RIEMANN'S METHOD AND THE CHARACTERISTIC VALUE AND CAUCHY PROBLEMS FOR THE DAMPED WAVE EQUATION†

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1. Introduction. Riemann's method for solving the Cauchy problem for hyperbolic differential equations in two independent variables has been extended in a number of papers [4], [5], [2] to the wave equation in space of higher dimensions. The method, which consists in the determination of a so-called Riemann function, hinges on the solution of a characteristic value problem. Accordingly, if Riemann's method is to be used in solving a characteristic value problem, one will have to consider another characteristic value problem and thus the process becomes circular. This difficulty was first overcome by Protter [7] in solving the characteristic value problem for the wave equation in three variables. There he employed a variation of Riemann's method developed by Martin [5]. Martin's result was later extended by Diaz and Martin [2] to the wave equation in an arbitrary number of variables. This made it possible to extend Protter's result to the wave equation in space of higher dimensions [8].

In this paper we show how the modified Riemann's method developed in [5] can be used to obtain an explicit solution to the characteristic value problem and the Cauchy problem for the damped wave equation

$$u_{tt}-u_{xx}-u_{yy}-4cu=0 \tag{1}$$

in three variables. At the end of the paper we make a remark concerning the extension to the case of the damped wave equation in an arbitrary number of variables.

2. The characteristic value problem. For convenience we suppose that the initial condition is prescribed on the direct characteristic cone

$$K \equiv t^2-x^2-y^2=0 \qquad (t \geqq 0)$$

with vertex at the origin, so that

$$u(x,y,r)=\psi(x,y), \qquad r=(x^2+y^2)^{\frac{1}{2}}. \tag{2}$$

The function $\psi(x,y)$ will be assumed to have continuous derivatives up to at least the second order. We then seek a solution of (1) at points interior to $K=0$ which assumes the prescribed values (2).

First we introduce the new coordinates α, β, ϕ defined by the relations

$$x=r\cos\phi, \qquad y=r\sin\phi, \qquad t=\tfrac{1}{2}(\alpha+\beta), \tag{3}$$

with $r=\frac{1}{2}(\alpha-\beta)$. The coordinates α, β are called characteristic since $\alpha=$ const., $\beta=$ const.

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represent characteristic cones with vertices on the t-axis. Under the transformation (3), equation (1) becomes

$$L(u) \equiv u_{\alpha\beta} - \frac{u_\alpha - u_\beta}{2(\alpha-\beta)} - \frac{u_{\phi\phi}}{(\alpha-\beta)^2} - cu = 0. \tag{4}$$

We associate with this the equation

$$M(v) \equiv v_{\alpha\beta} + \frac{v_\alpha - v_\beta}{2(\alpha-\beta)} - cv = 0 \tag{5}$$

for a function $v(\alpha, \beta)$ depending only on the two characteristic coordinates. This equation plays the role of the adjoint equation in our investigation.

If we write

$$A = u_\beta v_\beta + cuv, \quad B = -(u_\alpha v_\alpha + cuv), \quad C = \frac{v_\alpha - v_\beta}{(\alpha-\beta)^2} u_\phi,$$

it is easy to verify that

$$A_\alpha + B_\beta + C_\phi = (v_\beta - v_\alpha)L(u) + (u_\beta - u_\alpha)M(v).$$

Thus, if D is any domain in (α, β, ϕ) space bounded by a surface S, it follows by Green's theorem that

$$I \equiv \iint_S (A\nu_\alpha + B\nu_\beta + C\nu_\phi)\, dS = \iiint_D [(v_\beta - v_\alpha)L(u) + (u_\beta - u_\alpha)M(v)]\, d\alpha\, d\beta\, d\phi, \tag{6}$$

where $\nu_\alpha, \nu_\beta, \nu_\phi$ are the components of the unit outward normal to S.

It suffices to find the solution at points on the axis of the cone $K = 0$, for once the solution is obtained there we may employ a Lorentz transformation to obtain the solution at all interior points of the cone (see [6, p. 371]). Thus let $(0, 0, \tau)$ $(\tau \geqq 0)$ be the point at which we wish to determine the solution. Denote by D the domain bounded by the characteristic half cones

$$D: \begin{cases} x^2 + y^2 = t^2, & 0 \leqq t \leqq \frac{1}{2}\tau, \\ x^2 + y^2 = (t-\tau)^2, & \frac{1}{2}\tau \leqq t \leqq \tau. \end{cases}$$

Regarding (α, β, ϕ) as a rectangular coordinate system, we see that under the transformation of coordinates (3), D is transformed into the triangular prism

$$T: \begin{cases} 0 \leqq \beta \leqq \tau, \quad \beta \leqq \alpha \leqq \tau, \\ \qquad 0 \leqq \phi \leqq 2\pi, \end{cases}$$

in which the cone $K = 0$ becomes the plane $\beta = 0$, and the point $(0, 0, \tau)$ appears as the edge $\alpha = \beta = \tau$.

Now if u is a solution of (1) and if v is any regular function yet to be determined such that $M(v) = 0$, then formula (6) applied to the domain T yields

$$\underset{\beta=0}{I} + \underset{\alpha=\tau}{I} + \underset{\alpha=\beta}{I} + \underset{\phi=0}{I} + \underset{\phi=2\pi}{I} = 0.$$

The last two integrals above taken over the faces $\phi = 0$, $\phi = 2\pi$ cancel each other since u must be periodic of period 2π and the normals to these planes have opposite signs. Therefore, when the surface integrals in the first three terms are written out explicitly, we obtain

$$\int_0^{2\pi}\int_0^{\tau}[u_\alpha v_\alpha + cuv]_{\beta=0}\,d\alpha\,d\phi + \int_0^{2\pi}\int_0^{\tau}[u_\beta v_\beta + cuv]_{\alpha=\tau}\,d\beta\,d\phi$$
$$-\int_0^{2\pi}\int_0^{\tau}[u_\alpha v_\alpha + u_\beta v_\beta + 2cuv]_{\alpha=\beta}\,d\alpha\,d\phi. \qquad (7)$$

3. Determination of $v(\alpha, \beta)$ and the solution. We now try to determine the " Riemann function " $v(\alpha, \beta)$ satisfying the associate equation (5) and such that v and v_β both vanish when $\alpha = \tau$. Choose $v = V(\theta)$, where $\theta = (\tau-\alpha)(\tau-\beta)$ (see [9], p. 229). Then (5) becomes

$$M(v) \equiv v_{\alpha\beta} + \frac{v_\alpha - v_\beta}{2(\alpha-\beta)} - cv = \theta V'' + \tfrac{1}{2}V' - cV = 0,$$

where primes indicate differentiation with respect to θ. The last equation above is a Bessel equation whose two independent solutions are $z^{\frac{1}{2}}I_{\frac{1}{2}}(z)$ and $z^{\frac{1}{2}}I_{-\frac{1}{2}}(z)$, where $I_p(z)$ denotes the modified Bessel function of order p and $z = 2(c\theta)^{\frac{1}{2}}$. By a well known relation, the above solutions can be written as constant multiples of $\sinh z$ and $\cosh z$, respectively.

For our purpose we take $v = \sinh z$, $z = 2[c(\tau-\alpha)(\tau-\beta)]^{\frac{1}{2}}$. It is clear that v and v_β both vanish when $\alpha = \tau$. On $\alpha = \beta$, we note that $r = 0$, $\alpha = t$,

$$\tfrac{1}{2}(v_\alpha - v_\beta) = 0, \qquad \tfrac{1}{2}(v_\alpha + v_\beta) = -c^{\frac{1}{2}}\cosh[2c^{\frac{1}{2}}(\tau - t)],$$

and on $\beta = 0$ we have

$$v = \sinh[2\{c(\tau-\alpha)\tau\}^{\frac{1}{2}}],$$
$$v_\alpha = -\left[\frac{c\tau}{\tau-\alpha}\right]^{\frac{1}{2}}\cosh[2\{c(\tau-\alpha)\tau\}^{\frac{1}{2}}].$$

Moreover, from (3) we note that

$$u_\alpha = \tfrac{1}{2}(u_t + u_r) \qquad \text{and} \qquad u_\beta = \tfrac{1}{2}(u_t - u_r).$$

Substituting these values in (7), we then obtain

$$\int_0^{2\pi}\int_0^{\tau}[u_\alpha v_\alpha + cuv]_{\beta=0}\,d\alpha\,d\phi$$
$$= \int_0^{2\pi}\int_0^{\tau} -c^{\frac{1}{2}}u_t\cosh[2c^{\frac{1}{2}}(\tau-t)]\,dt\,d\phi + 2\int_0^{2\pi}\int_0^{\tau} u\sinh[2c^{\frac{1}{2}}(\tau-t)]\,dt\,d\phi.$$

The integration on the right above is performed along the t-axis. In the integral on the left, u and hence u_α are known on $\beta = 0$. Thus, after performing an integration by parts in the first integral on the right and rearranging terms, we have finally the formula

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$$u(0,0,\tau) = \psi(0,0)\cosh(2c^{\frac{1}{2}}\tau)$$
$$-\frac{1}{2\pi}\int_0^{2\pi}\int_0^{\tau}\left\{c^{\frac{1}{2}}\psi\sinh[2\{c(\tau-\alpha)\tau\}^{\frac{1}{2}}]-\left(\frac{\tau}{\tau-\alpha}\right)^{\frac{1}{2}}\psi_\alpha\cosh[2\{c(\tau-\alpha)\tau\}^{\frac{1}{2}}]\right\}d\alpha\, d\phi. \tag{8}$$

This gives the solution of the characteristic value problem (1), (2) for points on the t-axis. For a point (ξ,η,τ) inside $K = 0$ not lying on the t-axis we apply a Lorentz transformation [6, (1.6)] which takes (ξ,η,τ) into $(0,0,(\tau^2-\xi^2-\eta^2)^{\frac{1}{2}})$ and then use (8). That the solution obtained actually satisfies (1), (2) may be established in a manner similar to that performed by d'Adhemar [1].

When $c = 0$, (8) coincides with (1.4) of [6] provided we note that $2r = \alpha$ on $\beta = 0$.

4. The Cauchy problem. It is well known that the solution to the Cauchy problem

$$u(x,y,0) = f(x,y), \qquad u_t(x,y,0) = g(x,y) \tag{9}$$

for the damped wave equation (1) is given (see for example [3], p. 209) by

$$2\pi u(\xi,\eta,\tau) = F(g)+\frac{\partial}{\partial\tau}F(f), \tag{10}$$

where

$$F(h) = \int_0^{2\pi}\int_0^{\tau}\frac{h(\xi+r\cos\phi,\eta+r\sin\phi)\cosh 2[c(\tau^2-r^2)]^{\frac{1}{2}}}{(\tau^2-r^2)^{\frac{1}{2}}}\,r\,dr\,d\phi$$

with $(x-\xi)^2+(y-\eta)^2 = r^2$. In order to be able to compare this with the solution to be obtained here, we shall rewrite (10) as follows.

On setting $r = \rho\tau$ and differentiating under the integral sign, we find that

$$\frac{\partial}{\partial\tau}F(f) = \frac{1}{\tau}\int_0^{2\pi}\int_0^{\tau}\frac{f\cosh 2[c(\tau^2-r^2)]^{\frac{1}{2}}}{(\tau^2-r^2)^{\frac{1}{2}}}\,r\,dr\,d\phi$$
$$+\frac{1}{\tau}\int_0^{2\pi}\int_0^{\tau}\frac{f_r\cosh 2[c(\tau^2-r^2)]^{\frac{1}{2}}}{(\tau^2-r^2)^{\frac{1}{2}}}\,r^2\,dr\,d\phi$$
$$+\frac{4c}{\tau}\int_0^{2\pi}\int_0^{\tau}(\tau^2-r^2)^{\frac{1}{2}}f\sinh 2[c(\tau^2-r^2)]^{\frac{1}{2}}r\,dr\,d\phi.$$

After an integration by parts in the first term and a little simplification, this reduces to

$$\frac{\partial}{\partial\tau}F(f) = 2\pi f(\xi,\eta)\cosh 2c^{\frac{1}{2}}\tau+\int_0^{2\pi}\int_0^{\tau}\frac{f_r\cosh 2[c(\tau^2-r^2)]^{\frac{1}{2}}}{(\tau^2-r^2)^{\frac{1}{2}}}\,dr\,d\phi.$$

Thus the solution (10) takes the form

$$u(\xi,\eta,\tau) = f(\xi,\eta)\cosh 2c^{\frac{1}{2}}\tau+\frac{1}{2\pi}\int_0^{2\pi}\int_0^{\tau}\frac{\tau f_r+rg}{(\tau^2-r^2)^{\frac{1}{2}}}\,dr\,d\phi. \tag{11}$$

Now, to solve the problem (1), (9) by Riemann's method, we introduce the coordinates (α, β, ϕ) defined by the relations

$$x = \xi + \tfrac{1}{2}(\alpha-\beta)\cos\phi, \qquad y = \eta + \tfrac{1}{2}(\alpha-\beta)\sin\phi, \qquad t = \tfrac{1}{2}(\alpha+\beta). \tag{12}$$

If we write $r^2 = (x-\xi)^2 + (y-\eta)^2$, then we have $r = \frac{1}{2}(\alpha-\beta)$. Under (12) the characteristic retrogade cone

$$(t-\tau)^2 - (x-\xi)^2 - (y-\eta)^2 = 0 \qquad (\tau \geqq t \geqq 0)$$

is transformed in the (α, β, ϕ) space into the wedge

$$W\colon\ 0 \leqq \alpha \leqq \tau, \qquad -\alpha \leqq \beta \leqq \alpha, \qquad 0 \leqq \phi \leqq 2\pi,$$

with the face $\alpha = -\beta$ corresponding to the plane $t = 0$ on which the initial data are prescribed. Applying formula (6) over the wedge W and using the same argument as in Section 2, we obtain

$$\int_0^{2\pi}\int_{-\tau}^{\tau}[u_\beta v_\beta + cuv]_{\alpha=\tau}\,d\beta\,d\phi + \int_0^{2\pi}\int_0^{\tau}[u_\alpha v_\alpha - u_\beta v_\beta]_{\alpha=-\beta}\,d\alpha\,d\phi$$
$$-\int_0^{2\pi}\int_0^{\tau}[u_\alpha v_\alpha + u_\beta v_\beta + 2cuv]_{\alpha=\beta}\,d\alpha\,d\phi.$$

Again we take $v = \sinh z$, $z = 2\{c(\tau-\alpha)(\tau-\beta)\}^{\frac{1}{2}}$, so that the first term above drops out. On $\alpha = -\beta$, where the values of u are known, we note from (9) and (12) that

$$u_\alpha = \tfrac{1}{2}(u_t + u_r) = \tfrac{1}{2}(g + f_r)$$

and

$$u_\beta = \tfrac{1}{2}(u_t - u_r) = \tfrac{1}{2}(g - f_r).$$

Further, $\alpha = -\beta$ implies that $r = \alpha$ so that

$$\tfrac{1}{2}(v_\alpha - v_\beta) = -\frac{c^{\frac{1}{2}} r \cosh[2\{c(\tau^2 - r^2)\}^{\frac{1}{2}}]}{(\tau^2 - r^2)^{\frac{1}{2}}}$$

and

$$\tfrac{1}{2}(v_\alpha + v_\beta) = -\frac{c^{\frac{1}{2}} \tau \cosh[2\{c(\tau^2 - r^2)\}^{\frac{1}{2}}]}{(\tau^2 - r^2)^{\frac{1}{2}}}.$$

Substituting these values in (13), we finally obtain

$$u(\xi, \eta, \tau) = f(\xi, \eta)\cosh 2c^{\frac{1}{2}}\tau + \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\tau}\frac{\tau f_r + rg}{(\tau^2 - r^2)^{\frac{1}{2}}}\,dr\,d\phi,$$

which is (11).

5. Concluding remark. For the damped wave equation in more than three variables, the same problems may be solved by using the extension of Martin's result given by Diaz and Martin [2]. In this case the associate equation (5) becomes

$$v_{\alpha\beta} + \frac{\frac{1}{2}(n-1)}{(\alpha-\beta)^2}(v_\alpha - v_\beta) - cv = 0,$$

where n denotes the number of space variables, and the corresponding Riemann function $v(\alpha, \beta)$ is $z^{(n-1)/2}I_{(n-1)/2}(z)$. However, the solution $u(0,\tau)$ or $u(\xi,\tau)$, $\xi = (\xi_1, \ldots, \xi_n)$, must be obtained by solving an integral equation of Volterra type involving a Bessel function as a kernel.

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