Bounds for zeros of Meixner and Kravchuk polynomials

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ABSTRACT

The zeros of certain different sequences of orthogonal polynomials interlace in a well-defined way. The study of this phenomenon and the conditions under which it holds lead to a set of points that can be applied as bounds for the extreme zeros of the polynomials. We consider different sequences of the discrete orthogonal Meixner and Kravchuk polynomials and use mixed three-term recurrence relations, satisfied by the polynomials under consideration, to identify bounds for the extreme zeros of Meixner and Kravchuk polynomials.

1. *Introduction*

Consider a sequence $\{p_n\}_{n=0}^{\infty}$ of real polynomials, where p_n is of exact degree n, orthogonal with respect to an absolutely continuous measure that can be represented via a real weight function $w(x)$, always positive, on the interval (a, b), so that

$$\int_a^b p_n(x)p_m(x)w(x)\,dx = N_n^2 \delta_{mn}, \quad N_n \neq 0, m, n = 0, 1, \ldots \tag{1}$$

where δ_{mn} denotes the Kronecker delta. The weight function may be discrete and in this case, if $w_i > 0$ are the values of the weight at the distinct points x_i, $i = 0, 1, 2, \ldots, M$, M a positive integer, the orthogonality relation (1) becomes [**1**, p. 182, equation (1.4)]

$$\sum_{i=0}^{M} p_n(x_i)p_m(x_i)w_i = N_n^2 \delta_{mn}, \quad N_n \neq 0, m, n = 0, 1, \ldots.$$

One of the most important properties of orthogonal polynomials is that they satisfy a three-term recurrence relation of the form

$$p_{n+1}(x) = (\alpha_n x + \beta_n)p_n(x) - \gamma_n p_{n-1}(x), \quad n = 0, 1, \ldots,$$

where we set $p_{-1} = 0$. The coefficients α_n, β_n and γ_n, $n = 0, 1, 2, \ldots$, are real constants with

$$\alpha_n = \frac{k_{n+1}}{k_n}, \quad \gamma_{n+1} = \frac{\alpha_{n+1}}{\alpha_n} \frac{N_{n+1}^2}{N_n^2}$$

where k_n denotes the leading coefficient of p_n.

Let $x_{n,1} < x_{n,2} < \ldots < x_{n,n}$, $n = 1, 2, \ldots$, be the n real, distinct zeros of p_n, $a = x_{n,0}, b = x_{n,n+1}$. It is a classical result that each interval $(x_{n-1,j}, x_{n-1,j+1})$, $j = 0, 1, \ldots, n-1$, contains exactly one zero of p_n (cf. [**21**, Theorem 3.3.2]). Interlacing of zeros of polynomials from different sequences (of the same or adjacent degree) within the same family of orthogonal polynomials, where each sequence is generated by a different parameter, was first studied in 1967 by Levit (cf. [**16**]), who proved several separation results for the zeros of Hahn polynomials from different sequences. Using the limit relation between Hahn and Kravchuk polynomials (cf. [**15**, equation (9.5.16)]), one can obtain interlacing of zeros of certain Kravchuk polynomials

from different sequences (cf. [**16**, Theorem 6]), a result that was rediscovered independently by Chihara and Stanton in [**3**] and later by Jordaan and Toókos in [**14**].

Stieltjes (cf. [**21**, Theorem 3.3.3]) proved that, within the orthogonal sequence $\{p_n\}_{n=0}^{\infty}$, the zeros of p_n and p_{n-m}, $m = 2, 3, \ldots, n-1$, interlace in the following well-defined way: in each interval $(x_{n-m,j}, x_{n-m,j+1})$, $j = 0, 1, \ldots, n-m$, there is at least one zero of p_n. De Boor and Saff [**4**] and Vinet and Zhedanov [**23**] considered dual or, equivalently, associated polynomials, and the role of the zeros of these polynomials in completing the interlacing was explicitly stated by Beardon [**2**]. If p_{n-m} and p_n are co-prime, that is, they have no common zeros, there exists a real polynomial of degree $m-1$, completely determined by the coefficients in the three-term recurrence relation satisfied by the orthogonal sequence $\{p_n\}_{n=0}^{\infty}$, whose real simple zeros provide a set of points that complete the interlacing. We will refer to this as completed Stieltjes interlacing. An extension of this completed Stieltjes interlacing between polynomials belonging to the same orthogonal sequence, to polynomials from different orthogonal sequences, obtained by integer shifts of the appropriate parameter(s), was done in [**6**, **7**] and [**8**] for the Gegenbauer, Laguerre and Jacobi polynomials respectively.

In this paper we consider a sequence of polynomials $\{g_{n,k}\}_{n=0}^{\infty}$, depending on a nonnegative integer parameter k, that are orthogonal on (a, b) with respect to weight function $c_k(x)w(x) > 0$ where $c_k(x)$ represents a polynomial of degree k and prove that relations of finite-type (cf. [**18**, (1.7)]) involving $g_{n-2,k}$, p_n and p_{n-1}, necessary to obtain the completed Stieltjes interlacing, only hold for specific values of k. It is important to mention that existence of such sequences $\{g_{n,k}\}_{n=0}^{\infty}$ satisfying finite-type relations is not necessarily guaranteed, but necessary and sufficient conditions for existence are known, see for instance [**17**, (5.7)]. We apply our result to investigate the extent to which completed Stieltjes interlacing holds between the zeros of different sequences of the discrete orthogonal Meixner polynomials. Furthermore, we study completed Stieltjes interlacing for Kravchuk polynomials, a class of polynomials where the conditions of the general result are not always satisfied. In each case we identify the polynomial whose zeros complete the interlacing and we obtain new bounds for the extreme zeros of Meixner and Kravchuk polynomials. Outer bounds for the extreme zeros of, inter alia, Meixner polynomials were obtained in [**13**] by using recurrence coefficients. Knowledge of the location and behaviour of the zeros of Meixner polynomials is relevant in analysing discrete stochastic processes (cf. [**11**] and [**20**]) and the zeros of Kravchuk polynomials play a role in coding theory [**12**, p. 184] and graph theory [**5**, Chapter 11]. Numerical examples are given in order to illustrate the accuracy of our bounds.

2. Completed Stieltjes interlacing of different orthogonal polynomials

THEOREM 2.1. *Let* $\{p_n\}_{n=0}^{\infty}$ *be a sequence of polynomials orthogonal on the (finite or infinite) interval* (a, b) *with respect to* $w(x) > 0$. *Let* $k \in \mathbb{N}_0$ *be fixed and suppose* $\{g_{n,k}(x)\}_{n=0}^{\infty}$ *is a sequence of polynomials orthogonal with respect to* $c_k(x)w(x) > 0$ *on* (a, b), *where* $c_k(x)$ *is a polynomial of degree* k, *that satisfies*

$$A_n c_k(x) g_{n-2,k}(x) = a_{k-2}(x) p_n(x) - (x - B_n) p_{n-1}(x), \quad n = 1, 2, \ldots, \qquad (2)$$

with $g_{-1,k} = 0$, A_n, B_n, a_{-1}, a_{-2} *constants and* a_{k-2} *a polynomial of degree* $k-2$ *defined on* (a, b) *whenever* $k = 2, 3, \ldots$. *Then:*

  (i) $k \in \{0, 1, 2, 3, 4\}$;

 (ii) *the* $n-1$ *real, simple zeros of* $(x - B_n)g_{n-2,k}$ *interlace with the zeros of* p_n *and* B_n *is an upper bound for the smallest, as well as a lower bound for the largest zero of* p_n *if* $g_{n-2,k}$ *and* p_n *are co-prime;*

(iii) *if* $g_{n-2,k}$ *and* p_n *are not co-prime:*

(a) *they have one common zero that is equal to B_n and this common zero cannot be the largest or smallest zero of p_n;*
(b) *the $n - 2$ zeros of $g_{n-2,k}(x)$ interlace with the $n - 1$ non-common zeros of p_n;*
(c) *B_n is an upper bound for the smallest as well as a lower bound for the largest zero of p_n.*

Proof. (i) Let $j \in \{0, 1, \ldots, n - 3\}$. Since $g_{n,k}$ is orthogonal with respect to $c_k(x)w(x)$ we have

$$\int_a^b A_n x^j g_{n-2,k}(x) c_k(x) w(x) \, dx = 0 \quad \text{for } n = 2, 3, \ldots,$$

while it follows from (2) and the orthogonality of p_{n-1} with respect to $w(x)$ that

$$\int_a^b A_n x^j g_{n-2,k}(x) c_k(x) w(x) \, dx$$
$$= \int_a^b x^j (x - B_n) p_{n-1}(x) w(x) \, dx + \int_a^b x^j a_{k-2}(x) p_n(x) w(x) \, dx$$
$$= \int_a^b q(x) p_n(x) w(x) \, dx \quad \text{for } n = 2, 3, \ldots, \tag{3}$$

where $q(x) = x^j a_{k-2}(x)$ is a polynomial of degree $j + k - 2$ when $k \in \{2, 3, \ldots\}$ and a polynomial of degree j when $k \in \{0, 1\}$. Since p_n is orthogonal with respect to $w(x)$ on (a, b) and the integral in (3) vanishes, we have that $\deg(q(x)) = j + k - 2 \in \{0, 1, \ldots, n - 1\}$ and this is true for all $j \in \{0, 1, \ldots, n - 3\}$, which implies that $k \in \{2, 3, 4\}$. Furthermore, the integral in (3) will be zero for all values of k such that $\deg(q(x)) = j$, that is, when $k \in \{0, 1\}$.

(ii) See [**9**, Theorem 2.1 (i), Corollary 2.2 (i)] for a proof of the more general case when $g_{n-2,k}$ has any degree less than or equal to $n - 2$.

(iii) See [**9**, Theorem 2.1 (ii), Corollary 2.2 (ii)] where a proof of the more general case when $g_{n-2,k}$ has any degree less than or equal to $n - 2$ is given. $\square$

REMARK. (1) Theorem 2.1 also applies to finite orthogonal sequences $\{p_n\}_{n=0}^M$, $M \in \mathbb{N}$.

(2) In [**7**, Theorem 1], it was proved that, for $\alpha > -1$, the zeros of the Laguerre polynomials, $L_{n-2}^{\alpha+k}, k \in \{0, 1, 2, 3, 4\}$, together with a given extra-interlacing point, interlace with the zeros of L_n^α, and in [**8**, Theorems 2.1, 2.4 and 2.6] it was proved that, for $\alpha, \beta > -1$, completed Stieltjes interlacing holds between the zeros of the Jacobi polynomials $P_n^{\alpha,\beta}$ and the polynomials $P_{n-2}^{\alpha+k,\beta}$ and $P_{n-2}^{\alpha,\beta+k}$ for $k \in \{0, 1, 2, 3, 4\}$ as well as the polynomials $P_{n-2}^{\alpha+s,\beta+t}$, where $k = s + t \in \{0, 1, 2, 3, 4\}$. These papers used counter-examples to illustrate that the results are the best possible and that interlacing breaks down for $k \in \{5, 6, \ldots\}$. Theorem 2.1(i) shows clearly that the mixed three-term recurrence relations necessary to obtain completed Stieltjes interlacing hold only if $k \in \{0, 1, 2, 3, 4\}$. In [**6**, Theorem 1] it is proved that, for $\lambda > -\frac{1}{2}$, the zeros of Gegenbauer polynomials $C_{n-2}^{\lambda+k}, k \in \{0, 1, 2\}$, together with the point $x = 0$, interlace with the zeros of C_n^λ. Theorem 2.1 suggests that analogous results may hold for $k \in \{3, 4\}$.

(3) Theorem 2.1 extends a result by Gibson (cf. [**10**, p. 130]) that determines the maximum amount of common zeros of two polynomials from the same orthogonal sequence, to the zeros of any polynomials satisfying a recurrence relation of type (2) whose degrees differ by 2.

(4) Although condition (2) in Theorem 2.1 seems restrictive by not allowing more natural structural relations involving more than three terms for the orthogonal sequence $\{g_{n,k}\}_{n=0}^\infty$, it is important to point out that a three-term relation is essential in the proof of the completed Stieltjes interlacing which yields the bounds for the extreme zeros given in Theorem 2.1(ii) and (iii).

3. *Completed Stieltjes interlacing of zeros of different Meixner and Kravchuk polynomials*

Meixner polynomials may be defined in terms of the $_2F_1$ hypergeometric function (cf. [**15**, equation (9.10.1)])

$$M_n(x;\beta,c) = (\beta)_n \, _2F_1\left(-n,-x;\beta;1-\frac{1}{c}\right) \tag{4}$$

for $\beta, c \in \mathbb{R}$, $\beta \neq -1,-2,\ldots,-n+1$, $c \neq 0$, where the symbol $(\cdot)_k$ is the Pochhammer symbol [**12**, equation (1.3.6)], defined by

$$(\alpha)_k = \alpha(\alpha+1)\ldots(\alpha+k-1), \quad k \in \mathbb{N}$$
$$(\alpha)_0 = 1, \quad \alpha \neq 0.$$

For $0 < c < 1$ and $\beta > 0$, these polynomials are orthogonal on $(0,\infty)$ with respect to the weight function $\rho(x) = c^x(\beta)_x/x!$. We examine completed Stieltjes interlacing of the zeros of $M_n(x;\beta,c)$ and $M_{n-2}(x;\beta+k,c)$ for different integer values k with due attention to the possibility that these polynomials can have common zeros. The extra-interlacing points obtained are inner bounds for the extreme zeros of the Meixner polynomials and, in §4, we compare these points to identify the best upper (lower) bound for the smallest (largest) zero of the polynomials $M_n(x;\beta,c)$. Note that the polynomials $M_n(x;\beta,c)$ are orthogonal when the parameter c is restricted to the interval $(0,1)$ and therefore it does not make sense to consider Stieltjes interlacing for the zeros of Meixner polynomials where the parameter c is shifted by integer values.

THEOREM 3.1. *Let* $M_n(x;\beta,c)$, $\beta > 0$, $0 < c < 1$, $k,n \in \mathbb{N}_0$, *denote a Meixner polynomial of degree* n. *Then, for each fixed* $k \in \{0,1,2,3,4\}$:
(i) *the zeros of* $M_{n-2}(x;\beta+k,c)$, *together with the point* $B_n(k)$, *interlace with the zeros of* $M_n(x;\beta,c)$ *and each* $B_n(k)$ *is an upper bound for the smallest as well as a lower bound for the largest zero of* $M_n(x;\beta,c)$, *where*

$$B_n(k) = \frac{\beta c + (n-1)(1+(1-k)c)}{1-c} + \frac{k(1-k)(2-k)n(n-1)c^2}{6(1-c)(n+\beta)}$$

for $k = 0,1,2,3$ *and*

$$B_n(4) = \frac{(\beta+n)_2((n-1)+c(3+\beta-3n)) - c^2n(n-1)(c(\beta+n+1)-(3n+4\beta+3))}{(1-c)(\beta(2n+\beta+1)+n(n+1-(n-1)c^2))} \tag{5}$$

if $M_{n-2}(x;\beta+k,c)$ *and* $M_n(x;\beta,c)$ *are co-prime;*
(ii) *if* $M_{n-2}(x;\beta+k,c)$ *and* $M_n(x;\beta,c)$ *are not co-prime, then:*
  (a) *the two polynomials under consideration have one common zero located at the respective points identified in* (i);
  (b) *the* $n-2$ *zeros of* $M_{n-2}(x;\beta+k,c)$ *interlace with the remaining* $n-1$ (*non-common*) *zeros of* $M_n(x;\beta,c)$;
  (c) $B_n(k)$, *as identified in* (i), *is an upper bound for the smallest, as well as a lower bound for the largest zero of* p_n.

The orthogonality of a finite number of Meixner polynomials $M_n(x;\beta,c)$ when $c < 0$ and β is equal to a negative integer, say $\beta = -N$, $N \in \mathbb{N}$, is that of the Kravchuk polynomials, defined by (cf. [**15**, equation (9.11.1)])

$$K_n(x;p,N) = (-N)_n \, _2F_1\left(-n,-x;-N;\frac{1}{p}\right), \quad n = 0,1,\ldots,N, N \in \mathbb{N} \tag{6}$$

and these polynomials are orthogonal with respect to the finite binomial distribution $w(x;p,N) = \binom{N}{x}(p)^x(1-p)^{N-x}$ that is positive at the mass points $x = 0,1,\ldots,N$ of the

discrete distribution for $0 < p < 1$. This implies that, for $0 < p < 1$ and $n \leqslant N$, $n, N \in \mathbb{N}$, the zeros of $K_n(x; p, N)$, denoted by $x_{n,1}^N < x_{n,2}^N < \ldots < x_{n,n}^N$, are real, distinct, in the interval $(0, N)$ and separated by the mass points of the measure of orthogonality (cf. [**21**, Theorem 3.41.2]). In the particular case where $n = N$, the zeros of $K_N(x; p, N)$ interlace with the mass points as follows

$$0 < x_{N,1}^N < 1 < x_{N,2}^N < 2 < \ldots < x_{N,N}^N < N.$$

Completed Stieltjes interlacing between the zeros of Kravchuk polynomials $K_n(x; p, N)$ that lie in $(0, N)$ and the $n - 2$ zeros of $K_{n-2}(x; p, N - k)$, $k \in \{-1, 0, 1, 2\}$, that lie in $(0, N - k)$ are of particular interest since shifting the parameter N implies a change in the interval of orthogonality. Note that general results such as Theorem 2.1 or [**9**, Theorem 2.1] are then not immediately applicable. Additional restrictions on the parameter p will be necessary in some cases to obtain completed Stieltjes interlacing. We identify the extra-interlacing points that complete the interlacing and obtain new inner and outer bounds for the extreme zeros of the polynomials $K_n(x; p, N)$.

THEOREM 3.2. *Let $K_n(x; p, N)$, $0 < p < 1$, $n = 2, 3, \ldots, N$, $N \in \mathbb{N}$, denote the Kravchuk polynomials and let*

$$C_n(k) = Np + (n-1)(1 + kp - 2p) \quad \text{and} \quad p_n(k) = \frac{N - k - n + 1}{N + (k-2)(n-1)}, \quad k = 0, 1, 2.$$

Then we have the following.
  (i) *The zeros of $K_{n-2}(x; p, N)$, together with the point $C_n(0)$, interlace with the zeros of $K_n(x; p, N)$ if $K_{n-2}(x; p, N)$ and $K_n(x; p, N)$ are co-prime.*
  (ii) *The zeros of $K_{n-2}(x; p, N-1)$, together with the point $C_n(1)$, interlace with the zeros of $K_n(x; p, N)$, if $K_{n-2}(x; p, N-1)$ and $K_n(x; p, N)$ are co-prime. Furthermore:*
     (a) *if $0 < p_n(1) < p < 1$, then $x_{n-2,n-2}^{N-1} < x_{n,n-1}^N < N - 1 < C_n(1) < x_{n,n}^N < N$;*
     (b) *if $p = p_n(1) = (N-n)/(N-n+1)$, then $x_{n-2,n-2}^{N-1} < x_{n,n-1}^N < C_n(1) = N - 1 < x_{n,n}^N < N$.*
  (iii) *If $K_{n-2}(x; p, N-2)$ and $K_n(x; p, N)$ are co-prime, the zeros of $K_{n-2}(x; p, N-2)$, together with the point $C_n(2)$, interlace with the zeros of $K_n(x; p, N)$ for $p < 1 - n/N$, if and only if the zeros of $K_n(x; p, N)$ lie in $(0, N-1)$. Furthermore:*
     (a) *if $0 < p_n(2) < p < 1 - n/N$, then $x_{n-2,n-2}^{N-2} < x_{n,n-1}^N < N - 2 < C_n(2) < x_{n,n}^N < N - 1$;*
     (b) *if $p = p_n(2) = 1 - (n+1)/N$, then $x_{n-2,n-2}^{N-2} < x_{n,n-1}^N < C_n(2) = N - 2 < x_{n,n}^N < N - 1$.*
  (iv) *If $K_n(x; p, N)$ and $K_{n-2}(x; p, N - k)$, $k = 0, 1, 2$ have a zero in common:*
     (a) *$0 < p < 1$ when $k = 0$ and $0 < p < p_n(k) < 1$ for $k = 1, 2$;*
     (b) *the point $C_n(k)$ is the common zero of $K_{n-2}(x; p, N - k)$ and $K_n(x; p, N)$;*
     (c) *the $n - 2$ zeros of $K_{n-2}(x; p, N - k)$ interlace with the $n - 1$ (non-common) zeros of $K_n(x; p, N)$.*

Completed Stieltjes interlacing of the zeros of $K_n(x, p, N)$ and $K_{n-2}(x, p, N+1)$ is considered in the following theorem.

THEOREM 3.3. *Let $K_n(x; p, N)$, $0 < p < 1$, $n = 1, 2, \ldots, N$, $N \in \mathbb{N}$, denote the Kravchuk polynomial of degree n and let*

$$q = \frac{3(n-1) - N + \sqrt{9 - 22n + 17n^2 - 4n^3 + 6N - 10nN + 4n^2N + N^2}}{2n(n-1)}$$

$$X_1 = \frac{1}{2}\left(S_n - \sqrt{S_n^2 + 4((N+1)((n-1)(3p-1) - Np) - n(n-1)p^2)}\right)$$

and

$$X_2 = \frac{1}{2}\left(S_n + \sqrt{S_n^2 + 4((N+1)((n-1)(3p-1) - Np) - n(n-1)p^2)}\right)$$

where

$$S_n = N + n - 3(n-1)p + Np.$$

Then:

(i) $X_2 \in [N, N+1)$ *if* $p \leqslant q$;

(ii) *for each fixed* $n = 2, 3, \ldots, N$ *and* $p \leqslant q$, *the zeros of* $K_{n-2}(x; p, N+1)$ *are in* $(0, N)$ *and, together with the point* X_1, *they interlace with the zeros of* $K_n(x; p, N)$ *if* $K_{n-2}(x; p, N+1)$ *and* $K_n(x; p, N)$ *are co-prime;*

(iii) *for* $n = 2, 3, \ldots, N$ *and* p *fixed,* $p \leqslant q$, *if* $K_{n-2}(x; p, N+1)$ *and* $K_n(x; p, N)$ *are not co-prime:*

   (a) *they have one common zero at* $x = X_1$;

   (b) *the* $n - 2$ *zeros of* $K_{n-2}(x; p, N+1)$ *interlace with the* $n - 1$ *non-common zeros of* $K_n(x; p, N)$.

REMARK. The case $k = 0$ in Theorems 3.1(i) and 3.2(i) extend the classic result of Stieltjes (cf. [**21**, Theorem 3.3.3]) that between any two zeros of a polynomial $p_{n-2}(x)$, there is at least one zero of $p_n(x)$, by providing a formula for an extra-interlacing point.

4. Bounds for the extreme zeros of Meixner and Kravchuk polynomials

The inner bounds for the extreme zeros of Meixner polynomials $B_n(k)$, $k = 0, 1, \ldots, 4$, obtained in Theorem 3.1, satisfy

$$0 < x_{n,1} < B_n(4) < B_n(3) < \ldots < B_n(0) = \frac{(n-1)(1+c) + \beta c}{1 - c} < x_{n,n},$$

for all values of $\beta > 0$ and $0 < c < 1$ and it is clear that $B_n(0)$ is the best lower bound for the largest zero while $B_n(4)$, given in (5), is the most precise upper bound for the smallest zero of $M_n(x; \beta, c)$ obtained by using Theorem 2.1. In Table 1 we provide numerical examples in order to illustrate these bounds. The points $C_n(k) = Np + (n-1)(1 - (k-2)p)$, $k = 0, 1, 2$, obtained in Theorem 3.2 will be upper (lower) bounds for the smallest (largest) zero of the Kravchuk polynomial $K_n(x; p, N)$. For $p \in (0, 1)$ and $n = 2, 3, \ldots, N$ we have that

$$0 < x_{n,1}^N < C_n(0) < C_n(1) < x_{n,n}^N < N. \tag{7}$$

When $p \in (0, 1 - n/N) \subset (0, 1)$, we obtain a new upper bound for $x_{n,n}^N$ as well as a better lower bound for $x_{n,n}^N$ than the one given in (7), since it follows from Theorem 3.2 (iii), (iv) and the definition of $C_n(k)$, $k = 0, 1, 2$, that

$$0 < x_{n,1}^N < C_n(0) < C_n(1) < C_n(2) < x_{n,n}^N < N - 1.$$

TABLE 1. *Comparison of bounds for the extreme zeros of* $M_8(x; \beta, c)$ *for different values of* β *and* c.

β, c	$x_{8,1}$	$B_n(4)$	$B_n(0)$	$x_{8,8}$
$\beta = 0.09$, $c = 0.02$	2.9×10^{-15}	6.727	7.288	7.913
$\beta = 0.09$, $c = 0.99$	1.118	1.130	1401.91	2114.7
$\beta = 0.09$, $c = 0.5$	0.0004	2.195	21.09	31.082
$\beta = 20$, $c = 0.5$	5.234	16.474	41.00	65.935
$\beta = 20$, $c = 0.99$	892.097	1212.12	3373.00	5141.82

The lower bounds for the largest zeros, $C_n(1)$ and $C_n(2)$, are surprisingly good, particularly when $p \to 1$, which is consistent with the fact that

$$K_n(x; 1, N) = (-N)_n \,_2F_1(-n, -x; -N; 1) = (x - N)(x - N + 1) \dots (x - N + n - 1),$$

which vanishes when $x = N, N - 1, \dots, N - n + 1$. We provide some numerical examples in Table 2 to illustrate the new bounds.

5. Proofs

Gauss (cf. [19, p. 50]) defined as contiguous to $_2F_1(a, b; c; z)$, each of the six functions obtained by shifting one of the parameters by one unit and he proved that there is a relation between the function $_2F_1(a, b; c; z)$ and any two of its contiguous functions. The equation

$$\left(1 - \frac{n + b}{n + c} z\right) {}_2F_1(-n, b; c; z) = {}_2F_1(-n - 1, b; c; z) - \frac{(c - b)n}{(n + c)c} z {}_2F_1(-n + 1, b; c + 1; z) \tag{8}$$

follows from combining (2) and (4) in [19, p. 71]. In the following lemma we list the contiguous function relations that are used in our proofs. A useful algorithm for computing contiguous relations for $_2F_1$ Gauss hypergeometric series, written by Vidunas in 2002, is available as a computer package (cf. [22]). These relations can be easily verified by comparing the corresponding coefficients of equal powers.

LEMMA 5.1. *Let $F_n = {}_2F_1(-n, b; c; z)$ and denote $_2F_1(-n - 1, b + 1; c; z)$ by $F_{n+1}(b + 1)$, $_2F_1(-n + 1, b + 1; c - 3; z)$ by $F_{n-1}(b + 1, c - 3)$ and so on. Then*

$$(n + c - 1)F_n = nF_{n-1} + (c - 1)F_n(c - 1), \tag{9}$$

$$(bz + nz - n - c)F_n(c + 1) = -cF_{n+1} + n(z - 1)F_{n-1}(c + 1), \tag{10}$$

$$(z - 1)\left(z - \frac{c}{b + n}\right)F_n = \frac{c - z(c + n)}{b + n} F_{n+1} + \frac{n(c - b)(c - b + 1)}{(b + n)c(c + 1)} z^2 F_{n-1}(c + 2), \tag{11}$$

$$\left(\frac{-n(n + 1)}{1 + b - c} + (c + 3n)z - (b + n)z^2\right)F_n$$
$$= (c + n)\left(z - \frac{n}{1 + b - c}\right)F_{n+1} + \frac{n(c - 1)}{1 + b - c}(z - 1)^2 F_{n-1}(c - 1), \tag{12}$$

$$(z - 1)((n + 1)(b(z - 1) + nz)z + c(z - 1)(z(b + n) - c - 1))F_n$$
$$= -(c^2(z - 1)^2 + nz(z(n + 1) - b) + c(1 - (n + 2)z + (2n + 1)z^2))F_{n+1}$$
$$+ \frac{n(c - b)_3}{(c)_3}(c + n + 1)z^3 F_{n-1}(c + 3), \tag{13}$$

$$(1 - z)D_1 F_n = D_2 F_{n+1} + \frac{n(c - b)_4(n + c + 1)(n + c + 2)}{(c)_4} z^4 F_{n-1}(c + 4), \tag{14}$$

TABLE 2. *Comparison of bounds for the extreme zeros of $K_5(x; p, N)$, for different values of p and N.*

p, N	$x_{5,1}$	$C_n(0)$	$C_n(1)$	$C_n(2)$	$x_{5,5}$	Upper bound $N - 1$
$p = 0.1$, $N = 7$	0.0007	3.9	4.3	4.7	4.991	6
$p = 0.45$, $N = 10$	0.769	4.9	6.7	8.5	8.738	9
$p = 0.9$, $N = 7$	2.009	3.1	6.7	–	6.999	–
$p = 0.99$, $N = 8$	3.816	4	7.96	–	7.999	–
$p = 0.99$, $N = 14$	9.573	9.94	13.9	–	13.999	–

where

$$
\begin{aligned}
D_1 = {} & c^3(z-1)^2 - c^2(z-1)^2(bz+nz-3) - c((n+4+(2n+3)b)z \\
& - (2+3n+n^2+(4n+6)b)z^2 + (b+n)(2n+3)z^3 - 2) \\
& - (n+1)z(n(n+2)z^2 + b(2-2(n+2)z + (n+2)z^2)), \\
D_2 = {} & -c^3(z-1)^3 - c^2((n+9)z - 3(n+3)z^2 + 3(n+1)z^3 - 3) \\
& + c(2 - (6+n+2bn)z + (6+4(b+1)n+n^2)z^2 - (2+6n+3n^2)z^3) \\
& - nz(b^2z + (2+3n+n^2)z^2 + b(2-5z-2nz)).
\end{aligned}
$$

Proof of Theorem 3.1. Let $\beta > 0$, $0 < c < 1$, $k,n \in \mathbb{N}_0$. The mixed three-term recurrence relations

$$
\begin{aligned}
& (n-1)(x+\beta)M_{n-2}(x;\beta+1,c) \\
& = \left(\frac{c}{c-1}\right) M_n(x;\beta,c) - \left(x - \frac{1-n-\beta c}{c-1}\right) M_{n-1}(x;\beta,c),
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
& \frac{(n-1)(1-c)(x+\beta)_2}{\beta+n-1} M_{n-2}(x;\beta+2,c) \\
& = \frac{((n-1)(1-c)+\beta)c}{(c-1)(\beta+n-1)} M_n(x;\beta,c) - \left(x - \frac{(n-1)(1-c)+\beta c}{1-c}\right) M_{n-1}(x;\beta,c),
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
& \frac{(n-1)(1-c)^2(x+\beta)_3}{(\beta+n-1)_2} M_{n-2}(x;\beta+3,c) = \frac{cD_1(x)}{(c-1)(\beta+n-1)_2} M_n(x;\beta,c) \\
& - \left(x - \frac{(n-1)^2(c-1)^2 + (n-1)(c-1)(c-\beta-1) + (\beta)_2 c}{(1-c)(\beta+n)}\right) M_{n-1}(x;\beta,c),
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
& \frac{(n-1)(1-c)^3(x+\beta)_4}{(\beta+n-1)D_3} M_{n-2}(x;\beta+4,c) \\
& = \frac{cD_2(x)}{(c-1)(\beta+n-1)D_3} M_n(x;\beta,c) - (x-B_n(4))M_{n-1}(x;\beta,c),
\end{aligned} \tag{18}
$$

where

$$
\begin{aligned}
D_1(x) = {} & (n-1)^2(c-1)^2 + (\beta)_2 + (n-1)(c-1)((x+\beta+1)c - 2\beta - 1), \\
D_2(x) = {} & (n-1)^3(c-1)^3 - (\beta)_3 + (n-1)^2(c-1)^2((2x+2\beta+3)c - 3(\beta+1)) \\
& + (n-1)(c-1)(3\beta^2+6\beta+2 + ((\beta+1)_2 + x(x+2\beta+3))c^2 \\
& - (3\beta^2+8\beta+4 + x(x+4\beta+5))c), \\
D_3 = {} & (n-1)^2(1-c^2) + (n-1)(2\beta+3-c^2) + (\beta+1)_2
\end{aligned}
$$

can be obtained using (4), together with the three-term recurrence relation satisfied by Meixner polynomials [**15**, equation (9.10.3)]

$$
\begin{aligned}
& \frac{(n-1)(\beta+n-2)}{1-c} M_{n-2}(x;\beta,c) \\
& = \frac{c}{c-1} M_n(x;\beta,c) - \left(x - \frac{n-1+(\beta+n-1)c}{1-c}\right) M_{n-1}(x;\beta,c)
\end{aligned} \tag{19}
$$

and the identities (11), (13) and (14). Equations (19), (15), (16), (17) and (18) are of the general form (2) and satisfy the conditions given in Theorem 2.1. It follows from Theorem 2.1(i) that such equations involving $M_{n-2}(x;\beta+k,c)$ exist only for $k \in \{0,1,2,3,4\}$ and Theorem 2.1(ii) and (iii) immediately yield the stated results. □

Proof of Theorem 3.2.

(i) We assume that $K_{n-2}(x; p, N)$ and $K_n(x; p, N)$ do not have any zeros in common. By substituting n by $n-1$ in the three-term recurrence relation satisfied by Kravchuk polynomials [15, equation (9.11.3)] and applying Theorem 2.1(ii), we obtain the stated result.

(ii) Let $K_{n-2}(x; p, N-1)$ and $K_n(x; p, N)$ be co-prime. From (6) and the contiguous relation (10) we obtain the mixed recurrence relation

$$
\begin{aligned}
(n-1)(x-N)&K_{n-2}(x; p, N-1) \\
&= pK_n(x; p, N) - (x - Np + (n-1)(p-1))K_{n-1}(x; p, N).
\end{aligned}
\tag{20}
$$

The polynomial $x - N$ in (20) does not change sign on $(0, N)$ and from [9, Theorem 2.1(i)], we deduce that the zeros of $K_{n-2}(x; p, N-1)$, together with the point $C_n(1) = Np + (n-1)(1-p)$, interlace with the zeros of $K_n(x; p, N)$ on $(0, N)$.

Furthermore:

(a) if $p > p_n(1) = 1 - 1/(N - n + 1)$, we have $N - 1 < C_n(1) < x_{n,n}^N$ and since there is at most one zero of $K_n(x; p, N)$, $n = 1, 2, \ldots, N$ in between any two consecutive mass points $0, 1, 2 \ldots, N$, we have $x_{n,n-1}^N < N - 1$;

(b) if $p = 1 - 1/(N - n + 1)$, $C_n(1) = N - 1$ and hence $x_{n,n-1}^N < N - 1 < x_{n,n}^N < N$.

(iii) Let $K_{n-2}(x; p, N-2)$ and $K_n(x; p, N)$ be co-prime and consider the mixed three-term recurrence relation

$$
\begin{aligned}
(x-N)_2 K_{n-2}(x; p, N-2) = \frac{p(n-1+Np-N)}{n-1} &K_n(x; p, N) + (x - n + 1 - Np) \\
&\times \frac{(N-n+1)(1-p)}{n-1} K_{n-1}(x; p, N)
\end{aligned}
\tag{21}
$$

obtained using (6) and (11). Firstly, we assume that the zeros of $K_n(x; p, N)$ lie in $(0, N-1)$. The function $(x-N)(x-N+1)$ in (21) is defined and does not change sign on the interval $(0, N-1)$. The same proof as that of [9, Theorem 2.1(i)] can be used for the interval $(0, N-1)$ and it follows that the zeros of $K_{n-2}(x; p, N-2)$, together with the point $C_n(2)$, interlace with the zeros of $K_n(x; p, N)$ on $(0, N-1)$. Since $C_n(2) = Np + n - 1 < N - 1$, we deduce that $p < 1 - n/N$.

Next, assume that the zeros of $K_{n-2}(x; p, N-2)$, together with the point $C_n(2)$, interlace with the zeros of $K_n(x; p, N)$ and $p < 1 - n/N$, that is, $C_n(2) = Np + n - 1 < N - 1$.

Suppose $x_{n,n}^N > N - 1$. Evaluating (21) at $x_{n,n-1}^N$ and $x_{n,n}^N$, we obtain

$$
\begin{aligned}
\frac{K_n(x_{n,n-1}^N; p, N) K_n(x_{n,n}^N; p, N) d^2}{(x_{n,n-1}^N - N)(x_{n,n}^N - N)(x_{n,n-1}^N - N + 1)(x_{n,n}^N - N + 1)} \\
= \frac{K_{n-2}(x_{n,n-1}^N; p, N-2) K_{n-2}(x_{n,n}^N; p, N-2)}{(x_{n,n-1}^N - C_n(2))(x_{n,n}^N - C_n(2))},
\end{aligned}
\tag{22}
$$

where $d = (N - n + 1)(1 - p)/(n-1)$. The zeros of $K_n(x; p, N)$ lie in $(0, N)$ and there is only one zero of $K_n(x; p, N)$ between any two mass points, therefore $x_{n,n-1}^N < N - 1$. Consequently the denominator on the left-hand side of (22) is negative. The numerator is also negative, since the zeros of $K_n(x; p, N)$ and $K_{n-1}(x; p, N)$ interlace, which implies that the left-hand side of (22) is positive.

By assumption, either a zero of $K_{n-2}(x; p, N-2)$ or the point $C_n(2)$, needs to lie in the interval $(x_{n,n-1}^N, x_{n,n}^N)$. If $C_n(2) \in (x_{n,n-1}^N, x_{n,n}^N)$, the denominator on the right-hand side is negative. For the right-hand side of (22) to be positive, we need $K_{n-2}(x_{n,n-1}^N; p, N-2)K_{n-2}(x_{n,n}^N; p, N-2) < 0$, which means there is also a zero of $K_{n-2}(x; p, N-2)$ in $(x_{n,n-1}^N, x_{n,n}^N)$ and we have a contradiction.

Alternatively, if $K_{n-2}(x; p, N-2)$ has a zero in the interval $(x_{n,n-1}^N, x_{n,n}^N)$, the numerator on the right-hand side is negative and for the right-hand side of (22) to be positive, we need the denominator to be negative, that is, $C_n(2) \in (x_{n,n-1}^N, x_{n,n}^N)$ and again, this contradicts our assumption. We conclude that $x_{n,n}^N < N - 1$.

Furthermore:

  (a) if $0 < 1 - (n+1)/N < p < 1 - n/N$, then $N - 2 < C_n(2) < N - 1$ and, because of the completed Stieltjes interlacing, $N - 2 < C_n(2) < x_{n,n}^N < N - 1$ Since there is at most one zero of $K_n(x; p, N)$ in between any two consecutive mass points, we have $x_{n,n-1}^N < N - 2$;
  (b) if $p = 1 - (n+1)/N$, then $C_n(2) = N - 2$ and $x_{n,n-1}^N < N - 2 < x_{n,n}^N < N - 1$.

(iv) If, for each $k \in \{0, 1, 2\}$, $K_{n-2}(x; p, N-k)$ and $K_n(x; p, N)$ have zeros in common, it follows from the three-term recurrence relation (for $k = 0$) and the mixed recurrence relation (20) and (21) (for $k = 1, 2$ respectively), together with Theorem 2.1(iii) in [9], that they can only have one common zero, which is equal to the point $C_n(k)$. The common zero must lie in $(0, N - k)$ and consequently $p < p_n(k)$. □

Proof of Theorem 3.3. We use the contiguous relation (12) and the $_2F_1$ representation of the Kravchuk polynomials (6) to obtain the mixed three-term recurrence relation

$$\begin{aligned}(n - 1)(N - n + 2)_2(1 - p)^2 K_{n-2}(x; p, N + 1) \\ = p(x + (n-1)p - N - 1)K_n(x; p, N) - P_2(x)K_{n-1}(x; p, N)\end{aligned} \tag{23}$$

where

$$\begin{aligned}P_2(x) &= x^2 - (n + N - 3(n-1)p + Np)x + n(n-1)p^2 - (N+1)((n-1)(3p-1) - Np) \\ &= (x - X_1)(x - X_2).\end{aligned}$$

It is easy to show that $X_1 \in (0, N)$ and $X_2 \in (0, N + 1)$ for $0 < p < 1$. In order to apply [9, Theorem 2.1] to (23), we need to determine the parameter values for which $(x - X_2)$ does not change sign on $(0, N)$, that is, we need to find the conditions on p so that $N \leqslant X_2 < N+1$.

(i) A straight-forward calculation shows that $X_2 \geqslant N$ is equivalent to

$$(n^2 - n)p^2 + (N - 3n + 3)p + n - 1 - N \leqslant 0.$$

By solving this quadratic inequality and taking in consideration the assumption that $p > 0$, we find

$$0 < p \leqslant \frac{3(n - 1) - N + \sqrt{9 - 22n + 17n^2 - 4n^3 + 6N - 10nN + 4n^2N + N^2}}{2n(n-1)} = q.$$

(ii) Let $K_{n-2}(x; p, N + 1)$ and $K_n(x; p, N)$ be co-prime, n and p fixed and $p \leqslant q$. We apply [9, Theorem 2.1(i)] to (23) and conclude that for $p \leqslant q$ the $n-2$ zeros of $K_{n-2}(x; p, N+1)$, together with the point X_1, interlace with the zeros of $K_n(x; p, N)$. A direct consequence of this interlacing is that the zeros of $K_{n-2}(x; p, N + 1)$ lie in $(0, N)$ for the specified values of p, since $x_{n-2,n-2}^{N+1} < x_{n,n}^N < N$.

(iii) Let n and $p \leqslant q$ be fixed. When we assume that $K_{n-2}(x; p, N + 1)$ and $K_n(x; p, N)$ have a common zero, this zero must lie in $(0, N)$ and the stated result follows from [9, Theorem 2.1(ii)].

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