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ORLICZ SPACES WITHOUT STRONGLY EXTREME POINTS AND WITHOUT H-POINTS

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ABSTRACT. W. Kurc [5] has proved that in the unit sphere of Orlicz space $L^{\Phi}(\mu)$ generated by an Orlicz function Φ satisfying the suitable Δ_2-condition and equipped with the Luxemburg norm every extreme point is strongly extreme. In this paper it is proved in the case of a nonatomic measure μ that the unit sphere of the Orlicz space $L^{\Phi}(\mu)$ generated by an Orlicz function Φ which does not satisfy the suitable Δ_2-condition and equipped with the Luxemburg norm has no strongly extreme point and no H-point.

0. **Introduction.** In the sequel $\mathbb{R}$ denotes the reals, $\mathbb{R}_+$ denotes the nonnegative reals and Φ denotes an arbitrary *Orlicz function*, *i.e.* $\Phi\colon \mathbb{R} \longrightarrow \mathbb{R}_+$, $\Phi(0) = 0$, and Φ is even and convex. (T, Σ, μ) denotes a positive nonatomic (finite or infinite) measure space. $L^0(\mu)$ stands for the space of (equivalence classes of) all Σ-measurable real functions defined on T.

Given an Orlicz function Φ we define on $L^0(\mu)$ a convex functional I_Φ by

$$I_\Phi(x) = \int_T \Phi\big(x(t)\big)\, d\mu \quad \big(\forall x \in L^0(\mu)\big).$$

This functional is a convex modular on $L^0(\mu)$ (see [7]), *i.e.* $I_\Phi(0) = 0$, I_Φ is convex and even and $x = 0$ whenever $I_\Phi(\lambda x) = 0$ for any $\lambda > 0$. The *Orlicz space* $L^{\Phi}(\mu)$ generated by an Orlicz function Φ is defined to be the set of all $x \in L^0(\mu)$ such that $I_\Phi(\lambda x) < \infty$ for some $\lambda > 0$ depending on x. This space can be endowed with the norm

$$\|\varkappa\|_\Phi = \inf\{\lambda < 0 : I_\Phi(x/\lambda) \leq 1\} \quad \big(\forall x \in L^{\Phi}(\mu)\big),$$

called the *Luxemburg norm*. The couple $\big(L^{\Phi}(\mu), \|\ \|_\Phi\big)$ is a Banach space (see [4], [6] and [7]).

Recall that an Orlicz function Φ satisfies the Δ_2-*condition* for all $u \in \mathbb{R}$ (at infinity) if there are positive constants K and c such that $0 < \Phi(c) < +\infty$ and $\Phi(2u) \leq K\Phi(u)$ for any $u \in \mathbb{R}$ (for $u \in \mathbb{R}$ satisfying $|u| \geq c$).

We say that an Orlicz function Φ satisfies the *suitable* Δ_2-*condition* if Φ satisfies the Δ_2-condition for all $u \in \mathbb{R}$ whenever μ is infinite and Φ satisfies the Δ_2-condition at infinity whenever μ is finite.

For an arbitrary Banach space X, $S(X)$ denotes its unit sphere.

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A point $x \in S(X)$ is said to be *strongly extreme* (see [3] and [8]) if for any sequence (x_n) in X the conditions $\|x+x_n\| \longrightarrow 1$ and $\|x-x_n\| \longrightarrow 1$ imply that $\|x_n\| \longrightarrow 0$.

A point $x \in S(X)$ is said to be an *H-point* if for any sequence (x_n) in X such that $\|x_n\| \longrightarrow \|x\|$ and x_n tends weakly to x, we have $\|x-x_n\| \longrightarrow 0$.

1. **Results.** We start with the following:

THEOREM 1. *Let Φ be an Orlicz function which does not satisfy the suitable Δ_2-condition. Let us assume additionally in the case when μ is infinite that Φ vanishes only at zero. Then $S(L^\Phi)$ has no strongly extreme point.*

PROOF. It is known that every strongly extreme point is extreme and that under the assumptions concerning Φ, if $x \in S(L^\Phi)$ is extreme then it must be $I_\Phi(x) = 1$ (see [2]). Therefore, it suffices to consider only these points of $S(L^\Phi)$ for which $I_\Phi(x) = 1$.

Assume that $I_\Phi(x) = 1$, μ is finite (for infinite μ the proof is analogous) and Φ does not satisfy the Δ_2-condition at infinity. Then there exists a sequence (u_n) of positive reals with $u_n \longrightarrow \infty$ as $n \longrightarrow \infty$ and such that

$$\Phi\Big(\Big(1+\frac{1}{n}\Big)u_n\Big) > 2^{-n}\Phi(u_n).$$

Let $b > 0$ be large enough, so that the set $B = \{t \in T : b^{-1} \leq |x(t)| \leq b\}$ has positive measure. Let $A_n \subset B, A_n \in \Sigma\ (n = 1, 2, \ldots)$ be such that $\Phi(u_n)\mu(A_n) = 2^{-n}$ (if necessary we can pass to a subsequence). Of course, $\mu(A_n) \longrightarrow 0$ as $n \longrightarrow \infty$. Define

$$x_n = \frac{1}{2}u_n\chi_{A_n}\operatorname{sgn} x \quad (n = 1, 2, \ldots).$$

We have

$$\begin{aligned} I_\Phi(x+x_n) &= I_\Phi(x\chi_{T\setminus A_n}) + I_\Phi(x\chi_{A_n} + x_n) \\ &\leq I_\Phi(x\chi_{T\setminus A_n}) + \frac{1}{2}\{\Phi(2b)\mu(A_n) + \Phi(u_n)\mu(A_n)\} \\ &\longrightarrow I_\Phi(x) = 1. \end{aligned}$$

Moreover, $I_\Phi(x+x_n) \geq I_\Phi(x\chi_{T\setminus A_n}) \longrightarrow 1$. Thus, $I_\Phi(x+x_n) \longrightarrow 1$, whence it follows that $\|x+x_n\|_\Phi \longrightarrow 1$. We have also

$$I_\Phi(x-x_n) \leq I_\Phi(|x| + |x_n|) \longrightarrow 1$$

and

$$I_\Phi(x-x_n) \geq I_\Phi(x\chi_{T\setminus A_n}) \longrightarrow 1,$$

whence it follows that $I_\Phi(x-x_n) \longrightarrow 1$, *i.e.* $\|x-x_n\|_\Phi \longrightarrow 1$. On the other hand

$$I_\Phi\Big(2\Big(1+\frac{1}{n}\Big)x_n\Big) = \Phi\Big(\Big(1+\frac{1}{n}\Big)u_n\Big)\mu(A_n) > 2^n\Phi(u_n)\mu(A_n) = 1.$$

Therefore, $\|x_n\|_\Phi \geq \frac{1}{2}(1+\frac{1}{n})^{-1} \geq \frac{1}{4}$. This means that x is not a strongly extreme point. The proof is finished.

THEOREM 2. *Let Φ be an Orlicz function which does not satisfy the suitable Δ_2-condition. Let us assume additionally in the case when μ is infinite that Φ vanishes only at zero. Then $S(L^\Phi)$ has no H-point.*

PROOF. We will restrict ourselves only to finite measure. In the case of an infinite measure the proof is analogous. Take $d > 0$ large enough, so that defining $A = \{t \in T : d^{-1} \leq |x(t)| \leq d\}$ we have $I_\Phi(x\chi_A) \geq \frac{3}{4}I_\Phi(x)$. Next, take $C_n \subset A$, $C_n \in \Sigma$ such that

$$I_\Phi(x\chi_{C_n}) = 2^{-n}I_\Phi(x) \quad (n = 1, 2, \ldots).$$

Since Φ has only finite values by the definition and μ is finite, the condition $\|x\|_\Phi = 1$ yields $I_\Phi(x) > 0$ (note that for infinite μ it can be $I_\Phi(x) = 0$ even if $\|x\|_\Phi = 1$ whenever Φ vanishes outside zero). In fact, defining $\alpha = \sup\{u \geq 0 : \Phi(u) = 0\}$ the condition $I_\Phi(x) = 0$ yields $|x(t)| \leq \alpha$ for μ- a.e. $t \in T$. Next by the finiteness of μ we have $I_\Phi(\lambda x) < \infty$ for any $\lambda > 0$. This is a contradiction, because in the case when $I_\Phi(x) = 0$ the quality $\|x\|_\Phi = 1$ implies that $I_\Phi(\lambda x) = \infty$ for any $\lambda > 1$ (see [2]).

In view of the assumption that Φ does not satisfy the Δ_2-condition at infinity, there exists a sequence (u_n) of positive reals such that $u_n \longrightarrow \infty$ as $n \longrightarrow \infty$ and

$$\Phi\Big(\Big(1 + \frac{1}{n}\Big)u_n\Big) > 2^{2n}\Phi(u_n) \quad (n = 1, 2, \ldots).$$

Passing to subsequences of (u_n) and (C_n) if necessary, we can find a sequence (D_n) of measurable subsets of C_n such that

$$\Phi(u_n)\mu(D_n) = I_\Phi(x\chi_{C_n}) \quad (n = 1, 2, \ldots).$$

Define

$$x_n = x\chi_{T\setminus C_n} - u_n(\operatorname{sgn} x)\chi_{D_n}.$$

We have

$$I_\Phi(x_n) = I_\Phi(x\chi_{T\setminus C_n}) + \Phi(u_n)\mu(D_n).$$

If $I_\Phi(x) = 1$ then $I_\Phi(x_n) = 1$ for any $n \in \mathbb{N}$, whence $\|x_n\|_\Phi = 1$. If $I_\Phi(x) < 1$, then the equality $\|x\|_\Phi = 1$ yields $I_\Phi(\lambda x) = \infty$ for any $\lambda > 1$. Since x is uniformly bounded on the sets C_n, we get $I_\Phi(\lambda x_n) = \infty$ for any $\lambda > 1$ and $n \in \mathbb{N}$. Hence it follows that $\|x_n\|_\Phi = 1$ for any $n \in \mathbb{N}$. We have

$$x - x_n = x\chi_{C_n} + u_n\chi_{D_n} \operatorname{sgn} x.$$

The sequence $(x\chi_{C_n})$ is norm convergent to zero, because the function x is uniformly bounded on C_n, $n = 1, 2, \ldots$, and $\mu(C_n) \longrightarrow 0$ as $n \longrightarrow \infty$. Therefore, in order to prove that the sequence (x_n) is weakly convergent to x it suffices to prove that the sequence (y_n), where $y_n = u_n\chi_{D_n} \operatorname{sgn} x$, is weakly convergent to zero. We have $y_n \in E^\Phi(\mu)$ for any $n \in \mathbb{N}$, where $E^\Phi(\mu) = \{x \in L^0(\mu) : I_\Phi(\lambda x) < \infty \text{ for any } \lambda > 0\}$. Hence it follows that any linear continuous singular functional over $L^\Phi(\mu)$ vanishes at y_n, $n = 1, 2, \ldots$ (for the description of the dual space of $L^\Phi(\mu)$ see [1]). Thus, in order to prove that $y_n \longrightarrow$

0 weakly, it suffices to consider only regular (*i.e.* order continuous) linear continuous functionals over $L^{\Phi}(\mu)$. Take an arbitrary linear continuous regular functional ξ_f over $L^{\Phi}(\mu)$ generated by a function $f \in L^{\Phi^*}(\mu)$, where Φ^* is the function complementary to Φ in the sense of Young. Let $\lambda > 0$ be such that $I_{\Phi^*}(\lambda f) < \infty$. We have

$$\begin{aligned}|\xi_f(y_n)| &= \left|\int_T f(t)y_n(t)\,d\mu\right| = \frac{1}{\lambda}\left|\int_T \lambda f(t)y_n(t)\,d\mu\right| \\ &\leq \frac{1}{\lambda}\{I_{\Phi^*}(\lambda f\chi_{D_n}) + I_{\Phi}(y_n)\} \\ &\leq \frac{1}{\lambda}\{I_{\Phi^*}(\lambda f\chi_{D_n}) + 2^{-n}I_{\Phi}(x)\} \longrightarrow 0\end{aligned}$$

as $n \longrightarrow \infty$ because $\mu(D_n) \longrightarrow 0$ as $n \longrightarrow \infty$. Moreover,

$$\begin{aligned}I_{\Phi}\Big(\Big(1+\frac{1}{n}\Big)(x-x_n)\Big) &\geq I_{\Phi}\Big(\Big(1+\frac{1}{n}\Big)y_n\Big) = I_{\Phi}\Big(\Big(1+\frac{1}{n}\Big)u_n\Big)\mu(D_n) \\ &> 2^{2n}\Phi(u_n)\mu(D_n) = 2^n I_{\Phi}(x) \geq 1\end{aligned}$$

for $n \in \mathbb{N}$ large enough. Therefore

$$\|x - x_n\|_{\Phi} \geq 1\Big/\Big(1+\frac{1}{n}\Big) \geq 1/2$$

for sufficiently large $n \in \mathbb{N}$, which means that the sequence (x_n) is not norm convergent to x, *i.e.* x is not an H-point. Since $x \in S(L^{\Phi})$ was arbitrary, the proof is finished.

Recall that a point $x \in S(X)$ is said to be *strongly exposed* if there exists a functional $x^* \in S(X^*)$ such that $x^*(x) = 1$, and for any sequence (x_n) in X the condition $x^*(x-x_n) \longrightarrow 0$ implies that $\|x - x_n\| \longrightarrow 0$.

Since any strongly exposed point is strongly extreme, we obtain from Theorem 1 the following:

COROLLARY. *If Φ is an Orlicz function which does not satisfy the suitable Δ_2-condition and if additionally in the case when μ is infinite Φ vanishes only at zero, then $S(L^{\Phi})$ has no strongly exposed point.*

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