{b,c}$ and $\sigma_{b,c} \subseteq (\rho \vee \sigma)_{b,c}$, we have that $\rho_{b,c} \vee \sigma_{b,c} \subseteq (\rho \vee \sigma)_{b,c}$.

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Let now $b$ and $c$ be regular in $S^1$ and $b'$, $c'$ be certain inverses of $b$ and $c$ in $S^1$ respectively. To prove 3, in view of what has already been done, we are left to show that $(\rho \vee \sigma)_{b,c} \subseteq \rho_{b,c} \vee \sigma_{b,c}$. Suppose that $(x, y) \in (\rho \vee \sigma)_{b,c}$. Then $(bxc, byc) \in \rho \vee \sigma$ and so there are $p_1, \ldots, p_m \in S$ such that

$$(1) \qquad (bxc, p_1) \in \rho \cup \sigma, \ldots, (p_i, p_{i+1}) \in \rho \cup \sigma, \ldots, (p_m, byc) \in \rho \cup \sigma.$$

Clearly, $\rho \cup \sigma$ is left and right compatible, and therefore

$$(bxc, b \cdot b'p_1c' \cdot c) \in \rho \cup \sigma, \ldots, \\ (b \cdot b'p_ic' \cdot c, b \cdot b'p_{i+1}c' \cdot c) \in \rho \cup \sigma, \ldots, (b \cdot b'p_mc' \cdot c, byc) \in \rho \cup \sigma,$$

which yields

$$(x, b'p_1c') \in \rho_{b,c} \cup \sigma_{b,c}, \ldots, (b'p_ic', b'p_{i+1}c') \in \rho_{b,c} \cup \sigma_{b,c}, \ldots, (b'p_mc', y) \in \rho_{b,c} \cup \sigma_{b,c},$$

then $(x, y) \in \rho_{b,c} \vee \sigma_{b,c}$. Thus, 3 is proved.

Finally, assume that $bSc = S$. To prove that $\varphi_{b,c}$ is a homomorphism, it is enough to show that $(\rho \vee \sigma)_{b,c} \subseteq \rho_{b,c} \vee \sigma_{b,c}$. Take $(x, y) \in (\rho \vee \sigma)_{b,c}$. Then there are $p_1, \ldots, p_m \in S$ such that (1) holds. Then due to $bSc = S$ and the fact that $(bsc, btc) \in \rho \cup \sigma$ if and only if $(s, t) \in \rho_{b,c} \cup \sigma_{b,c}$ for all $s, t \in S$, we obtain that $(x, y) \in \rho_{b,c} \vee \sigma_{b,c}$. It remains to prove that $\varphi_{b,c}$ is injective. Suppose that $\rho_{b,c} = \sigma_{b,c}$. Take $(s, t) \in \rho$. There are $s_1$ and $t_1$ of $S$ such that $s = bs_1c$ and $t = bt_1c$. Then $(s_1, t_1) \in \rho_{b,c}$ which implies $(s_1, t_1) \in \sigma_{b,c}$, which, in turn, is equivalent to $(s, t) \in \sigma$. Thus, $\rho \subseteq \sigma$. Analogously, $\sigma \subseteq \rho$. So $\rho = \sigma$ and $\varphi_{b,c}$ is an injective homomorphism. □

We note that the converse inclusion of one in 2 of Theorem 6, namely $(\rho \vee \sigma)_{b,c} \subseteq \rho_{b,c} \vee \sigma_{b,c}$ for $\rho, \sigma \in \mathrm{Cong}(S)$, is not true in general as the following example illustrates.

EXAMPLE 3. Consider the free semigroup $\{a, b\}^+$ over the alphabet $\{a, b\}$. Let $I$ be the ideal consisting of all words from $\{a, b\}^+$ of length not less than 3. Set $S = \{a, b\}^+/I$ to be the Rees quotient semigroup.

Set $\rho = (ba, ab) \cup (ab, ba) \cup \iota$ and $\sigma = (ab, bb) \cup (bb, ab) \cup \iota$, where $\iota$ denotes the identity relation on $S$. It follows immediately from the construction of $S$ that $\rho, \sigma \in \mathrm{Cong}(S)$. Now we observe that $(a, b) \in (\rho \vee \sigma)_{b,1}$. Indeed, we have $(ba, ab) \in \rho$ and $(ab, bb) \in \sigma$, thus $(ba, bb) \in \rho \vee \sigma$ which is equivalent to $(a, b) \in (\rho \vee \sigma)_{b,1}$. But $(a, b) \notin \rho_{b,1} \vee \sigma_{b,1}$. Indeed, otherwise there would exist $t_1, \ldots, t_n \in S$ such that

$$(ba, bt_1) \in \rho \cup \sigma, \ldots, (bt_i, bt_{i+1}) \in \rho \cup \sigma, \ldots, (bt_n, bb) \in \rho \cup \sigma.$$

It follows from the construction of $\rho$ and $\sigma$ that $(ba, bt_1) \in \rho \cup \sigma$ implies $t_1 = a$. Now inductive arguments show that $t_i = a$ for all possible $i$. In particular, $t_n = a$. Then $(ba, bb) \in \rho \cup \sigma$, and we get a contradiction. Thus, $(\rho \vee \sigma)_{b,1} \not\subseteq \rho_{b,1} \vee \sigma_{b,1}$.

## 4. Congruences $\rho_a$

Let now $\rho$ be a congruence on a semigroup $S$ and $a \in S^1$. Then we can construct a congruence $\rho_a \in \mathrm{Cong}(S_a)$ as follows:

$$(s,t) \in \rho_a \text{ if and only if } (asa, ata) \in \rho \text{ for all } s,t \in S.$$

Thus, in terms of [3], $(s,t) \in \rho_a$ if and only if $(s\rho, t\rho) \in \delta^{a\rho}$.

If now one has $a = cb$, $b, c \in S^1$, then $\rho_{b,c} \subseteq \rho_a$. The following statement shows when the opposite inclusion holds in the case when $S/\rho$ is inverse.

**Proposition 5.** *Let $S$ be a semigroup, $b, c \in S^1$. Let also $\rho$ be a congruence on $S$ such that $S/\rho$ is inverse. Then $\rho_{b,c} = \rho_{cb}$ if and only if*

$$uxv = v^{-1}vu \cdot x \cdot vuu^{-1} \text{ for all } x \in S/\rho, \tag{2}$$

*where $u = b\rho$ and $v = c\rho$.*

Proof: The condition $\rho_{b,c} = \rho_{cb}$ is equivalent to $\rho_{cb} \subseteq \rho_{b,c}$, which is, in turn, equivalent to

$$v \cdot uxv \cdot u = v \cdot uyv \cdot u \Rightarrow uxv = uyv \text{ for all } x,y \in S/\rho. \tag{3}$$

Let us prove that the condition (3) is equivalent to (2).

Indeed, assume that (3) holds. Since

$$vu \cdot x \cdot vu = vu \cdot u^{-1}v^{-1}vu \cdot x \cdot vuu^{-1}v^{-1} \cdot vu \text{ for all } x \in S/\rho,$$

we have

$$u \cdot x \cdot v = u \cdot u^{-1}v^{-1}vu \cdot x \cdot vuu^{-1}v^{-1} \cdot v = v^{-1}vu \cdot x \cdot vuu^{-1},$$

due to the fact that all the idempotents of an inverse semigroup pairwise commute. That is, the condition (2) holds.

Assume now that (2) holds. Suppose that $v \cdot uxv \cdot u = v \cdot uyv \cdot u$ for some $x, y \in S/\rho$. Then

$$uxv = v^{-1}vu \cdot x \cdot vuu^{-1} = v^{-1}vu \cdot y \cdot vuu^{-1} = uyv,$$

then we obtain (3). $\square$

**Proposition 6.** *Let $S$ be a semigroup, $a \in S^1$, $\rho \in \mathrm{Cong}(S)$. Then*

$$S_a/\rho_a \cong aSa/\rho \cap (aSa \times aSa).$$

Proof: Set a map $\psi : S_a \to aSa/\rho \cap (aSa \times aSa)$ as follows:

$$\psi(x) = (axa)\rho \text{ for all } x \in S.$$

It remains to note that $\psi$ is an onto homomorphism and $\psi \circ \psi^{-1} = \rho_a$. $\square$

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