J. Appl. Probab. **61,** 1107–1126 (2024)
*doi:*10.1017/jpr.2024.15

BRANCHING PROCESSES IN NEARLY DEGENERATE VARYING ENVIRONMENT

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Abstract

We investigate branching processes in varying environment, for which $\bar{f}_n \to 1$ and $\sum_{n=1}^{\infty} (1-\bar{f}_n)_+ = \infty$, $\sum_{n=1}^{\infty} (\bar{f}_n - 1)_+ < \infty$, where $\bar{f}_n$ stands for the offspring mean in generation n. Since subcritical regimes dominate, such processes die out almost surely, therefore to obtain a nontrivial limit we consider two scenarios: conditioning on nonextinction, and adding immigration. In both cases we show that the process converges in distribution without normalization to a nondegenerate compound-Poisson limit law. The proofs rely on the shape function technique, worked out by Kersting (2020).

Keywords: Branching process in varying environment; Yaglom-type limit theorem; immigration; shape function

2020 Mathematics Subject Classification: Primary 60J80
Secondary 60F05

1. Introduction

A Galton–Watson branching process in varying environment (BPVE) $(X_n,\ n \geq 0)$ is defined as

$$X_0 = 1, \qquad X_n = \sum_{j=1}^{X_{n-1}} \xi_{n,j}, \quad n \in \mathbb{N} = \{1, 2, \ldots\}, \tag{1}$$

where $\{\xi_{n,j}\}_{n,j\in\mathbb{N}}$ are nonnegative independent random variables such that, for each n, $\{\xi_{n,j}\}_{j\in\mathbb{N}}$ are identically distributed; let ξ_n denote a generic copy. We can interpret X_n as the size of the nth generation of a population, and $\xi_{n,j}$ represents the number of offspring produced by the jth individual in generation $n-1$. These processes are natural extensions of usual homogeneous Galton–Watson processes, where the offspring distribution does not change with the generation.

The investigation of BPVE started in the early 1970s [4, 6]. There has been recent increasing interest in these processes, which was triggered by Kersting [15], who obtained a necessary and sufficient condition for almost-sure extinction for regular processes. In [15] he introduced the shape function of a generating function (g.f.), which turned out to be the appropriate tool for the analysis. Moreover, in [15] Yaglom-type limit theorems (i.e. conditioned on nonextinction) were obtained. In [2], under different regularity conditions, Yaglom-type limit theorems

Received 2 November 2022; accepted 23 January 2024.
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were proved in both discrete- and continuous-time settings, extending the results of [14]. For multitype processes these questions were investigated in [5]. In [3], the authors obtained probabilistic proofs of the results in [15] using spine decomposition. On the general theory of BPVE we refer to the recent monograph [16], and to the references therein.

Here we are interested in *branching processes in nearly degenerate varying environment*. Let $f_n(s) = \mathbb{E}s^{\xi_n}$, $s \in [0, 1]$, denote the g.f. of the offspring distribution of the nth generation, and put $\bar{f}_n := f_n'(1) = \mathbb{E}\xi_n$ for the offspring mean. We assume the following conditions:

(C1) $\lim_{n\to\infty} \bar{f}_n = 1$, $\sum_{n=1}^{\infty} (1-\bar{f}_n)_+ = \infty$, $\sum_{n=1}^{\infty} (\bar{f}_n - 1)_+ < \infty$.

(C2) $\lim_{n\to\infty, \bar{f}_n<1} \dfrac{f_n''(1)}{1-\bar{f}_n} = \nu \in [0, \infty)$, and $\dfrac{f_n''(1)}{|1-\bar{f}_n|}$ is bounded.

(C3) If $\nu > 0$, then $\lim_{n\to\infty, \bar{f}_n<1} \dfrac{f_n'''(1)}{1-\bar{f}_n} = 0$, and $\dfrac{f_n'''(1)}{|1-\bar{f}_n|}$ is bounded.

Here and later on, $\lim_{n\to\infty, \bar{f}_n<1}$ means that the convergence holds along the subsequence $\{n : \bar{f}_n < 1\}$, and $a_+ = \max\{a, 0\}$. Condition (C1) means that the process is nearly critical and the convergence $\bar{f}_n \to 1$ is not too fast. Note that we also allow supercritical generations, i.e. with $\bar{f}_n > 1$. However, subcritical regimes dominate since, by (C1), $\mathbb{E}X_n = \prod_{j=1}^{n} \bar{f}_j \to 0$, and thus the process dies out almost surely.

Condition (C2) implies that $f_n''(1) \to 0$, thus $f_n(s) \to s$, so the branching mechanism converges to degenerate branching. Therefore, this nearly critical model does not have a natural homogeneous counterpart. To obtain nontrivial limit theorems we can condition on nonextinction, or add immigration. Note that condition (C2) implies that the offspring distributions have finite second moment, which is assumed throughout the paper.

Conditioned on $X_n > 0$, in our Theorem 1 we prove that X_n converges in distribution without normalization to a geometric distribution. Although the process is nearly critical, in this sense its behavior is similar to a homogeneous subcritical process, where no normalization is needed to obtain a limit; see Yaglom's theorem [21] (or [1, Theorem I.8.1, Corollary I.8.1]). For a regular BPVE (in the sense of [15]), necessary and sufficient conditions were obtained for the tightness of X_n (without normalization) conditioned on $X_n > 0$ in [15, Corollary 2]. However, to the best of our knowledge, our result is the first proper limit theorem without normalization. Similarly to Yaglom's theorem in the homogeneous critical case (see [21] or [1, Theorem I.9.2]), exponential limits for properly normalized processes conditioned on non-extinction were obtained in several papers; see, e.g., [2, 14, 15].

Allowing immigration usually leads to similar behavior to conditioning on nonextinction. A branching process in varying environment with immigration $(Y_n,\ n \geq 0)$ is defined as

$$Y_0 = 0, \qquad Y_n = \sum_{j=1}^{Y_{n-1}} \xi_{n,j} + \varepsilon_n, \quad n \in \mathbb{N}, \tag{2}$$

where $\{\xi_n, \xi_{n,j}, \varepsilon_n\}_{n,j\in\mathbb{N}}$ are nonnegative, independent random variables such that $\{\xi_n, \xi_{n,j}\}_{j\in\mathbb{N}}$ are identically distributed. As before, $\xi_{n,j}$ is the number of offspring of the jth individual in generation $n-1$, and ε_n is the number of immigrants.

The study of nearly degenerate BPVE with immigration was initiated in [12], where it was assumed that the offspring have Bernoulli distribution. The most interesting finding in [12] is that the slow dying out effect can be balanced by a slow immigration rate to obtain a nontrivial distributional limit without normalization. In this case the resulting process is an

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inhomogeneous first-order integer-valued autoregressive process (INAR(1)). INAR processes are important in various fields of applied probability; for the theory and applications we refer to the survey in [20]. The setup of [12] was extended by [17], allowing more general offspring distributions. For INAR(1) processes the condition $\bar{f}_n < 1$ is automatic (being a probability), while in the more general setup of [17] it was assumed. Our conditions (C1) and (C2) do allow supercritical generations, i.e. $\bar{f}_n > 1$, with a dominating subcritical regime. The multitype case was studied in [11].

In general, Galton–Watson processes with immigration in varying environment (i.e. with time-dependent immigration) are less studied. In [8], the central limit theorem and law of the iterated logarithm were proved. In [13], a diffusion approximation was obtained in the strongly critical case, i.e. when, instead of (C1), the condition $\sum_{n=1}^{\infty} |1-\bar{f}_n| < \infty$ holds. [9] investigated almost-sure extinction, and obtained limit theorems for the properly normalized process. Since we prove limit theorems for the process without normalization, both our results and assumptions are rather different from those in the papers mentioned.

Generalizing the results in [17], in Theorems 2 and 3 we show that, under appropriate assumptions on the immigration, the slow extinction trend and the slow immigration trend are balanced and we get a nontrivial compound-Poisson limit distribution without normalization.

The rest of the paper is organized as follows. Section 2 contains the main results. All the proofs are gathered together in Section 3. The proofs are rather technical, and based on analysis of the composite g.f. We rely on the shape function technique worked out in [15].

2. Main results

2.1. Yaglom-type limit theorems

Consider the BPVE (X_n) in (1). Condition (C1) implies that the process dies out almost surely. In [15] a BPVE process (X_n) is called *regular* if

$$\mathbb{E}[\xi_n^2 \mathbf{1}(\xi_n \geq 2)] \leq c\mathbb{E}[\xi_n \mathbf{1}(\xi_n \geq 2)]\mathbb{E}[\xi_n \mid \xi_n \geq 1]$$

holds for some $c > 0$ and for all n, where $\mathbf{1}$ stands for the indicator function. In our setup, if (C1)–(C3) are satisfied with $\nu > 0$ then the process is regular, while if $\nu = 0$ there are nonregular examples (e.g. $\mathbb{P}(\xi_n = 0) = n^{-1}$, $\mathbb{P}(\xi_n = 1) = 1 - n^{-1} - n^{-4}$, $\mathbb{P}(\xi_n = n) = n^{-4}$ works). For regular processes the results in [15] apply. According to Kersting's classification of *regular* BPVEs [15, Proposition 1], a regular process satisfying (C1)–(C3) is subcritical. Indeed, $\mathbb{E}X_n \to 0$ and, by Lemma 2 and (C2),

$$\lim_{n\to\infty} \bar{f}_{0,n} \sum_{k=1}^{n} \frac{f_k''(1)}{\bar{f}_{0,k-1}\bar{f}_k^2} = \nu.$$

Furthermore, [15, Corollary 2] states that the sequence $\mathcal{L}(X_n \mid X_n > 0)$ is tight if and only if $\sup_{n\geq 0} \mathbb{E}[X_n \mid X_n > 0] < \infty$. Here, $\mathcal{L}(X_n \mid X_n > 0)$ stands for the law of X_n conditioned on nonextinction. In Lemma 4 we show that the latter condition holds; in fact the limit exists. Therefore, the sequence of conditional laws is tight. In the next result, we prove that the limit distribution also exists.

The random variable V has geometric distribution with parameter $p \in (0, 1]$, $V \sim \mathrm{Geom}(p)$, if $\mathbb{P}(V = k) = (1-p)^{k-1}p$, $k = 1, 2, \ldots$

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Theorem 1. *Assume that (C1)–(C3) are satisfied. Then, for the BPVE (X_n) in (1),*

$$\mathcal{L}(X_n|X_n > 0) \xrightarrow{\mathrm{D}} \mathrm{Geom}\left(\frac{2}{2+\nu}\right)$$

as $n \to \infty$, where $\xrightarrow{\mathrm{D}}$ denotes convergence in distribution.

The result holds with $\nu = 0$, in which case the sequence of random variables converges in probability to 1.

Example 1. Let $f_n(s) = f_n[0] + f_n[1]s + f_n[2]s^2$; then $\bar{f}_n = f_n[1] + 2f_n[2]$, $f_n''(1) = 2f_n[2]$, and $f_n'''(1) = 0$. Assuming that $f_n[0] + f_n[2] \to 0$, $f_n[0] > f_n[2]$, $\sum_{n=1}^{\infty}(f_n[0] - f_n[2]) = \infty$, and $2f_n[2]/(f_n[0] - f_n[2]) \to \nu \in [0, \infty)$ as $n \to \infty$, the conditions of Theorem 1 are fulfilled.

2.2. Nearly degenerate branching processes with immigration

Recall (Y_n), the BPVE with immigration from (2), and introduce the factorial moments of the immigration $m_{n,k} := \mathbb{E}[\varepsilon_n(\varepsilon_n - 1)\cdots(\varepsilon_n - k + 1)]$, $k \in \mathbb{N}$.

Theorem 2. *Assume that (C1)–(C3) are satisfied, and also*

(C4) $\lim_{n\to\infty, \bar{f}_n<1} m_{n,k}/(k!(1-\bar{f}_n)) = \lambda_k$, $k = 1, 2, \ldots, K$, *with* $\lambda_K = 0$, *and for each* $k \le K$ *the sequence* $(m_{n,k}/|1-\bar{f}_n|)_n$ *is bounded.*

Then, for the BPVE with immigration (Y_n) in (2), $Y_n \xrightarrow{\mathrm{D}} Y$ as $n \to \infty$, where the random variable Y has compound-Poisson distribution and, for its g.f. f_Y,

$$\log f_Y(s) = \begin{cases} -\sum_{k=1}^{K-1} \frac{2^k \lambda_k}{\nu^k}\left(\log\left(1 + \frac{\nu}{2}(1-s)\right) + \sum_{i=1}^{k-1} \frac{\nu^i}{i2^i}(s-1)^i\right), & \nu > 0, \\ \sum_{k=1}^{K-1} \frac{\lambda_k}{k}(s-1)^k, & \nu = 0. \end{cases}$$

Note that f_Y is continuous in $\nu \in [0, \infty)$. Under further assumptions we might allow infinitely many nonzero λs.

Theorem 3. *Assume that (C1)–(C3) are satisfied, and*

(C4′) $\lim_{n\to\infty, \bar{f}_n<1} m_{n,k}/(k!(1-\bar{f}_n)) = \lambda_k$, $k = 1, 2, \ldots$, *such that* $\limsup_{n\to\infty} \lambda_n^{1/n} \le 1$, *and* $(m_{n,k}/|1-\bar{f}_n|)_n$ *is bounded for each k.*

Then, for the BPVE with immigration (Y_n) in (2), $Y_n \xrightarrow{\mathrm{D}} Y$ as $n \to \infty$, where the random variable Y has compound-Poisson distribution and, for its g.f. f_Y,

$$\log f_Y(s) = \begin{cases} -\sum_{k=1}^{\infty} \frac{2^k \lambda_k}{\nu^k}\left(\log\left(1 + \frac{\nu}{2}(1-s)\right) + \sum_{i=1}^{k-1} \frac{\nu^i}{i2^i}(s-1)^i\right), & \nu > 0, \\ \sum_{k=1}^{\infty} \frac{\lambda_k}{k}(s-1)^k, & \nu = 0. \end{cases} \tag{3}$$

Remark 1. If $K = 2$ in Theorem 2 then Y has (generalized) negative binomial distribution with parameters $2\lambda_1/\nu$ (not necessarily integer) and $2/(2+\nu)$ as shown in [17, Theorem 5]. In particular, Theorems 2 and 3 are generalizations of [17, Theorem 5].

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If $\nu = 0$ in condition (C2), then the results in Theorems 2 and 3 follow from [17, Theorem 4] when $\bar{f}_n < 1$ for all n. Our results are new even in this special case.

Remark 2. Conditions (C4) and (C4′) mean that the immigration is of the proper order, i.e. $\lim_{n\to\infty, \bar{f}_n<1} \mathbb{P}(\varepsilon_n = j)/(1-\bar{f}_n)$ exists for each $j \geq 1$; see Theorem 4.

Conditions (C4) and (C4′) are not comparable in the sense that neither implies the other. Indeed, if (C4) holds with $K = 3$, then it is possible that fourth moments do not exist, while in (C4′) all moments have to be finite.

We can construct examples for which $\lambda_2 = 0$, $\lambda_3 = 1$, $\lambda_4 = \infty$. In this case (C4) in Theorem 2 holds with $K = 2$. On the other hand, it is easy to show that if $\lambda_{n_1} = \lambda_{n_2} = 0$ for some $n_1 < n_2$, then $\lambda_n = 0$ for $n_1 < n < n_2$.

The g.f. f_Y of the limit has a rather complicated form. In the proof, showing the pointwise convergence of the generating functions, we prove that the accompanying laws converge in distribution to Y. Since the accompanying laws are compound-Poisson, this implies (see, e.g., [19, Proposition 2.2]) that the limit Y is compound-Poisson too. That is, $Y = \sum_{i=1}^{N} Z_i$, where $Z, Z_1, Z_2, \ldots$ are independent and identically distributed nonnegative integer-valued random variables, and independently N has Poisson distribution. This is an important class of distributions, since it is exactly the class of infinitely divisible distributions on the nonnegative integers; see, e.g., [19, Theorem II.3.2]. Interestingly, from the form of the g.f. f_Y it is difficult to deduce that it is compound-Poisson, because the logarithm is given as a power series in $1-s$, while for the compound-Poisson this is a power series in s. Under some conditions on the sequence (λ_n) we can rewrite $\log f_Y$ to a series expansion in s, which allows us to understand the structure of the limit.

Theorem 4. *Assume one of the following:*

(i) the conditions of Theorem 2 hold; or

(ii) the conditions of Theorem 3 hold and $\limsup_{n\to\infty} \lambda_n^{1/n} \leq \frac{1}{2}$.

Then the limiting g.f. in Theorems 2 and 3 can be written as

$$f_Y(s) = \exp\left\{\sum_{n=1}^{\infty} A_n(s^n - 1)\right\},$$

where

$$A_n = \begin{cases} \dfrac{\nu^n}{n(2+\nu)^n} \displaystyle\sum_{j=1}^{\infty} q_j \left[\left(1+\frac{2}{\nu}\right)^{\min(j,n)} - 1\right], & \nu > 0, \\ \dfrac{1}{n} \displaystyle\sum_{j=n}^{\infty} q_j, & \nu = 0, \end{cases}$$

$$q_j = \lim_{n\to\infty, \bar{f}_n<1} \frac{\mathbb{P}(\varepsilon_n = j)}{1-\bar{f}_n}, \quad j = 1, 2, \ldots$$

Remark 3. For $\nu = 0$, the latter formula was obtained in [12, Remark 2]. In fact, the last limit exists without any extra conditions on the $(\lambda_n)_n$ sequence, see Lemma 8. Furthermore, it is clear that under (C4), $q_k = 0$ for $k \geq K$. The form of f_Y also implies that the limit has the representation $Y = \sum_{n=1}^{\infty} nN_n$, where (N_n) are independent Poisson random variables, such that N_n has parameter A_n.

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3. Proofs

3.1. Preparation

In the proofs, we analyze the g.f. of the underlying processes. To prove distributional convergence for nonnegative integer-valued processes, it is enough to prove the pointwise convergence of the g.f., and show that the limit is a g.f. as well [7, p. 280].

Recall that $f_n(s) = \mathbb{E}s^{\xi_n}$ represents the offspring g.f. in generation n. For the composite g.f., introduce the notation $f_{n,n}(s) = s$ and, for $j < n$, $f_{j,n}(s) := f_{j+1} \circ \cdots \circ f_n(s)$; and for the corresponding means $\bar{f}_{n,n} = 1$ and $\bar{f}_{j,n} := \bar{f}_{j+1} \cdots \bar{f}_n$, $j < n$. Then it is well known that $\mathbb{E}s^{X_n} = f_{0,n}(s)$ and $\mathbb{E}X_n = \bar{f}_{0,n}$.

For a g.f. f, with mean $\bar{f}$ and $f''(1) < \infty$, define the *shape function* as

$$\varphi(s) = \frac{1}{1-f(s)} - \frac{1}{\bar{f}(1-s)}, \quad 0 \le s < 1, \qquad \varphi(1) = \frac{f''(1)}{2\,(\bar{f})^2}. \tag{4}$$

Let φ_j be the shape function of f_j. By the definition of $f_{j,n}$,

$$\frac{1}{1-f_{j,n}(s)} = \frac{1}{\bar{f}_{j+1}(1-f_{j+1,n}(s))} + \varphi_{j+1}(f_{j+1,n}(s)).$$

Therefore, iteration gives ([15, Lemma 5], [16, Proposition 1.3])

$$\frac{1}{1-f_{j,n}(s)} = \frac{1}{\bar{f}_{j,n}(1-s)} + \varphi_{j,n}(s), \tag{5}$$

where

$$\varphi_{j,n}(s) := \sum_{k=j+1}^{n} \frac{\varphi_k(f_{k,n}(s))}{\bar{f}_{j,k-1}}. \tag{6}$$

The latter formulas show the usefulness of the shape function. The next statement gives precise upper and lower bounds on the shape function.

Lemma 1. ([15, Lemma 1], [16, Proposition 1.4].) *Assume* $0 < \bar{f} < \infty$, $f''(1) < \infty$, *and let* $\varphi(s)$ *be the shape function of* f. *Then, for* $0 \le s \le 1$, $\frac{1}{2}\varphi(0) \le \varphi(s) \le 2\varphi(1)$.

For further properties of shape functions we refer to [16, Chapter 1] and [15].

We frequently use the following extension of [12, Lemma 5], which is a version of the Silverman–Toeplitz theorem.

Lemma 2. *Let* $(\bar{f}_n)_{n\in\mathbb{N}}$ *be a sequence of positive real numbers satisfying (C1), and define*

$$a_{n,j}^{(k)} = (1-\bar{f}_j) \prod_{i=j+1}^{n} \bar{f}_i^k = (1-\bar{f}_j)\bar{f}_{j,n}^k, \qquad n, j, k \in \mathbb{N},\ j \le n-1,$$

$$a_{n,n}^{(k)} = 1 - \bar{f}_n.$$

If $(x_n)_{n\in\mathbb{N}}$ *is bounded and* $\lim_{n\to\infty, \bar{f}_n<1} x_n = x \in \mathbb{R}$, *then, for all* $k \in \mathbb{N}$,

$$\lim_{n\to\infty} \sum_{j=1}^{n} a_{n,j}^{(k)} x_j = \frac{x}{k}, \qquad \lim_{n\to\infty} \sum_{j=1}^{n} |a_{n,j}^{(k)}| x_j = \frac{x}{k}. \tag{7}$$

Proof. Let us define $A = \{j : \bar{f}_j > 1\}$, $A_n = A \cap \{1, \ldots, n\}$. First we show that the following conditions (similar to those of the Silverman–Toeplitz theorem) hold:

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(i) $\lim_{n\to\infty} a_{n,j}^{(k)} = 0$ for all $j \in \mathbb{N}$;

(ii) $\lim_{n\to\infty} \sum_{j=1}^{n} a_{n,j}^{(k)} = 1/k$;

(iii) $\sup_{n\in\mathbb{N}} \sum_{j=1}^{n} |a_{n,j}^{(k)}| < \infty$;

(iv) $\lim_{n\to\infty} \sum_{j\in A_n} |a_{n,j}^{(k)}| = 0$.

It is easy to see that (i) holds, as, by (C1),

$$|a_{n,j}^{(k)}| = |1-\bar{f}_j| \prod_{\ell=j+1}^{n} \bar{f}_\ell^k \le |1-\bar{f}_j| \exp\left\{-k \sum_{\ell=j+1}^{n} (1-\bar{f}_\ell)\right\} \to 0.$$

Here and later on, any nonspecified limit relation is meant as $n \to \infty$. Since

$$0 \le \bar{f}_{0,n} \le \exp\left\{-\sum_{j=1}^{n} (1-\bar{f}_j)\right\} \to 0,$$

we have

$$\sum_{j=1}^{n} a_{n,j}^{(1)} = \sum_{j=1}^{n} (1-\bar{f}_j)\bar{f}_{j,n} = \sum_{j=1}^{n} (\bar{f}_{j,n} - \bar{f}_{j-1,n}) = 1 - \bar{f}_{0,n} \to 1,$$

which is (ii) for $k = 1$. Furthermore,

$$\sum_{j=1}^{n} |a_{n,j}^{(1)}| = \sum_{j=1}^{n} |1-\bar{f}_j| \cdot \bar{f}_{j,n} = \sum_{j=1}^{n} \left[(1-\bar{f}_j) + 2(1-\bar{f}_j)_-\right]\bar{f}_{j,n} < \infty,$$

i.e. (iii) holds for $k = 1$. Note that $\sup_{j,n} \bar{f}_{j,n} < \infty$ by (C1). To see (iv), we have

$$\sum_{j\in A_n} |a_{n,j}^{(k)}| = \sum_{j=1}^{n} (\bar{f}_j - 1)_+ \bar{f}_{j,n}^k \to 0,$$

since $\lim_{n\to\infty} \bar{f}_{j,n} = 0$, and $\left((\bar{f}_j - 1)_+ \sup_{\ell,n} \bar{f}_{\ell,n}^k\right)_j$ is an integrable majorant, so Lebesgue's dominated convergence theorem applies.

Before proving (ii) and (iii) for $k \ge 2$ we show (7) for $k = 1$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence with the required properties. Then

$$\sum_{j=1}^{n} a_{n,j}^{(1)} x_j - x = \sum_{j=1}^{n} a_{n,j}^{(1)} (x_j - x) + x\left(\sum_{j=1}^{n} a_{n,j}^{(1)} - 1\right),$$

where the second term tends to 0 by (ii). For the first term we have

$$\begin{aligned}\left|\sum_{j=1}^{n} a_{n,j}^{(1)} (x_j - x)\right| &= \left|\sum_{j\in A_n} a_{n,j}^{(1)} (x_j - x) + \sum_{j\notin A_n} a_{n,j}^{(1)} (x_j - x)\right| \\ &\le 2 \sup_k |x_k| \cdot \sum_{j\in A_n} |a_{n,j}^{(1)}| + \sum_{j\notin A_n} |a_{n,j}^{(1)} (x_j - x)|,\end{aligned}$$

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where the first term tends to 0 by (iv), and the second tends to 0 by (i) and (iii). Thus, the first equation in (7) holds for $k=1$. Furthermore, since $a_{j,n}^{(k)} \le 0$ if and only if $j \in A_n$, conditions (i)–(iv) remain true for $|a_{n,j}^{(k)}|$, and thus

$$\lim_{n\to\infty} \sum_{j=1}^{n} |a_{n,j}^{(1)}| x_j = x. \tag{8}$$

Next, we prove (ii) and (iii) for arbitrary $k \ge 2$. By the binomial theorem,

$$1 - \bar{f}_{0,n}^{k} = k \sum_{j=1}^{n} a_{n,j}^{(k)} + \sum_{i=2}^{k} (-1)^{i+1} \binom{k}{i} \sum_{j=1}^{n} (1-\bar{f}_j)^{i-1} a_{n,j}^{(k)}. \tag{9}$$

Moreover, for $i \ge 2$,

$$\sum_{j=1}^{n} |(1-\bar{f}_j)^{i-1} a_{n,j}^{(k)}| = \sum_{j=1}^{n} |1-\bar{f}_j|^{i} \bar{f}_{j,n}^{k} \le (\sup_{\ell,n} \bar{f}_{\ell,n})^{k-1} \sum_{j=1}^{n} |a_{n,j}^{(1)}| |1-\bar{f}_j|^{i-1} \to 0$$

by (8). Thus, all the terms in the second sum on the right-hand side of (9) tend to 0. Since $\bar{f}_{0,n} \to 0$, (ii) follows. Then (iii) follows as for $k=1$. This completes the proof of (ii) and (iii) for $k \ge 2$. Then (7) for $k \ge 2$ follows exactly as for $k=1$. □

Lemma 3. *Let φ_n be the shape function of f_n. Then, under the conditions of Theorem 1 with $\nu > 0$,*

$$\lim_{n\to\infty, \bar{f}_n<1} \sup_{s\in[0,1]} \frac{|\varphi_n(1) - \varphi_n(s)|}{1-\bar{f}_n} = 0,$$

and the sequence $\sup_{s\in[0,1]} |\varphi_n(1) - \varphi_n(s)|/|1-\bar{f}_n|$ *is bounded.*

Proof. To ease the notation we suppress the lower index n. By the Taylor expansion,

$$f(s) = 1 + \bar{f}(s-1) + \frac{1}{2} f''(1)(s-1)^2 + \frac{1}{6} f'''(t)(s-1)^3$$

for some $t \in (s, 1)$. Thus, recalling (4), the shape function can be written in the form

$$\varphi(s) = \frac{\bar{f}(1-s) - 1 + f(s)}{\bar{f}(1-s)(1-f(s))} = \frac{f''(1)}{2\bar{f}^2} \frac{1 - (f'''(t)/(3f''(1)))(1-s)}{1 - (f''(1)/(2\bar{f}))(1-s) + (f'''(t)/(6\bar{f}))(1-s)^2}.$$

Therefore,

$$\frac{\varphi(1) - \varphi(s)}{1-\bar{f}} = \frac{f''(1)}{2\bar{f}^2(1-\bar{f})} \left(1 - \frac{1 - (f'''(t)/(3f''(1)))(1-s)}{1 - (f''(1)/(2\bar{f}))(1-s) + (f'''(t)/(6\bar{f}))(1-s)^2} \right). \tag{10}$$

Using the assumptions and the monotonicity of f''', uniformly in $s \in (0, 1]$,

$$\lim_{n\to\infty, \bar{f}_n<1} \left[\frac{f_n'''(t)}{f_n''(1)} + \frac{f_n''(1)}{\bar{f}_n} + \frac{f_n'''(t)}{\bar{f}_n} \right] = 0;$$

thus, convergence on $\{n : \bar{f}_n < 1\}$ follows.

The boundedness also follows from (10), since if $\bar{f}_n > 1$ then

$$f_n''(1) = \mathbb{E}(\xi_n(\xi_n - 1)) \ge \mathbb{E}(\xi_n - 1) = \bar{f}_n - 1,$$

and $f_n'''(1)/|1-\bar{f}_n|$ is bounded by assumption (C3). □

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3.2. Proof of Theorem 1

Lemma 4. *Under the conditions of Theorem 1, for $s \in [0, 1)$,*

$$\lim_{n\to\infty} \frac{\bar{f}_{0,n}}{1 - f_{0,n}(s)} = \frac{1}{1-s} + \frac{\nu}{2}.$$

Proof. Recalling (5) and (6), we have to show that

$$\bar{f}_{0,n}\varphi_{0,n}(s) = \sum_{j=1}^{n} \bar{f}_{j-1,n}\varphi_j(f_{j,n}(s)) \to \frac{\nu}{2}.$$

First, let $\nu = 0$. Using Lemmas 1 and 2,

$$\bar{f}_{0,n}\varphi_{0,n}(s) \le \sum_{j=1}^{n} \bar{f}_{j,n} \frac{f_j''(1)}{\bar{f}_j} = \sum_{j=1}^{n} |a_{n,j}^{(1)}| \frac{f_j''(1)}{\bar{f}_j|1-\bar{f}_j|} \to 0.$$

If $\bar{f}_j = 1$ then by (C2) necessarily $f_j''(1) = 0$, and $0/0$ in these types of sums are meant as 0.

For $\nu \in (0, \infty)$ write

$$\sum_{j=1}^{n} \bar{f}_{j-1,n}\varphi_j(f_{j,n}(s)) = \sum_{j=1}^{n} \bar{f}_{j-1,n}\varphi_j(1) - \sum_{j=1}^{n} \bar{f}_{j-1,n}(\varphi_j(1) - \varphi_j(f_{j,n}(s))). \tag{11}$$

By Lemma 2, for the first term we have

$$\sum_{j=1}^{n} \bar{f}_{j-1,n}\varphi_j(1) = \frac{1}{2}\sum_{j=1}^{n} a_{n,j}^{(1)} \frac{1}{\bar{f}_j} \frac{f_j''(1)}{1-\bar{f}_j} \to \frac{\nu}{2}. \tag{12}$$

For the second term in (11),

$$\left| \sum_{j=1}^{n} \bar{f}_{j-1,n}(\varphi_j(1) - \varphi_j(f_{j,n}(s))) \right| \le \sum_{j=1}^{n} a_{n,j}^{(1)} \bar{f}_j \sup_{s\in[0,1]} \frac{|\varphi_j(1) - \varphi_j(s)|}{|1-\bar{f}_j|} \to 0$$

according to Lemmas 2 and 3. Combining with (11) and (12), the statement follows. □

Proof of Theorem 1. We prove the convergence of the conditional g.f. For $s \in (0, 1)$ we have, by the previous lemma,

$$\mathbb{E}[s^{X_n} \mid X_n > 0] = \frac{f_{0,n}(s) - f_{0,n}(0)}{1 - f_{0,n}(0)} = 1 - \frac{1 - f_{0,n}(s)}{1 - f_{0,n}(0)} \to \frac{2}{2+\nu} \frac{s}{1 - (\nu/(\nu+2))s},$$

where the limit is the g.f. of the geometric distribution with parameter $2/(2+\nu)$. □

3.3. Proofs of Theorems 2 and 3

We need a strange version of the Riemann approximation, where the points in the partition are not necessarily increasing. It follows immediately from the uniform continuity.

Lemma 5. *Assume that $x_{n,j} \in [0, 1]$, $j = 0, \ldots, n$, $n \in \mathbb{N}$, such that $x_{n,0} = 0$, $x_{n,n} = 1$, $\lim_{n\to\infty} \sup_j |x_{n,j} - x_{n,j-1}| = 0$, and $\sup_n \sum_{j=1}^{n} |x_{n,j} - x_{n,j-1}| < \infty$. If f is continuous on $[0, 1]$ then*

$$\lim_{n\to\infty} \sum_{j=1}^{n} f(u_{n,j})(x_{n,j} - x_{n,j-1}) = \int_0^1 f(x)\,\mathrm{d}x,$$

with $u_{n,j} \in [\min(x_{n,j-1}, x_{n,j}), \max(x_{n,j-1}, x_{n,j})]$.

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We also need a simple lemma on an alternating sum involving binomial coefficients.

Lemma 6. *For $x \in \mathbb{R}$,*

$$\sum_{i=1}^{k-1} \binom{k-1}{i} (-1)^i \frac{(1+x)^i - 1}{i} = \sum_{i=1}^{k-1} (-1)^i \frac{x^i}{i}.$$

Proof. Both sides are polynomials of x, and equality holds for $x = 0$, therefore it is enough to show that the derivatives are equal. Differentiating the left-hand side we obtain

$$\sum_{i=1}^{k-1} \binom{k-1}{i} (-1)^i (1+x)^{i-1} = \frac{1}{1+x}[(1-(1+x))^{k-1} - 1].$$

Differentiating the right-hand side and multiplying by $(1+x)$, the statement follows. □

The next statement is an easy consequence of the Taylor expansion of the g.f.

Lemma 7. ([12, Lemma 6].) *Let ε be a nonnegative integer-valued random variable with factorial moments $m_k := \mathbb{E}[\varepsilon(\varepsilon-1)\cdots(\varepsilon-k+1)]$, $k \in \mathbb{N}$, $m_0 := 1$, and with g.f. $h(s) = \mathbb{E}s^{\varepsilon}$, $s \in [-1, 1]$. If $m_\ell < \infty$ for some $\ell \in \mathbb{N}$ with $|s| \le 1$, then*

$$h(s) = \sum_{k=0}^{\ell-1} \frac{m_k}{k!}(s-1)^k + R_\ell(s), \qquad |R_\ell(s)| \le \frac{m_\ell}{\ell!}|s-1|^\ell.$$

Proof of Theorem 2. Recall that $f_n(s) = \mathbb{E}s^{\xi_n}$, and let $g_n(s) = \mathbb{E}s^{Y_n}$ and $h_n(s) = \mathbb{E}s^{\varepsilon_n}$. Then the branching property gives (see, e.g., [9, Proposition 1]) $g_n(s) = \prod_{j=1}^n h_j(f_{j,n}(s))$. We prove the convergence of the g.f., i.e. $\lim_{n\to\infty} g_n(s) = f_Y(s)$, $s \in [0, 1]$. Fix $s \in [0, 1)$, and introduce $\widehat{g}_n(s) = \prod_{j=1}^n \mathrm{e}^{h_j(f_{j,n}(s))-1}$. Note that $\widehat{g}_n$ is a kind of accompanying law, its distribution is compound-Poisson, and therefore its limit is compound-Poisson too [19, Proposition 2.2]. By the convexity of the g.f.,

$$f_{j,n}(s) \ge 1 + \bar{f}_{j,n}(s-1). \tag{13}$$

If $|a_k|, |b_k| < 1$, $k = 1, \ldots, n$, then

$$\left| \prod_{k=1}^n a_k - \prod_{k=1}^n b_k \right| \le \sum_{k=1}^n |a_k - b_k|.$$

Consequently, using the latter inequality, the inequalities $|\mathrm{e}^u - 1 - u| \le u^2$ for $|u| \le 1$ and $0 \le 1 - h_j(s) \le m_{j,1}(1-s)$, and (13), we have

$$\begin{aligned} |g_n(s) - \widehat{g}_n(s)| &\le \sum_{j=1}^n |\mathrm{e}^{h_j(f_{j,n}(s))-1} - h_j(f_{j,n}(s))| \\ &\le \sum_{j=1}^n (h_j(f_{j,n}(s)) - 1)^2 \le \sum_{j=1}^n \frac{m_{j,1}^2}{|1-\bar{f}_j|} |a_{n,j}^{(2)}| \to 0, \end{aligned} \tag{14}$$

where we used Lemma 2, since

$$\lim_{n\to\infty, \bar{f}_n<1} \frac{m_{n,1}}{1-\bar{f}_n} = \lambda_1 \quad \text{implies} \quad \lim_{n\to\infty, \bar{f}_n<1} \frac{m_{n,1}^2}{1-\bar{f}_n} = 0.$$

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Therefore, we need to show that

$$\lim_{n\to\infty}\sum_{j=1}^{n}(h_j(f_{j,n}(s))-1)=\log f_Y(s). \tag{15}$$

By Lemma 7,

$$h_j(s)=\sum_{k=0}^{K-1}\frac{m_{j,k}}{k!}(s-1)^k+R_{j,K}(s), \tag{16}$$

where $|R_{j,K}(s)|\le(m_{j,K}/K!)(1-s)^K$, and thus

$$\begin{aligned}\sum_{j=1}^{n}(h_j(f_{j,n}(s))-1)&=\sum_{j=1}^{n}\left[\sum_{k=1}^{K-1}\frac{m_{j,k}}{k!}(f_{j,n}(s)-1)^k+R_{j,K}(f_{j,n}(s))\right]\\&=\sum_{k=1}^{K-1}(-1)^k\sum_{j=1}^{n}\frac{m_{j,k}}{k!(1-\bar f_j)}a_{n,j}^{(k)}\left(\frac{1-f_{j,n}(s)}{\bar f_{j,n}}\right)^k+\sum_{j=1}^{n}R_{j,K}(f_{j,n}(s)).\end{aligned} \tag{17}$$

By (13) and Lemma 2,

$$\begin{aligned}\left|\sum_{j=1}^{n}R_{j,K}(f_{j,n}(s))\right|&\le\sum_{j=1}^{n}\frac{m_{j,K}}{K!}|f_{j,n}(s)-1|^K\\&\le\sum_{j=1}^{n}\frac{m_{j,K}}{K!}\bar f_{j,n}^K(1-s)^K=(1-s)^K\sum_{j=1}^{n}\frac{m_{j,K}}{K!|1-\bar f_j|}|a_{n,j}^{(K)}|\to 0.\end{aligned} \tag{18}$$

Up to this point, everything works for $\nu\ge 0$. Now assume that $\nu>0$. Then, by Lemmas 2 and 3,

$$\left|\sum_{i=j+1}^{n}a_{n,i}^{(1)}\bar f_i\frac{\varphi_i(1)-\varphi_i(f_{i,n}(s))}{1-\bar f_i}\right|\le\sup_{k\ge1}\bar f_k\sum_{i=1}^{n}|a_{n,i}^{(1)}|\sup_{t\in[0,1]}\frac{|\varphi_i(1)-\varphi_i(t)|}{|1-\bar f_i|}\to 0, \tag{19}$$

and similarly

$$\left|\sum_{i=j+1}^{n}a_{n,i}^{(1)}\bar f_i\frac{\varphi_i(1)}{1-\bar f_i}-\frac{\nu}{2}\sum_{i=j+1}^{n}a_{n,i}^{(1)}\right|\le\sum_{i=1}^{n}|a_{n,i}^{(1)}|\left|\frac{1}{\bar f_i}\frac{f_i''(1)}{2(1-\bar f_i)}-\frac{\nu}{2}\right|\to 0. \tag{20}$$

Putting

$$\varepsilon_{j,n}=\sum_{i=j+1}^{n}a_{n,i}^{(1)}\bar f_i\frac{\varphi_i(f_{i,n}(s))}{1-\bar f_i}-\frac{\nu}{2}(1-\bar f_{j,n}), \tag{21}$$

by (5) and (6) we have

$$\frac{\bar f_{j,n}}{1-f_{j,n}(s)}=\frac{1}{1-s}+\sum_{i=j+1}^{n}a_{n,i}^{(1)}\bar f_i\frac{\varphi_i(f_{i,n}(s))}{1-\bar f_i}=\frac{1}{1-s}+\frac{\nu}{2}(1-\bar f_{j,n})+\varepsilon_{j,n}. \tag{22}$$

Noting that $\sum_{i=j+1}^{n}a_{n,i}^{(1)}=1-\bar f_{j,n}$, (19), (20), and the triangle inequality imply that, for $\varepsilon_{j,n}$ in (21),

$$\max_{j\le n}|\varepsilon_{j,n}|=\max_{j\le n}\left|\frac{\bar f_{j,n}}{1-f_{j,n}(s)}-\frac{1}{1-s}-\frac{\nu}{2}(1-\bar f_{j,n})\right|\to 0. \tag{23}$$

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The latter further implies that

$$\limsup_{n\to\infty} \max_{j\le n} \frac{1-f_{j,n}(s)}{\bar{f}_{j,n}} \le 1-s, \tag{24}$$

and that for n large enough, by the mean value theorem and (22),

$$\left|\left(\frac{\bar{f}_{j,n}}{1-f_{j,n}(s)}\right)^{-k} - \left(\frac{1}{1-s}+\frac{\nu}{2}(1-\bar{f}_{j,n})\right)^{-k}\right| \le k\varepsilon_{j,n}.$$

Thus, by (24),

$$\begin{aligned}
&\left|\sum_{j=1}^{n} a_{n,j}^{(k)}\left[\frac{m_{j,k}}{k!(1-\bar{f}_j)}\left(\frac{1-f_{j,n}(s)}{\bar{f}_{j,n}}\right)^{k} - \lambda_k\left(\frac{1}{1-s}+\frac{\nu}{2}(1-\bar{f}_{j,n})\right)^{-k}\right]\right| \\
&\quad\le \sum_{j=1}^{n} |a_{n,j}^{(k)}|\left[\left|\frac{m_{j,k}}{k!(1-\bar{f}_j)} - \lambda_k\right|\left(\frac{1-f_{j,n}(s)}{\bar{f}_{j,n}}\right)^{k}\right. \\
&\qquad\qquad \left. + \lambda_k\left|\left(\frac{1-f_{j,n}(s)}{\bar{f}_{j,n}}\right)^{k} - \left(\frac{1}{1-s}+\frac{\nu}{2}(1-\bar{f}_{j,n})\right)^{-k}\right|\right] \\
&\quad\le \sum_{j=1}^{n} |a_{n,j}^{(k)}|\left|\frac{m_{j,k}}{k!(1-\bar{f}_j)} - \lambda_k\right| + \lambda_k \sum_{j=1}^{n} |a_{n,j}^{(k)}| k \max_{j\le n} \varepsilon_{j,n} \to 0,
\end{aligned} \tag{25}$$

where the second inequality holds for n large enough.

Furthermore, by Lemma 5,

$$\lim_{n\to\infty} \sum_{j=1}^{n} a_{n,j}^{(k)}\left(\frac{1}{1-s}+\frac{\nu}{2}(1-\bar{f}_{j,n})\right)^{-k} = \int_0^1 \frac{y^{k-1}}{(1/(1-s)+(\nu/2)(1-y))^k}\,\mathrm{d}y, \tag{26}$$

since the left-hand side is a Riemann approximation of the right-hand side corresponding to the partition $\{\bar{f}_{j,n}\}_{j=-1}^{n}$, with $\bar{f}_{-1,n} := 0$ and $\bar{f}_{j,n} - \bar{f}_{j-1,n} = a_{n,j}^{(1)} \to 0$ uniformly in j according to Lemma 2, while $\bar{f}_{0,n} \to 0$ by (C1). Changing variables $u = \nu(1-s)(1-y)+2$ and using the binomial theorem,

$$\begin{aligned}
&\int_0^1 \frac{y^{k-1}}{(1/(1-s)+(\nu/2)(1-y))^k}\,\mathrm{d}y \\
&\quad= 2^k(1-s)^k \int_0^1 \frac{y^{k-1}}{(2+\nu(1-s)(1-y))^k}\,\mathrm{d}y \\
&\quad= \frac{2^k}{\nu^k}\int_2^{\nu(1-s)+2} \frac{(2+\nu(1-s)-u)^{k-1}}{u^k}\,\mathrm{d}u \\
&\quad= \frac{2^k}{\nu^k}\int_2^{\nu(1-s)+2} \frac{1}{u^k}\sum_{i=0}^{k-1}\left[\binom{k-1}{i}(2+\nu(1-s))^i(-u)^{k-1-i}\right]\mathrm{d}u \\
&\quad= \frac{2^k}{\nu^k}\sum_{i=0}^{k-1}\left[\binom{k-1}{i}(2+\nu(1-s))^i(-1)^{k-1-i}\int_2^{\nu(1-s)+2} u^{-(i+1)}\,\mathrm{d}u\right]
\end{aligned}$$

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$$= (-1)^{k+1}\frac{2^k}{\nu^k}\left(\log\left(1+\frac{\nu}{2}(1-s)\right)+\sum_{i=1}^{k-1}\binom{k-1}{i}(-1)^i\frac{(1+(\nu/2)(1-s))^i-1}{i}\right)$$
$$= (-1)^{k+1}\frac{2^k}{\nu^k}\left(\log\left(1+\frac{\nu}{2}(1-s)\right)+\sum_{i=1}^{k-1}(-1)^i\frac{\nu^i}{i2^i}(1-s)^i\right), \tag{27}$$

where the last equality follows from Lemma 6. Substituting back into (17), by (18), (25), (26), and (27) we obtain (15).

To finish the proof we need to handle the $\nu = 0$ case. Then the calculations are easier. We can still define $\varepsilon_{j,n}$ as in (21), and (20) remains true. Applying Lemma 1, we see that (23) and (24) hold true as well, therefore (25) follows with $\nu = 0$ everywhere. There is no need for the Riemann approximation, (15) follows from Lemma 2. □

Proof of Theorem 3. Here we consider only the $\nu > 0$ case. For $\nu = 0$ the calculations are easier, and follow similarly to the previous proof.

Similarly to the proof of Theorem 2, (14) holds, so it is enough to prove that

$$\sum_{j=1}^{n}(h_j(f_{j,n}(s))-1)\to \log f_Y(s).$$

Fix $s \in (0, 1)$. By Taylor's theorem, for some $\xi \in (0, \nu(1-s)/2)$,

$$\log\left(1+\frac{\nu}{2}(1-s)\right)+\sum_{i=1}^{k-1}(-1)^i\frac{\nu^i}{i2^i}(1-s)^i=\frac{\nu^k(1-s)^k}{2^k}\frac{1}{(1+\xi)^k}\frac{1}{k},$$

so

$$\frac{2^k}{\nu^k}\left|\log\left(1+\frac{\nu}{2}(1-s)\right)+\sum_{i=1}^{k-1}(-1)^i\frac{\nu^i}{i2^i}(1-s)^i\right|\le\frac{(1-s)^k}{k}.$$

Therefore, for any $\varepsilon > 0$ there exists ℓ large enough such that

$$\left|\sum_{k=\ell}^{\infty}\frac{2^k\lambda_k}{\nu^k}\left(\log\left(1+\frac{\nu}{2}(1-s)\right)+\sum_{i=1}^{k-1}(-1)^i\frac{\nu^i}{i2^i}(1-s)^i\right)\right|\le\sum_{k=\ell}^{\infty}\frac{\lambda_k(1-s)^k}{k}<\varepsilon. \tag{28}$$

By Lemma 7, (16) holds with $K = \ell$ and therefore

$$\sum_{j=1}^{n}(h_j(f_{j,n}(s))-1)=\sum_{j=1}^{n}\left[\sum_{k=1}^{\ell-1}\frac{m_{j,k}}{k!}(f_{j,n}(s)-1)^k+R_{j,\ell}(f_{j,n}(s))\right]$$
$$=\sum_{k=1}^{\ell-1}\sum_{j=1}^{n}\frac{m_{j,k}}{k!}(f_{j,n}(s)-1)^k+\sum_{j=1}^{n}R_{j,\ell}(f_{j,n}(s)).$$

Moreover, by (13) and Lemma 2,

$$\left|\sum_{j=1}^{n}R_{j,\ell}(f_{j,n}(s))\right|\le\sum_{j=1}^{n}\frac{m_{j,\ell}}{\ell!}|f_{j,n}(s)-1|^\ell\le\sum_{j=1}^{n}\frac{m_{j,\ell}}{\ell!}\bar{f}_{j,n}^{\ell}(1-s)^\ell\to\frac{(1-s)^\ell}{\ell}\lambda_\ell\le\varepsilon. \tag{29}$$

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Summarizing,

$$\begin{aligned}
&\left|\sum_{j=1}^{n}(h_j(f_{j,n}(s))-1)-\log f_Y(s)\right| \\
&\qquad\le\left|\sum_{k=1}^{\ell-1}\sum_{j=1}^{n}\frac{m_{j,k}}{k!}(f_{j,n}(s)-1)^k\right. \\
&\qquad\quad\left.+\sum_{k=1}^{\ell-1}\frac{2^k\lambda_k}{\nu^k}\left(\log\left(1+\frac{\nu}{2}(1-s)\right)-\sum_{i=1}^{k-1}(-1)^{i+1}\frac{\nu^i}{i2^i}(1-s)^i\right)\right| \\
&\qquad+\left|\sum_{k=\ell}^{\infty}\frac{2^k\lambda_k}{\nu^k}\left(\log\left(1+\frac{\nu}{2}(1-s)\right)-\sum_{i=1}^{k-1}(-1)^{i+1}\frac{\nu^i}{i2^i}(1-s)^i\right)\right| \\
&\qquad+\left|\sum_{j=1}^{n}R_{j,\ell}(f_{j,n}(s))\right|,
\end{aligned}$$

where the first term on the right-hand side converges to 0 by the previous result, while the second and third terms are small for n large by (28) and (29). As $\varepsilon>0$ is arbitrary, the proof is complete. □

3.4. Proof of Theorem 4

Before the proof, we need two auxiliary lemmas.

Lemma 8. *If condition (C4) of Theorem 2 or (C4′) of Theorem 3 hold, then*

$$q_i=\lim_{n\to\infty,\bar f_n<1}\frac{\mathbb{P}(\varepsilon_n=i)}{1-\bar f_n}$$

exists for each $i=1,2,\dots$

Proof. Without loss of generality we assume that $\bar f_n<1$ for each n. Otherwise, simply consider everything on the subsequence $\{n:\bar f_n<1\}$.

Let $h_n[i]=\mathbb{P}(\varepsilon_n=i)$. If (C4) holds, there are only finitely many λs, and the statement follows easily by backward induction. However, if (C4′) holds, a more involved argument is needed, which works in both cases.

The kth moment of a random variable can be expressed in terms of its factorial moments as

$$\mu_{n,k}:=\mathbb{E}\varepsilon_n^k=\sum_{i=1}^{k}\left\{{k \atop i}\right\}m_{n,i},$$

where

$$\left\{{k \atop i}\right\}=\frac{1}{i!}\sum_{j=0}^{i}(-1)^j\binom{i}{j}(i-j)^k$$

denotes the Stirling number of the second kind; on Stirling numbers, see [10, Section 6.1]. Therefore,

$$\sum_{i=1}^{\infty}\frac{i^kh_n[i]}{1-\bar f_n}=\frac{\mu_{n,k}}{1-\bar f_n}=\sum_{i=1}^{k}\left\{{k \atop i}\right\}\frac{m_{n,i}}{1-\bar f_n}\to\sum_{i=1}^{k}\left\{{k \atop i}\right\}i!\lambda_i=:\mu_k. \tag{30}$$

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Hence, the sequence $(h_n[i]/(1-\bar{f}_n))_{n\in\mathbb{N}}$ is bounded in n for all i. Therefore, any subsequence contains a further subsequence (n_ℓ) such that

$$\lim_{\ell\to\infty} \frac{h_{n_\ell}[i]}{1-\bar{f}_{n_\ell}} = q_i \quad \text{for all } i \in \mathbb{N}. \tag{31}$$

To prove the statement we have to show that the sequence (q_i) is unique, it does not depend on the subsequence. Note that the sequence $(q_i)_{i\in\mathbb{N}}$ is not necessarily a probability distribution.

By Fatou's lemma and (30),

$$\mu_k = \lim_{\ell\to\infty} \frac{\mu_{n_\ell,k}}{1-\bar{f}_{n_\ell}} \geq \sum_{i=1}^{\infty} \lim_{\ell\to\infty} \frac{h_{n_\ell}[i]}{1-\bar{f}_{n_\ell}} i^k = \sum_{i=1}^{\infty} q_i i^k. \tag{32}$$

Let $\varepsilon > 0$ and $k \in \mathbb{N}$ be arbitrary. Write $n_\ell = n$, and put $K := \lfloor \mu_{k+1}/\varepsilon \rfloor + 1$, with $\lfloor\cdot\rfloor$ representing the lower integer part. Then

$$\sum_{i=1}^{\infty} \frac{h_n[i]}{1-\bar{f}_n} i^{k+1} \geq \sum_{i=K+1}^{\infty} \frac{h_n[i]}{1-\bar{f}_n} i^k K,$$

and hence, by (30) and the definition of K,

$$\limsup_{n\to\infty} \sum_{i=K+1}^{\infty} \frac{h_n[i]}{1-\bar{f}_n} i^k \leq \varepsilon.$$

Therefore, by (31),

$$\mu_k = \lim_{n\to\infty} \sum_{i=1}^{\infty} \frac{h_n[i]}{1-\bar{f}_n} i^k \leq \limsup_{n\to\infty} \sum_{i=1}^{K} \frac{h_n[i]}{1-\bar{f}_n} i^k + \varepsilon \leq \sum_{i=1}^{\infty} q_i i^k + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, by (32),

$$\mu_k = \sum_{i=1}^{\infty} q_i i^k. \tag{33}$$

Using the Stieltjes moment problem we show that the sequence (μ_k) uniquely determines (q_i). In order to do that, it is enough to show that Carleman's condition [18, Theorem 5.6.6] is fulfilled, i.e. μ_k is not too large. Since $\limsup_{n\to\infty} \lambda_n^{1/n} \leq 1$, for n large, $\lambda_n \leq 2^n$. Furthermore, by trivial upper bounds,

$$\left\{ {k \atop i} \right\} i! = \sum_{\ell=0}^{i} (-1)^\ell \binom{i}{\ell} (i-\ell)^k \leq i^k 2^i,$$

and thus, by (30), for some $C > 0$,

$$\mu_k = \sum_{i=0}^{k} \left\{ {k \atop i} \right\} i! \lambda_i \leq C + \sum_{i=0}^{k} i^k 4^i \leq C(4k)^k \leq k^{2k}$$

for k large enough, showing that Carleman's condition holds. Hence the sequence $(q_i)_{i\in\mathbb{N}}$ is indeed unique, and the proof is complete. □

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Lemma 9. *For any $L \geq 1$, $n \geq 1$, and $x \in \mathbb{R}$,*

$$\sum_{j=1}^{L}\sum_{\ell=1}^{j}(-1)^{\ell+j}\binom{L+n}{j+n}\binom{j-\ell+n-1}{n-1}x^{\ell} = (1+x)^{L}-1.$$

Proof. Changing the order of summation, it is enough to prove that, for $L \geq 1$, $n \geq 1$, and $1 \leq \ell \leq L$,

$$\sum_{j=\ell}^{L}(-1)^{j+\ell}\binom{L+n}{j+n}\binom{j-\ell+n-1}{n-1} = \binom{L}{\ell}.$$

We prove this by induction on L. It holds for $L = 1$, and assuming for $L \geq 1$, for $L+1$ we have, by the induction hypothesis for (L, ℓ, n) and $(L, \ell-1, n)$,

$$\begin{aligned}
&\sum_{j=\ell}^{L+1}(-1)^{j+\ell}\binom{L+1+n}{j+n}\binom{j-\ell+n-1}{n-1}\\
&\qquad = \binom{L}{\ell} + \sum_{j=\ell}^{L+1}(-1)^{j+\ell}\binom{L+n}{j+n-1}\binom{j-\ell+n-1}{n-1}\\
&\qquad = \binom{L}{\ell} + \sum_{j=\ell-1}^{L}(-1)^{j+\ell-1}\binom{L+n}{j+n}\binom{j-(\ell-1)+n-1}{n-1}\\
&\qquad = \binom{L}{\ell} + \binom{L}{\ell-1} = \binom{L+1}{\ell},
\end{aligned}$$

as claimed. □

Proof of Theorem 4. In order to handle assumptions (i) and (ii) together, under (i) we use the notation $\lambda_k = 0$ for $k \geq K$. Then the limiting g.f. f_Y is given in (3).

First, assume that $\nu > 0$. Substituting the Taylor series of the logarithm,

$$\log\left(1+\frac{\nu}{2}(1-s)\right) = \log\left(1+\frac{\nu}{2}\right) - \sum_{n=1}^{\infty} n^{-1}\left(\frac{\nu}{2+\nu}\right)^{n} s^{n}, \qquad s \in (0, 1),$$

into (3), expanding $(1-s)^i$, and gathering together the powers of s, we have

$$\begin{aligned}
f_Y(s) &= \exp\left\{-\sum_{k=1}^{\infty}\frac{2^k\lambda_k}{\nu^k}\left(\log\left(1+\frac{\nu}{2}\right) - \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\nu}{2+\nu}\right)^{n}s^{n} + \sum_{i=1}^{k-1}(-1)^{i}\frac{\nu^i}{i2^i}\sum_{j=0}^{i}\binom{i}{j}(-s)^{j}\right)\right\}\\
&= \exp\left\{-\sum_{k=1}^{\infty}\frac{2^k\lambda_k}{\nu^k}\left(\log\left(1+\frac{\nu}{2}\right) + \sum_{i=1}^{k-1}(-1)^{i}\frac{\nu^i}{i2^i}\right)\right.\\
&\qquad \left. + \sum_{k=1}^{\infty}\frac{2^k\lambda_k}{\nu^k}\sum_{n=1}^{\infty}s^{n}\left(n^{-1}\left(\frac{\nu}{2+\nu}\right)^{n} - \sum_{i=n}^{k-1}(-1)^{i+n}\frac{\nu^i}{i2^i}\binom{i}{n}\right)\right\},
\end{aligned} \tag{34}$$

where the empty sum is defined to be 0. We claim that the order of summation in (34) with respect to k and n can be interchanged. Indeed, this is clear under (i), since the sum in k is in fact finite.

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Assume (ii). Then Taylor's theorem applied to the function $(1+x)^{-n}$ with $n \le k-1$ gives

$$(1+x)^{-n} = n\sum_{\ell=n}^{k-1}(-1)^{\ell+n}\binom{\ell}{n}\frac{x^{\ell-n}}{\ell} + (-1)^{k-n}\binom{k-1}{k-n}(1+\xi)^{-k}x^{k-n},$$

with $\xi \in (0, x)$. Therefore,

$$\left|\sum_{i=n}^{k-1}(-1)^{i+n}\binom{i}{n}\frac{x^{i-n}}{i} - n^{-1}(1+x)^{-n}\right| \le n^{-1}\binom{k-1}{k-n}x^{k-n}, \tag{35}$$

thus, with $x = \nu/2$,

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{2^k\lambda_k}{\nu^k}\sum_{n=1}^{k-1}s^n\left|\sum_{i=n}^{k-1}(-1)^{i+n}\binom{i}{n}\frac{\nu^i}{2^i i} - n^{-1}\left(\frac{\nu}{2+\nu}\right)^n\right| &\le \sum_{k=1}^{\infty}\frac{2^k\lambda_k}{\nu^k}\sum_{n=1}^{k-1}s^n\frac{\nu^n}{2^n}n^{-1}\binom{k-1}{k-n}\frac{\nu^{k-n}}{2^{k-n}}\\ &= \sum_{k=1}^{\infty}\lambda_k\sum_{n=1}^{k-1}s^n n^{-1}\binom{k-1}{k-n}\\ &\le \sum_{k=1}^{\infty}\lambda_k(1+s)^k,\end{aligned}$$

which is finite for $s \in (0, 1)$ if $\limsup \lambda_n^{1/n} \le \frac{1}{2}$. The other part is easy to handle, as

$$\sum_{k=1}^{\infty}\frac{2^k\lambda_k}{\nu^k}\sum_{n=k}^{\infty}s^n n^{-1}\left(\frac{\nu}{2+\nu}\right)^n \le \sum_{k=1}^{\infty}\frac{2^k\lambda_k}{\nu^k}s^k\left(\frac{\nu}{2+\nu}\right)^k = \sum_{k=1}^{\infty}\frac{2^k\lambda_k}{(2+\nu)^k}s^k,$$

which is summable.

Summarizing, in both cases the order of summation in (34) can be interchanged, and doing so we obtain $f_Y(s) = \exp\left\{-A_0 + \sum_{n=1}^{\infty}A_n s^n\right\}$, where

$$\begin{aligned}A_0 &= \sum_{k=1}^{\infty}\frac{2^k\lambda_k}{\nu^k}\left(\sum_{i=1}^{k-1}(-1)^i\frac{\nu^i}{i2^i} + \log\left(1+\frac{\nu}{2}\right)\right),\\ A_n &= \sum_{k=1}^{\infty}\frac{2^k\lambda_k}{\nu^k}\left(\left(\frac{\nu}{2+\nu}\right)^n n^{-1} - \sum_{i=n}^{k-1}(-1)^{i+n}\binom{i}{n}\frac{\nu^i}{i2^i}\right).\end{aligned} \tag{36}$$

Recall the notation from the proof of Lemma 8. By (30), using the inversion formula for Stirling numbers of the first and second kind (see, e.g., [10, Exercise 12, p. 310]), we have

$$k!\lambda_k = \sum_{i=0}^{k}(-1)^{k+i}\begin{bmatrix}k\\ i\end{bmatrix}\mu_k,$$

where $\left[{k \atop i}\right]$ represents the Stirling number of the first kind. Substituting (33), and using

$$\sum_{i=0}^{k}(-1)^{k+i}\begin{bmatrix}k\\ i\end{bmatrix}j^k = j(j-1)\cdots(j-k+1)$$

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(see, e.g., [10, p. 263, (6.13)]), we obtain the formula

$$\lambda_k = \sum_{j=k}^{\infty} \binom{j}{k} q_j. \tag{37}$$

We also see that $\lambda_k = 0$ implies $q_j = 0$ for all $j \geq k$. Substituting (37) back into (36) we claim that the order of summation with respect to k and j can be interchanged. This is again clear under (i), while under (ii), by (35),

$$\begin{aligned}\sum_{k=n+1}^{\infty} \sum_{j=k}^{\infty} \frac{2^k}{\nu^k} q_j \binom{j}{k} \left| \sum_{i=n}^{k-1} (-1)^{i+n} \binom{i}{n} \frac{\nu^i}{i2^i} - n^{-1} \frac{\nu^n}{(2+\nu)^n} \right| \\ \leq \sum_{k=n+1}^{\infty} \sum_{j=k}^{\infty} \frac{2^k}{\nu^k} q_j \binom{j}{k} \frac{\nu^n}{2^n} n^{-1} \binom{k-1}{k-n} \frac{\nu^{k-n}}{2^{k-n}} \\ = \sum_{k=n+1}^{\infty} \lambda_k n^{-1} \binom{k-1}{k-n} < \infty,\end{aligned}$$

since

$$n^{-1} \binom{k-1}{k-n} = k^{-1} \binom{k}{n} \leq k^n.$$

Thus, the order of summation is indeed interchangeable, and we obtain

$$\begin{aligned}A_n &= \sum_{j=1}^{\infty} q_j \left\{ \left[\left(1 + \frac{2}{\nu}\right)^j - 1 \right] \left(\frac{\nu}{2+\nu} \right)^n n^{-1} - \sum_{k=n+1}^{j} \frac{2^k}{\nu^k} \binom{j}{k} \sum_{i=n}^{k-1} (-1)^{i+n} \binom{i}{n} \frac{\nu^i}{i2^i} \right\} \\ &=: \sum_{j=1}^{\infty} q_j B(n, j).\end{aligned} \tag{38}$$

We claim that

$$B(n, j) = \frac{\nu^n}{(2+\nu)^n n} \left[\left(1 + \frac{2}{\nu}\right)^{n \wedge j} - 1 \right]. \tag{39}$$

This is clear for $j \leq n$. Writing $\ell = k - i$ in the summation and using Lemma 9 with $L = j - n$ we obtain

$$\begin{aligned}\sum_{k=n+1}^{j} \sum_{i=n}^{k-1} (-1)^i \binom{j}{k} \binom{i-1}{n-1} x^{k-i} &= \sum_{k=n+1}^{j} \sum_{\ell=1}^{k-n} (-1)^{k+\ell} \binom{j}{k} \binom{k-\ell-1}{n-1} x^{\ell} \\ &= (-1)^n [(1+x)^{j-n} - 1].\end{aligned}$$

Using that $n\binom{i}{n} = i\binom{i-1}{n-1}$, and substituting back into (38) with $x = 2/\nu$, we have

$$B(n, j) = n^{-1} \left[1 - \left(1 + \frac{2}{\nu}\right)^{j-n} \right] + \left[\left(1 + \frac{2}{\nu}\right)^j - 1 \right] \left(\frac{\nu}{2+\nu} \right)^n n^{-1},$$

which after simplification gives (39).

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Now let $\nu = 0$. In this case the calculations are again simpler. Expanding $(s-1)^i$ and changing the order of summation,

$$\log f_Y(s) = \sum_{k=1}^{\infty} (-1)^k \frac{\lambda_k}{k} + \sum_{n=1}^{\infty} s^n \sum_{k=n}^{\infty} (-1)^{n+k} \frac{\lambda_k}{k} \binom{k}{n} =: -A_0 + \sum_{n=1}^{\infty} s^n A_n.$$

Substituting (37) and changing the order of summation again we get

$$A_n = \sum_{j=n}^{\infty} q_j \sum_{k=n}^{j} \binom{k}{n} \binom{j}{k} (-1)^{k+n} \frac{1}{k}. \tag{40}$$

The changes of the order of summation can be justified as in the previous case. For the coefficient of q_j we have

$$\sum_{k=n}^{j} \binom{k}{n} \binom{j}{k} (-1)^{k+n} \frac{1}{k} = \binom{j}{n} \sum_{k=n}^{j} \frac{1}{k} \binom{j-n}{k-n} (-1)^{k+n} = \frac{1}{n},$$

where we used that for $j \geq n$,

$$\sum_{k=0}^{j-n} (-1)^k \frac{j}{k+n} \binom{j-n}{k} = \binom{j-1}{n-1}^{-1}.$$

The latter formula follows by induction on j. Substituting back into (40), the proof is complete. □

Acknowledgement

We are grateful to Mátyás Barczy for discussions on the topic. We thank the anonymous referees for useful comments and remarks, and in particular for suggesting allowing supercritical generations, which led to more general results.

Funding information

This research was supported by the Ministry of Culture and Innovation of Hungary from the National Research, Development and Innovation Fund, project no. TKP2021-NVA-09. Kevei's research was partially supported by the János Bolyai Research Scholarship of the Hungarian Academy of Sciences.

Competing interests

There were no competing interests to declare which arose during the preparation or publication process of this article.

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