VARIATION-DIMINISHING TRANSFORMATIONS AND GENERAL ORTHOGONAL POLYNOMIALS

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1. Introduction. Let $\alpha(dx)$ be a finite measure defined on the Borel subsets of $[-1, 1]$, the spectrum of which is infinite. Let $\{P(n, x)\}_0^\infty$ be the family of orthonormal polynomials associated with α, so that

$$\int_{-1}^{1} P(n, x)P(m, x)\alpha(dx) = \delta(n, m).$$

The $\{P(n, x)\}_0^\infty$ are uniquely determined by this and by the condition

$$p(n, n) > 0, \qquad n = 0, 1, \ldots,$$

where

$$P(n, x) = \sum_{\nu=0}^{n} p(n, \nu)x^\nu.$$

We denote by l^2 the space of those real functions $F(n)$ defined for $n = 0, 1, \ldots$ and such that

$$||F|| = \left\{ \sum_{n=0}^{\infty} F(n)^2 \right\}^{1/2} < \infty,$$

and we denote by L^2 the space of those real Borel measurable functions $f(x)$ defined for $[-1, 1]$ and such that

$$||f|| = \left\{ \int_{-1}^{1} f(x)^2 \alpha(dx) \right\}^{1/2} < \infty.$$

For $F \in l^2$ we define

$$F^\wedge(x) = \sum_{n=0}^{\infty} F(n)P(n, x).$$

Here the partial sums of the series on the right converge in the metric of L^2 and we have $||F^\wedge|| = ||F||$. For $f \in L^2$ we define

$$f^\vee(n) = \int_{-1}^{1} f(x)P(n, x)\alpha(dx).$$

and here $||f^\vee|| = ||f||$. Finally, it is well known that

$$\begin{aligned} (F^\wedge)^\vee &= F \qquad \text{for } F \in l^2, \\ (f^\vee)^\wedge &= f \qquad \text{for } f \in L^2. \end{aligned}$$

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Let $M(x)$ be a real bounded Borel measurable function on $[-1, 1]$, and let T_M be the associated multiplier transformation defined by

$$(T_M F)(n) = (F^\wedge M)^\vee (n).$$

Clearly T_M is a bounded self-adjoint transformation of l^2 into itself with $||T_M|| = ||M||_\infty$.

For $F(n)$ any real function defined for $n = 0, 1, \ldots$ we say that $F(n)$ has at least m changes of sign if there exist integers $0 \leqslant n_0 < n_1 < \ldots < n_m < \infty$ such that

$$F(n_{k-1})\, F(n_k) < 0, \qquad k = 1, \ldots, m.$$

$F(n)$ has m changes of sign, $\mathbf{V}[F] = m$, if F has at least m changes of sign but not at least $m + 1$ changes of sign. If F has at least m changes of sign for every m, then $\mathbf{V}[F] = +\infty$.

Definition 1*a*. *A multiplier $M(x)$ is said to be variation-diminishing if*

$$\mathbf{V}[T_M F] \leqslant \mathbf{V}[F]$$

for every $F \in l^2$.

Our objective in the present paper, which continues the work begun in **(4–7)**, is to obtain, under fairly general hypotheses on the measure $\alpha(dx)$, necessary and sufficient conditions that T_M be variation-diminishing. Let

$$\alpha(dx) = \alpha_s(dx) + \alpha_c(x)dx \tag{1}$$

be the decomposition of α into a singular part and an absolutely continuous part.

Definition 1*b*. *The measure $\alpha(dx)$ will be said to satisfy the conditions* **S** *if*:

$$\int_{-1}^{1} \log[\alpha_c(x)](1 - x^2)^{-1/2} dx > -\infty.$$

Our principal result is the following theorem.

Theorem 1*c*. *Let $\alpha(dx)$ satisfy condition* **S**. *Then $M(x)$ is a variation-diminishing multiplier if and only if $M(x)$ is of the form*

$$M(x) = de^{cx} \prod_k (1 + a_k x) \prod_k (1 - b_k x)^{-1}, \tag{2}$$

where d is real and

$$c \geqslant 0, \quad 1 \geqslant a_k > 0, \quad 1 > b_k > 0, \quad \textstyle\sum_k (a_k + b_k) < \infty. \tag{3}$$

Of course (2) need hold only almost everywhere with respect to $\alpha(dx)$.

The sufficiency of the conditions (2) and (3) is rather easy to prove and holds without restriction on α. The difficulty lies in the proof that these conditions are necessary. This was previously known for the special cases of Jacobi, Laguerre, and Hermite polynomials. In the present note the methods of **(6)** are extended to yield the general result described above.

2. Some preliminary results. In this section we shall collect some (though not all) of the information from other sources which we shall need later.

DEFINITION 2a. *A real matrix* $[A(n,m)]$, $n_1 < n < n_2$, $m_1 < m < m_2$, *is said to be variation-diminishing if*

$$h(n) = \sum_{m_1<m<m_2} A(n,m)g(m), \qquad n_1 < n < n_2,$$

implies $\mathbf{V}[h] \leqslant \mathbf{V}[g]$ *for every real function* $g(m)$, $m_1 < m < m_2$, *which is zero except for finitely many* m.

Here m_1 and n_1 can be $-\infty$ or finite and m_2 and n_2 can be finite or $+\infty$. For a demonstration of the following result see (**8**, Chap. 4) or (**2**, Chap. 5).

THEOREM 2b (Schoenberg). *If* $[A(n,m)]$, $n_1 < n < n_2$, $m_1 < m < m_2$, *is totally non-negative, then it is variation-diminishing.*

Note, a matrix $[A(n,m)]$ is said to be totally non-negative if, whenever

$$n_1 < \alpha_1 < \alpha_2 < \ldots < \alpha_r < n_2, \qquad m_1 < \beta_1 < \beta_2 < \ldots < \beta_r < m_2,$$

we have

$$\begin{vmatrix} A(\alpha_1,\beta_1) & A(\alpha_1,\beta_2) & \cdots & A(\alpha_1,\beta_r) \\ A(\alpha_2,\beta_1) & A(\alpha_2,\beta_1) & \cdots & A(\alpha_2,\beta_r) \\ \vdots & \vdots & & \vdots \\ A(\alpha_r,\beta_1) & A(\alpha_r,\beta_2) & \cdots & A(\alpha_r,\beta_r) \end{vmatrix} \geqslant 0.$$

A symmetric $(N+1) \times (N+1)$ matrix

$$J^{(N)} = [j(r,s)]_{r,s=0,\ldots,N}$$

is called a Jacobi matrix if

$$j(r,s) = 0 \qquad \text{whenever } |r-s| \geqslant 2.$$

Suppose that in addition the condition

$$(1) \qquad j(r,s) \geqslant 0 \qquad \text{for } |r-s| = 1$$

is satisfied.

LEMMA 2c. *If* $J^{(N)}$, *a symmetric Jacobi matrix satisfying* (1) *is positive semi-definite, it is totally non-negative.*

This is an immediate consequence of (**2**, p. 92, formula (38)).

A symmetric $(N+1) \times (N+1)$ matrix

$$U^{(N)} = [u(r,s)]_{r,s=0,\ldots,N}$$

is called one-paired if there exist sets of numbers $\psi(1),\ldots,\psi(n)$ and $\chi(1),\ldots,\chi(n)$ such that

$$u(r,s) = \psi(\min[r,s])\,\chi(\max[r,s]).$$

We further assume that

(2) $$\psi(k) > 0, \qquad k = 1, \ldots, n.$$

LEMMA 2d. *If the symmetric one-paired matrix $U^{(N)}$ satisfying* (2) *is positive definite it is totally non-negative.*

This follows from **(2**, p. 90, formula (28)**)**.

Finally, we need the following result, which unlike the preceding results has a long and difficult demonstration; see **(1)**.

THEOREM 2e (Edrei). *Let $M(e^{i\theta}) \in L^1(-\pi, \pi)$ and let $M(e^{i\theta})^* = M(e^{-i\theta})$. We define*

$$L(n) = \int_{-\pi}^{\pi} M(e^{i\theta}) e^{-in\theta} d\theta, \qquad -\infty < n < \infty.$$

If the infinite matrix $[L(n-m)]$, $-\infty < n, m < \infty$, is variation-diminishing, then writing z for $e^{i\theta}$ we have

$$M(z) = \xi z^N \exp(\epsilon_1 z + \epsilon_{-1} z^{-1}) \frac{\prod_k (1+\alpha_k z) \prod_k (1+\beta_k z^{-1})}{\prod_k (1-\gamma_k z) \prod_k (1-\delta_k z^{-1})},$$

where the α's, β's, γ's, δ's, ϵ's, and ξ are real, and $1 \geqslant \alpha_k > 0$, $1 \geqslant \beta_k > 0$, $1 > \gamma_k > 0$, $1 > \delta_k > 0$, $\epsilon_1 \geqslant 0$, $\epsilon_{-1} \geqslant 0$, *and finally N is an integer.*

Here "*" is the operation of complex conjugation.

3. Sufficiency. Our principal result in the present section is the following theorem.

THEOREM 3a. *If $\alpha(dx)$ is a finite measure on the Borel subsets of $[-1, 1]$ and if $M(x)$ is given by* (2) *and* (3) *of* § 1, *then $M(x)$ is variation-diminishing.*

Note that in this sufficiency proof it is *not necessary* to assume that $\alpha(dx) \in \mathbf{S}$.

We begin by listing several general principles.

(i) *If $\{G_k(n)\}_{k=1}^{\infty}$ and $G(n)$ are real functions defined for $n = 0, 1, 2, \ldots,$ and if*

$$G(n) = \lim_{k\to\infty} G_k(n)$$

for each $n = 0, 1, 2, \ldots,$ then

$$\mathbf{V}[G] \leqslant \varliminf_{k\to\infty} \mathbf{V}[G_k(n)].$$

(ii) *If $\{M_k\}_1^{\infty}$ are variation-diminishing multipliers and if*

$$\lim_{k\to\infty} ||M(x) - M_k(x)||_\infty = 0,$$

then $M(x)$ is a variation-diminishing multiplier. Here $||\cdot||_\infty$ is the uniform norm on $[-1, 1]$.

To see this, let $G = T_M F$, $G_k = T_{M_k} F$ for any $F \in l^2$ and apply (i).

(iii) *If $M_1(x)$ and $M_2(x)$ are variation-diminishing multipliers, then so is $M_1(x)M_2(x)$.*

Here we assume, of course, that $M_1(x)$ and $M_2(x)$ are bounded Borel measurable functions. Because the polynomials $P(n, x)$ are complete in $L^2(\alpha)$ it follows that $T_{M_1,M_2} = T_{M_1}T_{M_2}$ and thus

$$\mathbf{V}[T_{M_1M_2}F] = \mathbf{V}[T_{M_1}(T_{M_2}F)] \leqslant \mathbf{V}[T_{M_2}F] \leqslant \mathbf{V}[F].$$

Because of these principles it is clearly sufficient for the proof of Theorem 3a to validate the following two assertions:

I. *$a + x$ is a variation-diminishing multiplier if $a \geqslant 1$.*
II. *$(b - x)^{-1}$ is a variation-diminishing multiplier if $b > 1$.*

Let B be any bounded linear transformation of l^2 into itself. Let l_N^2 be the subspace of l^2 generated by those functions $F \in l^2$ for which $F(n) = 0$ for $n > N$, and let E_N be the projection of l^2 onto l_N^2. Finally let us denote by $B^{(N)}$ the restriction of $E_N BE_N$ to l_N^2. It is obvious that $B \geqslant 0$ implies $B^{(N)} \geqslant 0$ and $B > 0$ implies $B^{(N)} > 0$.

Proof of I. It is well known (see **11** or **12**) that the $P(n, x)$'s satisfy a recursion formula

$$xP(n, x) = A(n)P(n-1, x) + B(n)P(n, x) + C(n)P(n+1, x),$$

where, with the normalization we have used,

$$A(n+1) = C(n) > 0, \qquad n = 0, 1, \ldots. \tag{1}$$

If $a \geqslant 1$, then $a + x \geqslant 0$ on $[-1, 1]$ and therefore $T_{(a+x)} \geqslant 0$. By the remark above $T_{(a+x)}^{(N)} \geqslant 0$. However, the action of $T_{(a+x)}$ on l_N^2 is by means of the matrix

$$J^{(N)} = [j(r, s)]_{r,s=0,\ldots,N},$$

where

$$j(r, s) = \int_{-1}^{1} (a + x)P(r, x)P(s, x)\alpha(dx).$$

Using (1) it is evident that $J^{(N)}$ satisfies (1) of § 2. Therefore, by Lemma 2c and Theorem 2b, $J^{(N)}$ is variation-diminishing. For $F \in l^2$ we have

$$T_{(a+x)}F(n) = \lim_{N\to\infty} T_{(a+x)}^{(N)}F\cdot(n), \qquad n = 0, 1, \ldots;$$

appealing to principle (i), our proof is complete.

Proof of II. Let us define

$$Q(n, x) = \int_{-1}^{1} (x - y)^{-1}P(n, y)\alpha(dy), \qquad n = 0, 1, \ldots.$$

We assert that

$$(2) \qquad \int_{-1}^{1} (b-x)^{-1} P(r,x)P(s,x)\alpha(dx) = P(\min[r,s], b)Q(\max[r,s], b).$$

We have, if $r \leqslant s$,

$$\int_{-1}^{1} (b-x)^{-1} P(r,x)P(s,x)\alpha(dx) = \int_{-1}^{1} \frac{P(r,x) - P(r,b)}{(b-x)} P(s,x)\alpha(dx)$$
$$+ P(r,b) \int_{-1}^{1} (b-x)^{-1} P(s,x)\alpha(dx).$$

Since $[P(r,x) - P(r,b)]\,(b-x)^{-1}$ is a polynomial of degree $r-1 < s$, the first integral on the right vanishes, and so on. If $b > 1$, then $(b-x)^{-1} > 0$ on $[-1, 1]$ and therefore $T_{(b-x)^{-1}} > 0$. By a remark above $T^{(N)}_{(b-x)^{-1}} > 0$. The action of $T^{(N)}_{(b-x)^{-1}}$ on l_N^2 is by means of the matrix

$$U^{(N)} = [u(r,s)]_{r,s=0,\ldots,N},$$

where

$$u(r,s) = \int_{-1}^{1} (b-x)^{-1} P(r,x)P(s,x)\alpha(dx).$$

Using (2) it is evident that $U^{(N)}$ satisfies (2) of § 2. (Note that $P(n,b) > 0$, $n = 0, 1, \ldots$, if $b > 1$.) Therefore, by Lemma 2d and Theorem 2b, $U^{(N)}$ is variation-diminishing, and so on.

4. Necessity. Extensive use will be made in this section of Szegö's results given in **(3**, Chap. 2**)** and **(12**, Chaps. 11 and 12**)**. Let $\alpha(dx)$ be a measure on the Borel subsets of $[-1, 1]$. We define a measure $\omega(d\theta)$ on $T = (-\pi, \pi]$ by setting

$$(1) \qquad \omega(E) = \alpha(A(E \cap U)) + \alpha(A(E \cap L)),$$

where $A(\theta) = \cos\theta$ and where $U = \{\theta | \, 0 < \theta \leqslant \pi\}$, $L = \{\theta | -\pi < \theta \leqslant 0\}$. Let $\Phi(n, z)$ be the Szegö polynomials with respect to $\omega(d\theta)$; that is

$$\Phi(n,z) = \sum_{\nu=0}^{n} \phi(n,\nu) z^{\nu}, \qquad n = 0, 1, \ldots,$$

$$(2) \qquad \phi(n,n) > 0,$$

$$\int_T \Phi(n, e^{i\theta}) \Phi(m, e^{i\theta})^* \omega(d\theta) = \delta(n,m).$$

Here "*" represents complex conjugation. Note that because ω is "essentially" even

$$\Phi(n, e^{i\theta})^* = \Phi(n, e^{-i\theta}).$$

The following connection exists between the $P(n,x)$'s and the $\Phi(n,z)$'s; see **(12**, p. 287**)**. If $x = \frac{1}{2}(z + z^{-1})$, then

$$(3) \quad P(n,x) = \left\{ \frac{\phi(2n, 2n)}{2\pi(\phi(2n,2n) + \phi(2n, 0))} \right\}^{1/2} \{z^{-n}\Phi(2n, z) + z^{n}\Phi(2n, z^{-1})\}.$$

From this point on we assume that $\alpha(dx)$ satisfies condition **S**. If

$$\omega(d\theta) = \omega_s(d\theta) + \omega_c(\theta)d\theta$$

is the decomposition of $\omega(d\theta)$ into its singular and absolutely continuous parts, then

$$\int_T \log \omega_c(\theta)d\theta = 2 \int_{-1}^{1} \log \alpha_c(x)(1 - x^2)^{-1/2}dx > -\infty.$$

Consequently we may define, for $|z| < 1$,

$$\log g(z) = (4\pi)^{-1} \int_T \log \omega_c(\psi) \frac{1 + ze^{-i\psi}}{1 - ze^{-i\psi}} d\psi. \tag{4}$$

It can be verified that $g(z)$ is analytic for $|z| < 1$, that

$$g(e^{i\theta}) = \lim_{r \to 1-} g(re^{i\theta}) \tag{5}$$

exists for almost all θ (Lebesgue measure), and that for almost all θ

$$|g(e^{i\theta})|^2 = \omega_c(\theta). \tag{6}$$

Furthermore,

$$\lim_{n\to\infty} \int_T |\Phi(n, e^{i\theta})e^{-in\theta}g(e^{i\theta})^* - 1|^2\omega_c(\theta)d\theta = 0, \tag{7}$$

$$\lim_{n\to\infty} \int_T |\Phi(n, e^{i\theta})|^2\omega_s(d\theta) = 0. \tag{8}$$

Finally, we note that

$$\lim_{n\to\infty} \phi(n, n) = \phi, \tag{9}$$

where

$$\phi = \exp\left\{\frac{1}{4\pi} \int_T \log \omega_c(\theta)d\theta\right\}$$

and

$$\lim_{n\to\infty} \phi(n, 0) = 0. \tag{10}$$

For all such results see (**12**, Chap. 11).

THEOREM 4a. *If $\alpha(dx) \in$ **S** and if $M(x)$ is a variation-diminishing multiplier, then $M(x)$ has the form* (2) *and* (3) *of* § 1.

We introduce the matrix $[K(n, m)]_{n,m=0,1,\ldots}$, where

$$K(n, m) = \int_{-1}^{1} P(n, x)P(m, x)M(x)\alpha(dx). \tag{11}$$

We also set for $-\infty < n, m < \infty$

$$L(n - m) = 4a \int_0^\pi \cos[(n - m)\theta]M(\cos\theta)d\theta, \tag{12}$$

where a is a positive constant described below. Note that if r is sufficiently large and positive, then $n + r \geqslant 0$, $m + r \geqslant 0$. We shall show that

$$\lim_{r\to\infty} K(n + r, m + r) = L(n - m), \qquad -\infty < n, m < \infty. \tag{13}$$

We have, by an evident change of variables,

$$\begin{aligned} K(n + r, m + r) &= \frac{1}{2} \int_T P(n + r, \cos\theta)P(m + r, \cos\theta)M(\cos\theta)\omega(d\theta) \\ &= a(n, m, r) \int_T \{e^{-i(n+r)\theta}\Phi(2n + 2r, e^{i\theta}) + e^{i(n+r)\theta}\Phi(2n + 2r, e^{-i\theta})\} \\ &\qquad \times\{e^{-i(m+r)\theta}\Phi(2m + 2r, e^{i\theta}) + e^{i(m+r)\theta}\Phi(2m + 2r, e^{-i\theta}\} \\ &\qquad \times M(\cos\theta)\omega(d\theta). \end{aligned}$$

Using (3), (9), and (10) we see that

$$\lim_{r\to\infty} a(n, m, r) = a,$$

where a is a positive constant independent of m and n. We have

$$K(n + r, m + r)a(n, m, r)^{-1} = I + I_1 + I_2 + I_3 + I_4,$$

where

$$I = \int_T \{\cdots\}\{\cdots\}M(\cos\theta)\omega_s(d\theta),$$

and

$$I_1 = \int_T e^{-i(n+r)\theta}\Phi(2n + 2r, e^{i\theta})e^{-i(m+r)\theta}\Phi(2m + 2r, e^{i\theta})M(\cos\theta)\omega_c(\theta)d\theta,$$

$$I_2 = \int_T e^{-i(n+r)\theta}\Phi(2n + 2r, e^{i\theta})e^{i(m+r)\theta}\Phi(2m + 2r, e^{-i\theta})M(\cos\theta)\omega_c(\theta)d\theta,$$

$$I_3 = \int_T e^{i(n+r)\theta}\Phi(2n + 2r, e^{-i\theta})e^{-i(m+r)\theta}\Phi(2m + 2r, e^{i\theta})M(\cos\theta)\omega_c(\theta)d\theta,$$

$$I_4 = \int_T e^{i(n+r)\theta}\Phi(2n + 2r, e^{-i\theta})e^{i(m+r)\theta}\Phi(2m + 2r, e^{-i\theta})M(\cos\theta)\omega_c(\theta)d\theta.$$

It follows from (8) that

$$\lim_{r\to\infty} I = 0.$$

Recalling (6) we find that

$$\begin{aligned} I_1 = \int_T e^{i(n+m+2r)\theta}&\{\Phi(2n + 2r, e^{i\theta})e^{-2i(n+r)\theta}g(e^{i\theta})^*\} \\ &\times\{\Phi(2m + 2r, e^{i\theta})e^{-2i(m+r)\theta}g(e^{i\theta})^*\} \\ &\times|g(e^{i\theta})||g(e^{i\theta})^*|^{-1}M(\cos\theta)d\theta. \end{aligned}$$

By (7) and the Riemann-Lebesgue theorem, we have

$$\lim_{r\to\infty} I_1 = 0,$$

and similarly

$$\lim_{r\to\infty} I_4 = 0.$$

Again,

$$I_2 = \int_T e^{i(n-m)\theta}\{\Phi(2n+2r, e^{i\theta})e^{-2i(n+r)\theta}g(e^{i\theta})^*\} \times\{\Phi(2m+2r, e^{-i\theta})e^{2i(m+r)\theta}g(e^{i\theta})\} \times M(\cos\theta)d\theta.$$

Using (7) it follows that

$$\lim_{r\to\infty} I_2 = \int_T e^{i(n-m)\theta}M(\cos\theta)d\theta,$$

and similarly

$$\lim_{r\to\infty} I_3 = \int_T e^{-i(n-m)\theta}M(\cos\theta)d\theta.$$

Combining these results we see that

$$\lim_{r\to\infty} K(n+r, m+r) = 4a\int_0^\pi \cos[(n-m)\theta]M(\cos\theta)d\theta$$

and we have proved (12). Since the limit of variation-diminishing matrices is again variation-diminishing, we see that $[L(n,m)]$, $-\infty < n, m < \infty$ is variation-diminishing. Using Theorem 2e and taking into account the evenness of $L(n)$, we see that

$$M(\cos\theta) = \xi\exp[\epsilon(e^{i\theta}+e^{-i\theta})]\frac{\prod_k(1+\alpha_k e^{i\theta})\prod_k(1+\alpha_k e^{-i\theta})}{\prod_k(1-\gamma_k e^{i\theta})\prod_k(1-\gamma_k e^{-i\theta})},$$

where $\epsilon \geqslant 0$, $0\leqslant\alpha_k\leqslant 1$, $0\leqslant\gamma_k<1$, and $\sum_k(\alpha_k+\gamma_k)<\infty$. Setting $d=\xi$, $c=2\epsilon$, $2\alpha_k(1+\alpha_k^2)^{-1}=a_k$, $2\gamma_k(1+\gamma_k^2)^{-1}=b_k$, and $x=\cos\theta$, we obtain our desired result.

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