CONNECTEDNESS IN THE SCALE OF UNIFORM SUBSPACES OF R

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Kent [4] showed that each uniform space $(S, \mathscr{U})$ could be embedded in a complete, uniform lattice, called the scale of $(S, \mathscr{U})$. The scale was first introduced by Bushaw [3] for studying stability in topological dynamics. In [5], the notions of connectedness and local connectedness were studied. This note is a follow-up of [5]; the purpose being to characterize the uniform subspaces of the reals, R, which have connected (locally connected) scales. The reader is asked to refer to [5] for definitions and notation not given here.

PROPOSITION 0. *In the scale $(P, \mathscr{V})$ of a metrizable uniform space, the various forms of connectedness are related as follows: arcwise connectedness $\Leftrightarrow$ connectedness $\Rightarrow$ local connectedness $\Leftrightarrow$ local arcwise connectedness.*

Let $(S, \mathscr{U})$ be metrizable; then by construction the retracted scale (see [4]) is metrizable. As mentioned in [5], the scale possesses any of the topological properties of Proposition 0 when and only when the retracted scale possesses this property. Hence, Proposition 0 follows from the fact that a connected, locally connected, complete metric space is arcwise connected (5.4 of [6]), and from the results of [5] which imply that the scale is locally connected whenever it is connected.

PROPOSITION 1. *The following statements about a uniform subspace of R are equivalent:*

(1) *$P(\mathscr{V})$ is connected;*
(2) *$(S, \mathscr{U})$ is bounded;*
(3) *$\bar{S}$ is a closed interval in R.*

PROOF. That (1) implies (2) follows from Theorem 1 of [5]). Assume condition (2); let $a = \inf S$, $b = \sup S$. Then S bounded, implies that it has no "gaps," and thus S is dense in $[a, b]$.

Assume condition (3), and let $U_\varepsilon = \{(x, y) \mid x, y \in S \text{ and } |x - y| < \varepsilon\}$. Then it is straightforward to show that $U_\varepsilon \circ U_\delta = U_{\varepsilon+\delta}$, for each positive ε, δ. Thus (3) implies (1) from Theorem 5 of [5].

We remark that in Proposition 5 of [5], "$V^n \subset U$" should be replaced by "$U \subset V^n$." This leads to the following definition:

Definition. *A uniform space $(S, \mathscr{U})$ is called partially bounded whenever there is an entourage U such that for any entourage V, V^n contains U, for some n.*

The notion of boundedness was studied in [2], in which it was shown that for compact spaces, boundedness and connectedness are equivalent. We use this result to prove the following:

Proposition 2. *A compact Hausdorff uniform space is partially bounded if and only if each point has a connected neighborhood.*

Proof. Let $(S, \mathscr{U})$ be compact, partially bounded, with U having the property mentioned in the above definition. It is well known that the component of a point x in S is $A_x = \cap \{A_{x,V} \mid V \in \mathscr{U}\}$, where $A_{x,V} = \cup \{V^n(x) \mid n \geqq 1\}$ (e.g. see [1]). Thus A_x contains $U(x)$, and so A_x is a connected neighborhood of x.

Conversely, let A_x denote a connected neighborhood of x, for each x in S. Since $(S, \mathscr{U})$ is compact Hausdorff, then there is an entourage of the form. $U = \cup \{A_i \times A_i \mid 1 \leqq i \leqq n\}$. Since A_i is compact and connected, then from the above remarks, the uniform subspace $(A_i, \mathscr{U}_i)$ is bounded, and so $(S, \mathscr{U})$ is partially bounded.

Partial boundedness is a necessary condition for the scale to be locally connected (Proposition 5 of [5]). As the next proposition points out, the converse is also true for uniform subspaces of R. We delete the proof since the details are similar to those given in Proposition 1 above. However, it should be noted that in Theorem 5 of [5], condition B can be relaxed to $U_\epsilon \circ U_\delta = U_{\varepsilon+\delta}$, for ε, δ sufficiently small, and used to prove that (3) implies (1) below.

Proposition 3. *The following statements about a uniform subspace of R are equivalent:*

(1) *$(P, \mathscr{V})$ is locally connected;*
(2) *$(S, \mathscr{U})$ is partially bounded;*
(3) *$\exists \varepsilon > 0$ such that for each $x, y \in S$ with $|x - y| < \varepsilon$, $[x,y] \subset \bar{S}$.*

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