Ergod. Th. & Dynam. Sys., (2023), **43**, 2111–2136 © The Author(s), 2022. Published by Cambridge University Press.
doi:10.1017/etds.2022.30

Relative pressure functions and their equilibrium states

YUKI YAYAMA

Centro de Ciencias Exactas and Grupo de investigación en Sistemas Dinámicos y Aplicaciones (GISDA), Departamento de Ciencias Básicas, Facultad de Ciencias, Universidad del Bío-Bío, Avenida Andrés Bello 720, Casilla 447, Chillán, Chile
(e-mail: yyayama@ubiobio.cl)

(*Received* 1 *October* 2021 *and accepted in revised form* 7 *March* 2022)

Abstract. For a subshift (X, σ_X) and a subadditive sequence $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ on X, we study equivalent conditions for the existence of $h \in C(X)$ such that $\lim_{n\to\infty}(1/n) \int \log f_n \, d\mu = \int h \, d\mu$ for every invariant measure μ on X. For this purpose, we first we study necessary and sufficient conditions for $\mathcal{F}$ to be an asymptotically additive sequence in terms of certain properties for periodic points. For a factor map $\pi : X \to Y$, where (X, σ_X) is an irreducible shift of finite type and (Y, σ_Y) is a subshift, applying our results and the results obtained by Cuneo [Additive, almost additive and asymptotically additive potential sequences are equivalent. *Comm. Math. Phys.* **37** (3) (2020), 2579–2595] on asymptotically additive sequences, we study the existence of h with regard to a subadditive sequence associated to a relative pressure function. This leads to a characterization of the existence of a certain type of continuous compensation function for a factor map between subshifts. As an application, we study the projection $\pi\mu$ of an invariant weak Gibbs measure μ for a continuous function on an irreducible shift of finite type.

Key words: relative pressure, equilibrium states, subadditive potentials
2020 Mathematics Subject Classification: 37B10 (Primary); 37D35 (Secondary)

1. *Introduction*

The thermodynamic formalism for sequences of continuous functions generalizes the formalism for continuous functions and has been applied to solve some dimension problems in non-conformal dynamical systems. The equilibrium states for sequences of continuous functions are the equilibrium states for Borel measurable functions in general. In [10] Falconer introduced the thermodynamic formalism for subadditive sequences to study repellers of non-conformal transformations. Cao, Feng and Huang in [6] established the theory for subadditive sequences wherein the variational principle was obtained for compact dynamical systems. Asymptotically additive sequences, which generalize the almost additive sequences studied by Barreira [2] and Mummert [18], were also introduced by Feng and Huang [13]. The properties of equilibrium states for sequences of continuous functions, such as uniqueness, the (generalized) Gibbs property and mixing properties, have been also studied (see, for example, [2, 12, 18]). Here, a natural question arises.

Question 1. Given a subadditive sequence $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ on a compact metric space X, what are necessary and sufficient conditions for the existence of a continuous function h on X such that

$$\lim_{n\to\infty} \frac{1}{n} \int \log f_n \, d\mu = \int h \, d\mu \tag{1}$$

for every invariant Borel probability measure μ on X?

If such an h exists, then the thermodynamic formalism for such sequences $\mathcal{F}$ reduces to the formalism for continuous functions. Cuneo [9, Theorem 1.2] proved that if a sequence of continuous functions is asymptotically additive (see (4) for the definition), then there always exists $h \in C(X)$ satisfying (1) for every invariant measure μ on X. In this paper, we study necessary conditions for a subadditive sequence $\mathcal{F}$ on an irreducible subshift (X, σ_X) to have a continuous function $h \in C(X)$ satisfying (1) for every invariant measure μ on X. Using our results and the result obtained by Cuneo [9, Theorem 1.2], we give some answers to Question 1 (Theorems 4.3, 6.9 and 7.8). Towards this end, we first study conditions for a subadditive sequence on a subshift to be an asymptotically additive sequence in terms of certain properties for periodic points. Given a subadditive sequence $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ on X, if (1) holds for every invariant Borel probability measure μ on X, then the sequence $\tilde{\mathcal{F}} = \{(1/n) \log(f_n/e^{S_n h})\}_{n=1}^{\infty}$ converges (pointwise) to the zero function 0 for every periodic point of σ_X (see Proposition 3.1). We show in Theorems 4.3 and 4.4 that if the sequence $\tilde{\mathcal{F}}$ converges (pointwise) to 0 for every periodic point of σ_X and $\mathcal{F}$ satisfies a particular property for certain periodic points then $\tilde{\mathcal{F}}$ converges to 0 everywhere; moreover, it converges uniformly to 0 on X. This gives the asymptotic additivity of $\mathcal{F}$. We apply Theorem 4.3 when we study Question 1 with regard to a relative pressure function of a continuous function (Theorems 6.9 and 7.8). In Proposition 3.1, Question 1 is studied in a general form. Note that subadditive sequences are not asymptotically additive in general (see Example 7.2 in §7).

In §6, we consider relative pressure functions in relation to compensation functions. Let (X, σ_X), (Y, σ_Y) be subshifts and $\pi : X \to Y$ be a factor map. Let $f \in C(X)$, $n \in \mathbb{N}$ and $\delta > 0$. For each $y \in Y$, define

$$P_n(\sigma_X, \pi, f, \delta)(y) = \sup \left\{ \sum_{x\in E} e^{(S_n f)(x)} : E \text{ is an } (n, \delta) \text{ separated subset of } \pi^{-1}(\{y\}) \right\},$$

$$P(\sigma_X, \pi, f, \delta)(y) = \limsup_{n\to\infty} \frac{1}{n} \log P_n(\sigma_X, \pi, f, \delta)(y),$$

$$P(\sigma_X, \pi, f)(y) = \lim_{\delta\to 0} P(\sigma_X, \pi, f, \delta)(y).$$

The function $P(\sigma_X, \pi, f) : Y \to \mathbb{R}$ is the *relative pressure* function of $f \in C(X)$ with respect to $(\sigma_X, \sigma_Y, \pi)$. In general it is merely Borel measurable. In Theorem 6.9, for an irreducible shift of finite type (X, σ_X), we study equivalent conditions for a relative pressure function $P(\sigma_X, \pi, f)$ on Y to have a function $h \in C(Y)$ such that

$$\int P(\sigma_X, \pi, f)\, d\mu = \int h\, d\mu \quad \text{for every } \mu \in M(Y, \sigma_Y) \tag{2}$$

where $M(Y, \sigma_Y)$ is the set of invariant Borel probability measures on Y. In general, a relative pressure function $P(\sigma_X, \pi, f)$ is represented by a subadditive sequence $\mathcal{G} = \{\log g_n\}_{n=1}^{\infty}$ of continuous functions on Y (see (32) for g_n), that is, $P(\sigma_X, \pi, f) = \lim_{n\to\infty}(1/n) \log g_n$ almost everywhere with respect to every $\mu \in M(Y, \sigma_Y)$. The sequence $\mathcal{G}$ satisfies an additional condition (see (D2) in §2.2) weaker than almost additivity and it is not asymptotically additive in general. We prove that the subadditive sequence $\mathcal{G}$ on Y associated to $P(\sigma_X, \pi, f)$ satisfies the particular property for certain periodic points in Lemma 4.1(ii). Applying Theorem 4.3, we obtain in Theorem 6.9 that, for $h \in C(Y)$, uniform convergence of $\tilde{\mathcal{G}} = \{(1/n)\log(g_n/e^{S_n h})\}_{n=1}^{\infty}$ to 0 on Y is equivalent to pointwise convergence of $\tilde{\mathcal{G}}$ to 0 for every periodic point of σ_Y. In particular, we obtain that (2) holds if and only if the sequence $\mathcal{G}$ associated to $P(\sigma_X, \pi, f)$ is asymptotically additive. Moreover, if there exists an invariant weak Gibbs measure m for $f \in C(X)$, then (2) holds if and only if πm is an invariant weak Gibbs measure for some continuous function on Y (Theorem 7.8). The properties of the sequence $\mathcal{G}$ associated to $P(\sigma_X, \pi, f)$ under the existence of h in (2) are studied and a condition of non-existence of such a continuous function is also studied (Corollary 6.11). These results are applied directly to study the projection of an invariant weak Gibbs measure for a continuous function on X in §7 (see Theorem 7.6 and Corollary 7.9). Note that in general if there exists an invariant weak Gibbs measure m for $f \in C(X)$, then πm is a weak Gibbs equilibrium state for the subadditive sequence $\mathcal{G}$ associated to $P(\sigma_X, \pi, f)$.

On the other hand, relative pressure functions are connected with compensation functions. Given $f \in C(X)$, Theorem 6.9 relates the question on the existence of h in (2) with the existence of a compensation function $f - h \circ \pi$ for some $h \in C(Y)$. A function $F \in C(X)$ is a compensation function for a factor map π if

$$\sup_{\mu \in M(X,\sigma_X)} \left\{ h_\mu(\sigma_X) + \int F\, d\mu + \int \phi \circ \pi\, d\mu \right\} = \sup_{\nu \in M(Y,\sigma_Y)} \left\{ h_\nu(\sigma_Y) + \int \phi\, d\nu \right\} \tag{3}$$

for every $\phi \in C(Y)$. If $F = G \circ \pi$, $G \in C(Y)$, then $G \circ \pi$ is a saturated compensation function. The concept of compensation functions was introduced by Boyle and Tuncel [**5**], and their properties were studied by Walters [**28**] in relation to relative pressure. The existence of compensation functions has been studied [**1**, **24**–**26**]. Shin [**25**, **26**] proved that a saturated compensation function does not always exist and gave a characterization for the existence of a saturated compensation function for factor maps between shifts of finite type. A function $-h \circ \pi \in C(X)$ is a saturated compensation function if and only if (2) holds for $f = 0$. Our results connect the result obtained by Shin with the asymptotic additivity of the sequence associated to $P(\sigma_X, \pi, 0)$ (see Remark 6.10 and Corollary 7.1). Since saturated compensation functions were applied to study the measures of full Hausdorff dimension

https://doi.org/10.1017/etds.2022.30 Published online by Cambridge University Press

of non-conformal repellers, studying the properties of equilibrium states for h in (2) would help in the further study of certain dimension problems (see Example 7.4).

Section 5 deals with a particular class of subadditive sequences on subshifts satisfying an additional property (see condition (C2)) weaker than almost additivity but stronger than (D2). The result of Feng [**12**, Theorem 5.5] implies that there is a unique (generalized) Gibbs equilibrium state for a subadditive sequence with bounded variation satisfying property (C2). We study equivalent conditions for this type of sequence $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ on a subshift X to have a continuous function for which the unique Gibbs equilibrium state is a weak Gibbs measure (Theorem 5.1). In this case, we obtain that for $h \in C(X)$ uniform convergence of the sequence of functions $\{(1/n)\log(f_n/e^{S_n h})\}_{n=1}^{\infty}$ to 0 on X is equivalent to pointwise convergence of the sequence to 0 on X. We note that it is not clear that the condition for certain periodic points in Theorem 4.4(ii) is satisfied for this type of sequence in general.

2. *Background*

2.1. *Shift spaces.* We give a brief summary of the basic definitions in symbolic dynamics. (X, σ_X) is a *one-sided subshift* if X is a closed shift-invariant subset of $\{1, \ldots, k\}^{\mathbb{N}}$ for some $k \geq 1$, that is, $\sigma_X(X) \subseteq X$, where the shift $\sigma_X : X \to X$ is defined by $(\sigma_X(x))_i = x_{i+1}$ for all $i \in \mathbb{N}$, $x = (x_n)_{n=1}^{\infty} \in X$. Define a metric d on X by $d(x, x') = 1/2^k$ if $x_i = x'_i$ for all $1 \leq i \leq k$ and $x_{k+1} \neq x'_{k+1}$, $d(x, x') = 1$ if $x_1 \neq x'_1$, and $d(x, x') = 0$ otherwise. Throughout this paper, we consider one-sided subshifts. Define a cylinder set $[x_1 \ldots x_n]$ of length n in X by $[x_1 \ldots x_n] = \{(z_i)_{i=1}^{\infty} \in X : z_i = x_i \text{ for all } 1 \leq i \leq n\}$. For each $n \in \mathbb{N}$, denote by $B_n(X)$ the set of all n-blocks that appear in points in X. Define $B_0(X) = \{\epsilon\}$, where ϵ is the empty word of length 0. The language of X is the set $B(X) = \bigcup_{n=0}^{\infty} B_n(X)$. A subshift (X, σ_X) is *irreducible* if for any allowable words $u, v \in B(X)$, there exists $w \in B(X)$ such that $uwv \in B(X)$. A subshift has the *weak specification property* if there exists $p \in \mathbb{N}$ such that for any allowable words $u, v \in B(X)$, there exist $0 \leq k \leq p$ and $w \in B_k(X)$ such that $uwv \in B(X)$. We call such p a weak specification number. A point $x \in X$ is a periodic point of σ_X if there exists $l \in \mathbb{N}$ such that $\sigma_X^l(x) = x$.

Let (X, σ_X) and (Y, σ_Y) be subshifts. A shift of finite type (X, σ_X) is *one-step* if there exists a set F of forbidden blocks of length less than or equal to 2 such that $X = \{x \in \{1, \ldots, k\}^{\mathbb{N}} : \omega \text{ does not appear in } x \text{ for any } \omega \in F\}$. A map $\pi : X \to Y$ is a *factor map* if it is continuous, surjective and satisfies $\pi \circ \sigma_X = \sigma_Y \circ \pi$. If, in addition, the ith position of the image of x under π depends only on x_i, then π is a *one-block factor map*. Throughout the paper we assume that a shift of finite type (X, σ_X) is one-step and π is a one-block factor map. Denote by $M(X, \sigma_X)$ the collection of all σ_X-invariant Borel probability measures on X and by $\mathrm{Erg}(X, \sigma_X)$ all ergodic members of $M(X, \sigma_X)$.

2.2. *Sequences of continuous functions.* We give a brief summary on the basic results on the sequences of continuous functions considered in this paper. Let (X, σ_X) be a subshift on finitely many symbols. For each $n \in \mathbb{N}$, let $f_n : X \to \mathbb{R}^+$ be a continuous function. A sequence $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ is *almost additive* if there exists a constant $C \geq 0$ such that $e^{-C} f_n(x) f_m(\sigma_X^n x) \leq f_{n+m}(x) \leq e^C f_n(x) f_m(\sigma_X^n x)$. In particular, if $C = 0$, then

$\mathcal{F}$ is additive. The thermodynamic formalism for almost additive sequences was studied in Barrera [2] and Mummert [18]. More generally, Feng and Huang [13] introduced asymptotically additive sequences which generalize almost additive sequences. A sequence $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ is *asymptotically additive* on X if for every $\epsilon > 0$ there exists a continuous function ρ_ϵ such that

$$\limsup_{n\to\infty} \frac{1}{n} \|\log f_n - S_n\rho_\epsilon\|_\infty < \epsilon, \tag{4}$$

where $\|\cdot\|_\infty$ is the supremum norm and $(S_n\rho_\epsilon)(x) = \sum_{i=0}^{n-1} \rho_\epsilon(\sigma^i(x))$ for each $x \in X$. A sequence $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ is *subadditive* if $\mathcal{F}$ satisfies $f_{n+m}(x) \le f_n(x) f_m(\sigma_X^n x)$. The thermodynamic formalism for subadditive sequences was studied by Cao, Feng and Huang [6].

We assume certain regularity conditions on sequences. A sequence $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ has *bounded variation* if there exists $M \in \mathbb{R}^+$ such that $\sup\{M_n : n \in \mathbb{N}\} \le M$ where

$$M_n = \sup\left\{\frac{f_n(x)}{f_n(y)} : x, y \in X, x_i = y_i \text{ for } 1 \le i \le n\right\}. \tag{5}$$

More generally, if $\lim_{n\to\infty}(1/n)\log M_n = 0$, then we say that $\mathcal{F}$ has *tempered variation*. Without loss of generality, we assume $M_n \le M_{n+1}$ for all $n \in \mathbb{N}$.

A function $f \in C(X)$ belongs to the *Bowen class* if the sequence $\mathcal{F}$ formed by setting $f_n = e^{S_n(f)}$ has bounded variation [29]. A function of summable variation belongs to the Bowen class. In this paper, we consider the sequences $\mathcal{F}$ satisfying the following properties.

(C1) The sequence $\mathcal{F}' := \{\log(f_n e^C)\}_{n=1}^{\infty}$ is subadditive for some $C \ge 0$.

(C2) There exist $p \in \mathbb{N}$ and $D > 0$ such that, given any $u \in B_n(X)$, $v \in B_m(X)$, $n, m \in \mathbb{N}$, there exist $0 \le k \le p$ and $w \in B_k(X)$ such that $uwv \in B_{n+m+k}(X)$ and

$$\sup\{f_{n+m+k}(x) : x \in [uwv]\} \ge D \sup\{f_n(x) : x \in [u]\} \sup\{f_m(x) : x \in [v]\}.$$

More generally, we have the following property.

(D2) There exist $p \in \mathbb{N}$ and a positive sequence $\{D_{n,m}\}_{(n,m)\in\mathbb{N}\times\mathbb{N}}$ such that, given any $u \in B_n(X)$, $v \in B_m(X)$, $n, m \in \mathbb{N}$, there exist $0 \le k \le p$ and $w \in B_k(X)$ such that $uwv \in B_{n+m+k}(X)$ and

$$\sup\{f_{n+m+k}(x) : x \in [uwv]\} \ge D_{n,m} \sup\{f_n(x) : x \in [u]\} \sup\{f_m(x) : x \in [v]\},$$

where $\lim_{n\to\infty}(1/n)\log D_{n,m} = \lim_{m\to\infty}(1/m)\log D_{n,m} = 0$. Without loss of generality, we assume that $D_{n,m} \ge D_{n,m+1}$ and $D_{n,m} \ge D_{n+1,m}$.

Remark 2.1. A sequence $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ satisfying (C1) is not always asymptotically additive (see §7). Condition (C2) was introduced by Feng [11] where the thermodynamic formalism of products of matrices was studied. The sequences satisfying (C1) and (C2) with bounded variation generalize almost additive sequences with bounded variation on subshifts with the weak specification property and have been applied to solve questions concerning the Hausdorff dimensions of non-conformal repellers [12, 31]. See [14, 15] for the non-compact case. We will study the sequences satisfying (C1) and (D2) in §§6 and 7.

Let (X, σ_X) be a subshift and $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ be a subadditive sequence of continuous functions on X. For each $n \in \mathbb{N}$, define

$$P_n(\mathcal{F}, \delta) = \sup_E \left\{ \sum_{x \in E} f_n(x) : E \text{ is an } (n, \delta) \text{ separated subset of } X \right\}.$$

The *topological pressure* for $\mathcal{F}$ is defined by

$$P(\mathcal{F}) = \lim_{\delta \to 0} \limsup_{n \to \infty} \frac{1}{n} \log P_n(\mathcal{F}, \delta). \tag{6}$$

THEOREM 2.2. [6] *Let (X, σ_X) be a subshift and $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ be a subadditive sequence on X. Then*

$$P(\mathcal{F}) = \sup_{\mu \in M(X, \sigma_X)} \left\{ h_\mu(X) + \lim_{n \to \infty} \frac{1}{n} \int \log f_n \, d\mu \right\}. \tag{7}$$

A measure $m \in M(X, \sigma_X)$ is an *equilibrium state* for $\mathcal{F}$ if the supremum in (7) is attained at m.

Definition 2.3. Let (X, σ_X) be a subshift and $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ be a subadditive sequence on X satisfying $P(\mathcal{F}) \neq -\infty$. A measure $\mu \in M(X, \sigma_X)$ is a *weak Gibbs measure* for $\mathcal{F}$ if there exists $C_n > 0$ such that

$$\frac{1}{C_n} < \frac{\mu[x_1 \ldots x_n]}{e^{-nP(\mathcal{F})} f_n(x)} < C_n$$

where $\lim_{n \to \infty} (1/n) \log C_n = 0$, for every $x \in X$ and $n \in \mathbb{N}$. If there exists $C > 0$ such that $C = C_n$ for all $n \in \mathbb{N}$, then μ is a *Gibbs measure*.

If μ is an invariant weak Gibbs measure for a subadditive sequence $\mathcal{F}$, then it is an equilibrium state for $\mathcal{F}$. The result of Feng [12, Theorem 5.5] implies the uniqueness of equilibrium states for a class of sequences satisfying (C1) and (C2).

THEOREM 2.4. [12] *Let (X, σ_X) be a subshift and $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ be a sequence on X satisfying (C1) and (C2) with bounded variation. Then there is a unique invariant Gibbs measure for $\mathcal{F}$ and it is the unique equilibrium state for $\mathcal{F}$.*

Cuneo [9] showed that finding equilibrium states for asymptotically additive sequences is equivalent to that for continuous functions.

THEOREM 2.5. (Special case of [9, Theorem 1.2]) *Let (X, σ_X) be a subshift and $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ be an asymptotically additive sequence on X. Then there exists $f \in C(X)$ such that*

$$\lim_{n \to \infty} \frac{1}{n} \|\log f_n - S_n f\|_\infty = 0. \tag{8}$$

Hence, if $\mathcal{F}$ is asymptotically additive, then there exists $f \in C(X)$ such that $\lim_{n \to \infty} (1/n) \int \log f_n \, d\mu = \int f \, d\mu$ for every $\mu \in M(X, \sigma_X)$. It is clear that (8) implies that $\mathcal{F}$ is asymptotically additive.

https://doi.org/10.1017/etds.2022.30 Published online by Cambridge University Press

3. *Subadditive sequences*

In this section, we consider Question 1 from §1. Proposition 3.1 is valid for the case when X is a compact metric space and $T : X \to X$ is a continuous transformation of X. Proposition 3.1 will be applied in the next sections.

PROPOSITION 3.1. *Let (X, σ_X) be a subshift and $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ a subadditive sequence on X. For $h \in C(X)$, the following conditions are equivalent.*

(i)

$$\lim_{n\to\infty} \frac{1}{n} \int \log f_n \, d\mu = \int h \, d\mu$$

for every $\mu \in M(X, \sigma_X)$.

(ii)

$$\lim_{n\to\infty} \frac{1}{n} \int \log f_n \, d\mu = \int h \, d\mu$$

for every $\mu \in \mathrm{Erg}(X, \sigma_X)$.

(iii)

$$\lim_{n\to\infty} \frac{1}{n} \log\left(\frac{f_n(x)}{e^{(S_n h)(x)}}\right) = 0$$

μ-almost everywhere on X, for every $\mu \in \mathrm{Erg}(X, \sigma_X)$.

Remark 3.2. Proposition 3.1 holds for a sequence $\mathcal{F}$ satisfying (C1) because $\{\log(e^C f_n)\}_{n=1}^{\infty}$ is a subadditive sequence.

Proof. It is clear that (i) implies (ii). By the ergodic decomposition (see [**13**, Proposition A.1(c)]), (ii) implies (i). Now we assume that (ii) holds. For a measure $\mu \in \mathrm{Erg}(X, \sigma_X)$, we obtain

$$\lim_{n\to\infty} \frac{1}{n} \int \log\left(\frac{f_n}{e^{(S_n h)}}\right) d\mu = 0.$$

To see that this implies (iii), define $r_n(x) := f_n(x)/e^{(S_n h)(x)}$. Then $\log r_n \in L_1(\mu)$ and $\{\log r_n\}_{n=1}^{\infty}$ is a subadditive sequence of continuous functions on X. Since μ is an ergodic measure, by Kingman's subadditive ergodic theorem, we obtain that $\lim_{n\to\infty}(1/n) \log r_n(x) = \lim_{n\to\infty}(1/n) \int \log r_n \, d\mu = 0$ μ-almost everywhere on X. Now we assume that (iii) holds. Given $\mu \in \mathrm{Erg}(X, \sigma_X)$, applying the subadditive ergodic theorem to the sequence $\{\log r_n\}_{n=1}^{\infty}$, we obtain

$$\int \lim_{n\to\infty} \frac{1}{n} \log\left(\frac{f_n}{e^{(S_n h)}}\right) d\mu = \lim_{n\to\infty} \frac{1}{n} \int \log\left(\frac{f_n}{e^{(S_n h)}}\right) d\mu$$
$$= \lim_{n\to\infty} \left(\frac{1}{n} \int \log f_n \, d\mu - \int h \, d\mu\right).$$

Hence, we obtain (ii). □

https://doi.org/10.1017/etds.2022.30 Published online by Cambridge University Press

4. *Subadditive sequences which are asymptotically additive*

Subadditive sequences are not always asymptotically additive. In this section we study a class of subadditive sequences on shift spaces (compact spaces) which are also asymptotically additive. The goal of this section is to characterize such sequences using a particular property for periodic points. The results in this section are applied in §§6 and 7 to study relative pressure functions.

LEMMA 4.1. *Let (X, σ_X) be a subshift and $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ be a sequence on X satisfying (C1) with tempered variation. Suppose that $\mathcal{F}$ satisfies the following two conditions (i) and (ii).*

(i) *There exists $h \in C(X)$ such that*

$$\lim_{n\to\infty} \frac{1}{n} \log\left(\frac{f_n(x)}{e^{(S_n h)(x)}}\right) = 0$$

for every periodic point $x \in X$.

(ii) *There exist $k, N \in \mathbb{N}$ and a sequence $\{M_n\}_{n=1}^{\infty}$ of positive real numbers satisfying $\lim_{n\to\infty}(1/n)\log M_n = 0$ such that, for given any $u \in B_n(X)$, $n \geq N$, there exist $0 \leq q \leq k$ and $w \in B_q(X)$ such that $z := (uw)^{\infty}$ is a point in X satisfying*

$$f_{j(n+q)}(z) \geq (M_n \sup\{f_n(x) : x \in [u]\})^j \tag{9}$$

for every $j \in \mathbb{N}$.

Then $\mathcal{F}$ is an asymptotically additive sequence on X.

Remark 4.2. Let (X, σ_X) be an irreducible shift of finite type and k be a weak specification number. Then for each $u \in B_n(X)$ there exist $0 \leq q \leq k$ and $w \in B_q(X)$ such that $(uw)^{\infty} \in X$.

Proof. Suppose that (i) and (ii) hold. We will show that

$$\lim_{n\to\infty} \frac{1}{n}\left\| \log\left(\frac{f_n}{e^{(S_n h)}}\right)\right\|_{\infty} = 0. \tag{10}$$

Let k, M_n, N be defined as in (ii). For $h \in C(X)$, let

$$M_n^h := \sup\left\{\frac{e^{(S_n h)(x)}}{e^{(S_n h)(x')}} : x_i = x'_i, 1 \leq i \leq n\right\} \tag{11}$$

for each $n \in \mathbb{N}$ and $C_h := \max_{0\leq i\leq k}\{(S_i h)(x) : x \in X\}$, where $(S_0 h)(x) := 1$ for every $x \in X$. Let $\epsilon > 0$. Take $N_1 \in \mathbb{N}$ large enough so that

$$\frac{1}{n}|\log(M_n^h e^{C_h})| < \epsilon, \quad \frac{1}{n}|\log M_n| < \epsilon \quad \text{and} \quad \frac{n}{n+k} > \frac{1}{2}$$

for all $n > N_1$. Let $N_2 = \max\{N, N_1\}$ and let $n \geq N_2$. Then, for $x_1 \ldots x_n \in B_n(X)$, there exists $w \in B_q(X)$, $0 \leq q \leq k$, such that $y^* := (x_1, \ldots, x_n, w)^{\infty} \in X$ satisfying (9). Since y^* is a periodic point, (i) implies that there exists $N(y^*) \in \mathbb{N}$ such that

$$\frac{1}{i}\left|\log\left(\frac{f_i(y^*)}{e^{(S_i h)(y^*)}}\right)\right| < \epsilon$$

for all $i > N(y^*)$. Take $j > N(y^*)$. By (ii), for $z \in [x_1 \ldots x_n]$, we obtain

$$\begin{aligned}\epsilon &> \frac{1}{j(n+q)} \log\left(\frac{f_{j(n+q)}(y^*)}{e^{(S_{j(n+q)}h)(y^*)}}\right) \geq \frac{1}{j(n+q)} \log\left(\frac{M_n f_n(z)}{M_n^h e^{(S_n h)(z)} e^{C_h}}\right)^j \\ &= \frac{1}{(n+q)} \log M_n + \frac{1}{(n+q)} \log\left(\frac{f_n(z)}{e^{(S_n h)(z)}}\right) - \frac{1}{(n+q)} \log(M_n^h e^{C_h}) \\ &> -2\epsilon + \frac{1}{n+q} \log\left(\frac{f_n(z)}{e^{(S_n h)(z)}}\right) > -2\epsilon + \frac{n}{n+q}\left(\frac{1}{n} \log\left(\frac{f_n(z)}{e^{(S_n h)(z)}}\right)\right).\end{aligned}$$

Without loss of generality assume $(1/n)\log(f_n(z)/e^{(S_n h)(z)}) > 0$. Hence, for any $x_1 \ldots x_n \in B_n(X), n \geq N_2, z \in [x_1 \ldots x_n]$, we obtain that $(1/n)\log(f_n(z)/e^{(S_n h)(z)}) < 6\epsilon$.

Next we show that there exists $N' \in \mathbb{N}$ such that, for all $z \in [x_1, \ldots, x_n], n \geq N'$, $(1/n)\log(f_n(z)/e^{(S_n h)(z)}) > -4\epsilon$. Since $\mathcal{F}$ has tempered variation, for each $n \in \mathbb{N}$, let $M_n^{\mathcal{F}} := \sup\{f_n(x)/f_n(x') : x_i = x_i', 1 \leq i \leq n\}$. Let $C_{\mathcal{F}} := \max_{0 \leq i \leq k}\{f_i(x) : x \in X\}$, where $f_0(x) := 1$ for every $x \in X$, and $\bar{C}_h := \min_{0 \leq i \leq k}\{(S_i h)(x) : x \in X\}$. Let C be defined as in (C1). Take $N_3 \in \mathbb{N}$ large enough so that

$$\frac{1}{n}|\log(M_n^{\mathcal{F}} M_n^h C_{\mathcal{F}} e^{-\bar{C}_h + 2C})| < \epsilon \quad \text{and} \quad \frac{n}{n+k} > \frac{1}{2}$$

for all $n > N_3$. Since $\mathcal{F}$ satisfies (C1), we obtain that

$$\begin{aligned}\frac{f_{j(n+q)}(y^*)}{e^{(S_{j(n+q)}h)(y^*)}} &\leq \left(\frac{C_{\mathcal{F}} M_n^h e^{2C} \sup\{f_n(y) : y \in [x_1 \ldots x_n]\}}{e^{\bar{C}_h} \sup\{e^{(S_n h)(y)} : y \in [x_1 \ldots x_n]\}}\right)^j \\ &\leq \left(\frac{C_{\mathcal{F}} M_n^h M_n^{\mathcal{F}} e^{2C} f_n(z)}{e^{\bar{C}_h + (S_n h)(z)}}\right)^j,\end{aligned}$$

where in the last inequality z is a point from the cylinder set $[x_1 \ldots x_n]$. Hence, for $j > N(y^*)$,

$$\begin{aligned}-\epsilon &< \frac{1}{j(n+q)} \log\left(\frac{f_{j(n+q)}(y^*)}{e^{(S_{j(n+q)}h)(y^*)}}\right) \\ &< \frac{1}{n+q} \log\left(\frac{f_n(z)}{e^{(S_n h)(z)}}\right) + \frac{1}{n+q} \log(M_n^{\mathcal{F}} M_n^h C_{\mathcal{F}} e^{-\bar{C}_h + 2C}) \\ &< \frac{1}{n+q} \log\left(\frac{f_n(z)}{e^{(S_n h)(z)}}\right) + \epsilon \\ &= \frac{n}{n+q}\left(\frac{1}{n} \log\left(\frac{f_n(z)}{e^{(S_n h)(z)}}\right)\right) + \epsilon.\end{aligned}$$

Without loss of generality assume $(1/n)\log(f_n(z)/e^{(S_n h)(z)}) < 0$. For all $z \in [x_1 \ldots x_n]$, $n \geq N_3$, we obtain that $(1/n)\log(f_n(z)/e^{(S_n h)(z)}) > -4\epsilon$. Hence, we obtain (10). □

By Lemma 4.1, we obtain some conditions for a sequence $\mathcal{F}$ satisfying (C1) to be asymptotically additive, assuming that Lemma 4.1(ii) is satisfied.

https://doi.org/10.1017/etds.2022.30 Published online by Cambridge University Press

THEOREM 4.3. *Let (X, σ_X) be a subshift. Let $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ be a sequence on X satisfying (C1) with tempered variation and Lemma 4.1(ii). Then the following statements are equivalent for $h \in C(X)$.*

(i) *$\mathcal{F}$ is asymptotically additive on X satisfying*

$$\lim_{n\to\infty} \frac{1}{n} \left\| \log\left(\frac{f_n}{e^{(S_n h)}}\right) \right\|_{\infty} = 0.$$

(ii)

$$\lim_{n\to\infty} \frac{1}{n} \int \log f_n \, d\mu = \int h \, d\mu$$

for every $\mu \in M(X, \sigma_X)$.

(iii)

$$\lim_{n\to\infty} \frac{1}{n} \log\left(\frac{f_n(x)}{e^{(S_n h)(x)}}\right) = 0$$

for every periodic point $x \in X$.

(iv)

$$\lim_{n\to\infty} \frac{1}{n} \log\left(\frac{f_n(x)}{e^{(S_n h)(x)}}\right) = 0$$

for every $x \in X$.

Proof. The implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) are clear by applying Theorem 2.5 and Proposition 3.1. To see (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (i) , we apply Lemma 4.1. □

In the next theorem we study an equivalent condition for a subadditive sequence $\mathcal{F}$ to be an asymptotically additive sequence.

THEOREM 4.4. *Let (X, σ_X) be an irreducible shift of finite type and $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ be a sequence on X satisfying (C1) with tempered variation. Then $\mathcal{F}$ is asymptotically additive on X if and only if the following two conditions hold.*

(i) *There exists $h \in C(X)$ such that*

$$\lim_{n\to\infty} \frac{1}{n} \log\left(\frac{f_n(x)}{e^{(S_n h)(x)}}\right) = 0$$

for every periodic point $x \in X$.

(ii) *There exist $k \in \mathbb{N}$, $c \geq 0$ and a sequence $\{M_n\}_{n=1}^{\infty}$ of positive real numbers satisfying $\lim_{n\to\infty}(1/n) \log M_n = 0$ such that the following property, which we refer to as property (P), holds. For every $0 < \epsilon < 1$, there exists $N \in \mathbb{N}$ such that, given any $u \in B_n(X)$, $n \geq N$, there exist $0 \leq q \leq k$ and $w \in B_q(X)$ such that $z := (uw)^{\infty}$ is a point in X satisfying*

$$f_{j(n+q)}(z) \geq (M_n e^{-cn\epsilon})^j (\sup\{f_n(x) : x \in [u]\})^j \tag{12}$$

for every $j \in \mathbb{N}$.

Theorem 4.3 holds if we replace Lemma 4.1(ii) by condition (ii) above.

https://doi.org/10.1017/etds.2022.30 Published online by Cambridge University Press

Remark 4.5. Theorem 4.4(ii) is a generalization of Lemma 4.1(ii). If we set $c = 0$ in (12), we obtain (9).

Proof. Assume that $\mathcal{F}$ is asymptotically additive. Then (i) is obvious and for a given $0 < \epsilon < 1$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$e^{-n\epsilon+(S_n h)(x)} < f_n(x) < e^{n\epsilon+(S_n h)(x)} \tag{13}$$

for all $x \in X$. Since (X, σ_X) is an irreducible shift of finite type, let k be a weak specification number. Then for $x_1 \ldots x_n \in B_n(X)$, $n \geq N$, there exists $w \in B_q(X)$, $0 \leq q \leq k$, such that $y^* := (x_1, \ldots, x_n, w)^\infty \in X$. Let $\bar{C}_h$, M_n^h and $M_n^{\mathcal{F}}$ be defined as in the proof of Lemma 4.1. Then for any $z \in [x_1 \ldots x_n]$, $j \in \mathbb{N}$,

$$\begin{aligned} f_{(n+q)j}(y^*) &\geq e^{-j(n+q)\epsilon+(S_{j(n+q)}h)(y^*)} \geq e^{-j(n+q)\epsilon} \cdot \left(\frac{1}{M_n^h} e^{(S_n h)(z)} e^{\bar{C}_h}\right)^j \\ &\geq \left(\frac{1}{M_n^h} e^{-2\epsilon n - k\epsilon + \bar{C}_h}\right)^j f_n^j(z) \geq \left(\frac{1}{M_n^h} e^{-2\epsilon n - k + \bar{C}_h}\right)^j f_n^j(z). \end{aligned}$$

Setting $c = 2$ and $M_n = e^{-k+\bar{C}_h}/(M_n^{\mathcal{F}} M_n^h)$, we obtain (ii). Now we show the reverse implication. We slightly modify the proof of Lemma 4.1 by taking account of property (P). We only consider the case when $c > 0$. Let C_h and M_n^h be defined as in the proof of Lemma 4.1. Let $0 < \epsilon < 1$ be fixed. By (ii), there exists $N' \in \mathbb{N}$ such that

$$-\frac{3c}{2}\epsilon < \frac{1}{n+i}\log(e^{-nc\epsilon} M_n) < -\frac{c}{2}\epsilon, \quad \frac{1}{n}|\log(M_n^h e^{C_h})| < \epsilon \quad \text{and} \quad \frac{n}{n+k} > \frac{1}{2}$$

for all $n > N'$, $0 \leq i \leq k$. In the proof of Lemma 4.1, define $N_2 := \max\{N, N'\}$. Replacing M_n by $e^{-nc\epsilon} M_n$ in the proof of Lemma 4.1, we obtain that for any $x_1 \ldots x_n \in B_n(X)$, $n \geq N_2$, $z \in [x_1 \ldots x_n]$,

$$\frac{1}{n}\log\left(\frac{f_n(z)}{e^{(S_n h)(z)}}\right) < (4 + 3c)\epsilon.$$

Using the latter part of the proof of Lemma 4.1, we obtain the results. □

5. *Asymptotically additive sequences and subadditive sequences satisfying (C1) and (C2)*

In this section, we study the sequences $\mathcal{F}$ on subshifts X with bounded variation satisfying (C1) and (C2). Since there exists a unique Gibbs equilibrium state m for such a sequence $\mathcal{F}$ (Theorem 2.4), we study the condition for m to be an invariant Gibbs measure for some continuous function. In Theorem 5.6, we also characterize the form of sequences $\mathcal{F}$ in terms of the properties of equilibrium states.

THEOREM 5.1. *Let (X, σ_X) be a subshift and $\mathcal{F} = \{\log f_n\}_{n=1}^\infty$ be a sequence on X satisfying (C1) and (C2) with bounded variation. Let m be the unique invariant Gibbs measure for $\mathcal{F}$. Then the following statements are equivalent.*

(i) *There exists $h \in C(X)$ such that*

$$\lim_{n\to\infty} \frac{1}{n}\log\left(\frac{f_n(x)}{e^{(S_n h)(x)}}\right) = 0$$

for every $x \in X$.

(ii) *$\mathcal{F}$ is asymptotically additive on X.*

(iii) *The measure m is an invariant weak Gibbs measure for a continuous function on X.*

Remark 5.2.

(1) There exists a sequence $\mathcal{F}$ which satisfies (C1), (C2) with bounded variation satisfying Theorem 5.1(ii). On the other hand, there exists a sequence $\mathcal{F}$ with bounded variation satisfying (C1) and (C2) without being asymptotically additive (see §7).

(2) If $h \in C(X)$ in (i) exists, then m is a unique equilibrium state for h.

To prove Theorem 5.1, we apply the following lemmas. We continue to use $\mathcal{F}$ and m defined as in Theorem 5.1. In the next lemma we first study the relation between Theorem 5.1(i) and (ii).

LEMMA 5.3. *Let (X, σ_X) be a subshift and $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ be a sequence on X satisfying (C1) and (C2) with bounded variation. If there exists $h \in C(X)$ such that*

$$\lim_{n\to\infty} \frac{1}{n} \log \left(\frac{f_n(x)}{e^{(S_n h)(x)}} \right) = 0 \tag{14}$$

for every $x \in X$, then $\mathcal{F}$ is asymptotically additive on X.

Remark 5.4. Lemma 5.3 implies that if $\mathcal{F}$ satisfies the assumptions of the lemma then uniform convergence of the sequence of functions $\{1/n \log(f_n/e^{(S_n h)})\}_{n=1}^{\infty}$ is equivalent to pointwise convergence of the sequence of functions.

Proof. Let $\epsilon > 0$. It is enough to show that there exists $N \in \mathbb{N}$ such that for any $z \in [u], u \in B_n(X), n > N$,

$$-\epsilon < \frac{1}{n} \log \left(\frac{f_n(z)}{e^{(S_n h)(z)}} \right) < \epsilon.$$

Let p be defined as in (C2). Let $\overline{m}_h := \max_{0\le l\le p}\{e^{(S_l h)(x)} : x \in X\}$, where $(S_0 h)(x) := 1$ for every $x \in X$. Since h has tempered variation, let M_n^h be defined as in (11). Let M be a constant defined as in the definition of bounded variation and D be defined as in (C2). Then there exists $N_1 \in \mathbb{N}$ such that

$$\frac{1}{n} \log M < \epsilon, \quad \frac{1}{n} \log M_n^h < \epsilon, \quad \frac{1}{n}\left| \log \frac{1}{\overline{m}_h} \right| < \epsilon, \quad \frac{1}{n}| \log D| < \epsilon \quad \text{and} \quad \frac{n}{n+p} > \frac{1}{2}, \tag{15}$$

for all $n > N_1$. Take $n > N_1$. Condition (C2) implies that for a given $u \in B_n(X)$, there exists $w_1 \in B_{l_1}(X), 0 \le l_1 \le p$ such that for any $x \in [uw_1u], z \in [u]$,

$$\sup\{f_{2n+l_1}(x) : x \in [uw_1u]\} \ge D(\sup\{f_n(x) : x \in [u]\})^2 \ge Df_n^2(z).$$

Repeating this, given $j \ge 2, u \in B_n(X)$, there exist allowable words w_i of length l_i, $1 \le i \le j-1, 0 \le l_i \le p$, such that $uw_1uw_2u \ldots uw_{j-1}u$ is an allowable word of length $jn + \sum_{i=1}^{j-1} l_i$ satisfying that, for any $x \in [uw_1uw_2u \ldots uw_{j-1}u]$ and $z \in [u]$,

$$Mf_{jn+\sum_{i=1}^{j-1} l_i}(x) \ge \sup\{f_{jn+\sum_{i=1}^{j-1} l_i}(x) : x \in [uw_1uw_2u \ldots uw_{j-1}u]\} \ge D^{j-1} f_n(z)^j. \tag{16}$$

By the additivity of the sequence $\{S_n h\}_{n=1}^{\infty}$,

$$e^{(S_{jn+\sum_{i=1}^{j-1} l_i} h)(x)} \leq (M_n^h e^{S_n h(z)})^j \overline{m}_h{}^{j-1}. \tag{17}$$

Hence, by (16) and (17) we obtain for $j \geq 2$, $x \in [uw_1uw_2u \ldots uw_{j-1}u]$ and $z \in [u]$,

$$\frac{f_{jn+\sum_{i=1}^{j-1} l_i}(x)}{e^{(S_{jn+\sum_{i=1}^{j-1} l_i} h)(x)}} \geq \left(\frac{1}{M_n^h}\right)^j \left(\frac{f_n(z)}{e^{(S_n h)(z)}}\right)^j \left(\frac{D}{\overline{m}_h}\right)^{j-1} \cdot \frac{1}{M}. \tag{18}$$

Let $c_1 = [uw_1u], \ldots, c_i = [uw_1uw_2u \ldots uw_iu]$, $i \in \mathbb{N}$. Then by Cantor's intersection theorem $\bigcap_{i\in\mathbb{N}} c_i \neq \emptyset$ and it consists of exactly one point in X. We call it $x^* \in X$. For each $y \in X$, define $A_n(y) := f_n(y)/e^{(S_n h)(y)}$. By assumption (14), there exists $t(x^*) \in \mathbb{N}$, which depends on x^* such that for all $i \geq t(x^*)$,

$$-\epsilon < \frac{1}{i} \log A_i(x^*) < \epsilon.$$

Letting $s(u, j) := \sum_{i=1}^{j-1} l_i$, for $j \geq t(x^*) \geq 2$, and using (15) and (18), we obtain

$$\begin{aligned}
\epsilon &> \frac{1}{jn + s(u, j)} \log A_{jn+s(u,j)}(x^*) \\
&\geq \frac{1}{n + (1/j)s(u, j)} \log \frac{1}{M_n^h} + \frac{1 - 1/j}{n + (1/j)s(u, j)} \log \frac{1}{\overline{m}_h} + \frac{1}{jn + s(u, j)} \log \frac{1}{M} \\
&\quad + \frac{n(j-1)}{jn + s(u, j)} \cdot \frac{1}{n} \log D + \frac{n}{n + (1/j)s(u, j)} \cdot \frac{1}{n} \log A_n(z).
\end{aligned}$$

Without loss of generality, assume $\log A_n(z) > 0$. By a simple calculation, we obtain that

$$\frac{1}{n} \log A_n(z) < 10\epsilon \tag{19}$$

for all $n > N_1$, $z \in [u]$, for any $u \in B_n(X)$.

Next we will show that there exists $N_2 \in \mathbb{N}$ such that

$$-6\epsilon < \frac{1}{n} \log A_n(z) \tag{20}$$

for all $n > N_2$, $z \in [u]$ for any $u \in B_n(X)$. Define $f_0(x) := 1$. Let $\overline{M} := \max_{0\leq i\leq p}\{f_i(x) : x \in X\}$ and $\overline{m}_1 := \min_{0\leq k\leq p}\{e^{(S_k h)(x)} : x \in X\}$. Take N_2 so that

$$\frac{1}{n}|\log(MM_n^h)| < \epsilon, \ \frac{1}{n}\left|\log\left(\frac{\overline{M}e^{2C}}{\overline{m}_1}\right)\right| < \epsilon, \ \frac{n}{n+p} > \frac{1}{2} \tag{21}$$

for all $n > N_2$. For $n > N_2$, let $u \in B_n(X)$. Construct $x \in [uw_1uw_2 \ldots uw_{j-1}u]$, $j \geq 2$, as in the above argument and let $z \in [u]$. Using (C1), it is easy to obtain for each $j \geq 2$,

$$f_{jn+\sum_{i=1}^{j-1} l_i}(x) \leq (\overline{M}e^{2C})^{j-1}(Mf_n(z))^j \tag{22}$$

and

$$e^{(S_{jn+\sum_{i=1}^{j-1} l_i} h)(x)} \geq \left(\frac{e^{(S_n h)(z)}}{M_n^h}\right)^j (\overline{m}_1)^{j-1}. \tag{23}$$

https://doi.org/10.1017/etds.2022.30 Published online by Cambridge University Press

Define $x^* \in X$ as before. For all $j \geq t(x^*)$, by using (21), (22) and (23), we obtain

$$-\epsilon < \frac{1}{jn+s(u,j)} \log A_{jn+s(u,j)}(x^*) < 2\epsilon + \frac{n}{n+(1/j)s(u,j)} \cdot \frac{1}{n} \log A_n(z).$$

Without loss of generality, assuming that $\log A_n(z) < 0$, we obtain (20) for all $n > N_2$, each $z \in [u]$, $u \in B_n(X)$. The result follows by (19) and (20). □

LEMMA 5.5. *Under the assumptions of Theorem 5.1, $\mathcal{F}$ is asymptotically additive if and only if there exists a continuous function for which m is an invariant weak Gibbs measure.*

Proof. Suppose $\mathcal{F}$ is asymptotically additive. Then by [9, Theorem 1.2] there exist $h, u_n \in C(X)$, $n \in \mathbb{N}$, such that $f_n(x) = e^{(S_n h)(x)+u_n(x)}$ satisfying $\lim_{n\to\infty}(1/n)||u_n||_\infty = 0$. Since there exists a constant $C > 0$ such that

$$\frac{1}{C} \leq \frac{m[x_1 \ldots x_n]}{e^{-nP(\mathcal{F})} f_n(x)} \leq C \tag{24}$$

for each $x \in [x_1 \ldots x_n]$, replacing $f_n(x)$ by $e^{(S_n h)(x)+u_n(x)}$, we obtain

$$\frac{1}{Ce^{||u_n||_\infty}} \leq \frac{e^{u_n(x)}}{C} \leq \frac{m[x_1 \ldots x_n]}{e^{-nP(\mathcal{F})+(S_n h)(x)}} \leq Ce^{u_n(x)} \leq Ce^{||u_n||_\infty}.$$

Set $A_n = Ce^{||u_n||_\infty}$. Since $\lim_{n\to\infty}(1/n) \int \log f_n \, d\mu = \int h \, d\mu$ for every $\mu \in M(X, \sigma_X)$, we obtain that $P(\mathcal{F}) = P(h)$. Conversely, assume that m is an invariant weak Gibbs measure for $\tilde{h} \in C(X)$. Hence, there exists $C_n > 0$ such that

$$\frac{1}{C_n} \leq \frac{m[x_1 \ldots x_n]}{e^{-nP(\tilde{h})+(S_n\tilde{h})(x)}} \leq C_n \tag{25}$$

for all $x \in [x_1 \ldots x_n]$, where $\lim_{n\to\infty}(1/n)\log C_n = 0$. Since m is the Gibbs measure for $\mathcal{F}$,

$$\frac{1}{C} \leq \frac{m[x_1 \ldots x_n]}{e^{-nP(\mathcal{F})} f_n(x)} \leq C \tag{26}$$

for some $C > 0$. Using (25) and (26), we obtain

$$\frac{1}{C_n C} \leq \frac{f_n(x)}{e^{(S_n(\tilde{h}-P(\tilde{h})+P(\mathcal{F})))(x)}} \leq C_n C, \tag{27}$$

for all $x \in [x_1 \ldots x_n]$. Hence, by [9, Theorem 1.2], $\mathcal{F}$ is an asymptotically additive sequence. □

Proof of Theorem 5.1. By [9, Theorem 1.2], (ii) implies (i). Theorem 5.1 follows by Lemmas 5.3 and 5.5.

THEOREM 5.6. *Let (X, σ_X) be a subshift and $\mathcal{F} = \{\log f_n\}_{n=1}^\infty$ be a sequence on X satisfying (C1) and (C2) with bounded variation. Let m be the unique invariant Gibbs measure for $\mathcal{F}$. Suppose that one of the equivalent statements in Theorem 5.1 holds. Then the following statements hold.*

https://doi.org/10.1017/etds.2022.30 Published online by Cambridge University Press

(i) *There exits a sequence* $\{C_{n,m}\}_{n,m\in\mathbb{N}}$ *such that*

$$\frac{1}{C_{n,m}} \le \frac{f_{n+m}(x)}{f_n(x) f_m(\sigma_X^n x)} \le C_{n,m}, \quad \text{where } \lim_{n\to\infty} \frac{1}{n} \log C_{n,m} = \lim_{m\to\infty} \frac{1}{m} \log C_{n,m} = 0. \tag{28}$$

(ii) *If m is a Gibbs measure for a continuous function, then $\mathcal{F}$ is an almost additive sequence on X.*

Hence, if there is no sequence $\{C_{n,m}\}_{n,m\in\mathbb{N}}$ *satisfying (28), then there exists no continuous function for which m is an invariant weak Gibbs measure.*

Remark 5.7.

(1) In Example 7.2, we study a sequence which satisfies (C1) and (C2) without (28).
(2) See [**3**, Theorem 1.14(ii)] for the result related to (i). (ii) was also obtained in [**9**, §4.1] since m satisfies the quasi-Bernoulli property (see [**9**]).

Proof. Let h be defined as in Theorem 5.1(i). By the proofs of Lemmas 5.3 and 5.5, we obtain that $P(\mathcal{F}) = P(h)$ and m is an invariant weak Gibbs measure for h. Replacing $\tilde{h}$ by h in (27), we obtain that

$$\frac{1}{C^3 C_n C_m C_{n+m}} \le \frac{f_{n+m}(x)}{f_n(x) f_m(\sigma_X^n x)} \le C^3 C_{n+m} C_n C_m. \tag{29}$$

Since $\lim_{n\to\infty}(1/n)\log C_n = 0$, by setting $C_{n,m} := C^3 C_n C_m C_{n+m}$ we obtain the first statement. To obtain the second statement, we apply the latter part of the proof of Lemma 5.5. By replacing C_n in (25) and (27) by a constant, we obtain the second statement. The last statement follows from Theorem 5.1. □

6. *Relation between the existence of a continuous compensation function and an asymptotically additive sequence*

In this section we consider relative pressure functions $P(\sigma_X, \pi, f)$, where $f \in C(X)$. In general we can represent $P(\sigma_X, \pi, f)$ by using a subadditive sequence satisfying (D2). What are necessary and sufficient conditions for the existence of $h \in C(Y)$ satisfying $\int P(\sigma_X, \pi, f)\, dm = \int h\, dm$ for each $m \in M(Y, \sigma_Y)$? By [**9**, Theorem 2.1], if $P(\sigma_X, \pi, f)$ is represented by an asymptotically additive sequence then we can find such a function h. We will study necessary conditions for the existence of such a function h and relate them with the existence of a compensation function for a factor map between subshifts. To this end, we will apply the results from §4. We will study the property for periodic points from Lemma 4.1(ii).

THEOREM 6.1. (Relativized variational principle [**17**]) *Let (X, σ_X) and (Y, σ_Y) be subshifts and $\pi : X \to Y$ be a one-block factor map. Let $f \in C(X)$. Then for $m \in M(Y, \sigma_Y)$,*

$$\int P(\sigma_X, \pi, f)\, dm = \sup\left\{h_\mu(\sigma_X) - h_m(\sigma_Y) + \int f\, d\mu : \mu \in M(X, \sigma_X), \pi\mu = m\right\}. \tag{30}$$

Applying the relativized variational principle, we first study Borel measurable compensation functions for factor maps between subshifts.

PROPOSITION 6.2. *Let (X, σ_X) and (Y, σ_Y) be subshifts and $\pi : X \to Y$ be a one-block factor map. For each $f \in C(X)$, $f - P(\sigma_X, \pi, f) \circ \pi$ is a Borel measurable compensation function for π.*

Remark 6.3. In general, $f - P(\sigma_X, \pi, f) \circ \pi$ is not continuous on X.

Proof. Let $m \in M(Y, \sigma_Y)$ and $\phi \in C(Y)$. Applying Theorem 6.1, we obtain

$$\sup\left\{h_\mu(\sigma_X) - \int P(\sigma_X, \pi, f) \circ \pi \, d\mu + \int f \, d\mu + \int \phi \circ \pi \, d\mu : \mu \in M(X, \sigma_X), \pi\mu = m\right\}$$
$$= \sup\left\{h_\mu(\sigma_X) + \int f \, d\mu : \mu \in M(X, \sigma_X), \pi\mu = m\right\} - \int P(\sigma_X, \pi, f) \, dm + \int \phi \, dm$$
$$= h_m(\sigma_Y) + \int \phi \, dm.$$

Taking the supremum over $m \in M(Y, \sigma_Y)$, we obtain

$$\sup\left\{h_\mu(\sigma_X) - \int P(\sigma_X, \pi, f) \circ \pi \, d\mu + \int f \, d\mu + \int \phi \circ \pi \, d\mu : \mu \in M(X, \sigma_X)\right\}$$
$$= \sup\left\{h_m(\sigma_Y) + \int \phi \, dm : m \in M(Y, \sigma_Y)\right\}. \tag{31}$$

□

Let $\pi : X \to Y$ be a one-block factor map between subshifts. For $y = (y_i)_{i=1}^{\infty}$, let $E_n(y)$ be a set consisting of exactly one point from each cylinder $[x_1 \ldots x_n]$ in X such that $\pi(x_1 \ldots x_n) = y_1 \ldots y_n$. For $n \in \mathbb{N}$ and $f \in C(X)$, define

$$g_n(y) = \sup_{E_n(y)} \left\{ \sum_{x \in E_n(y)} e^{(S_n f)(x)} \right\}. \tag{32}$$

The following result can be deduced by [12, Proposition 3.7(i)]. If (X, σ_X) and (Y, σ_Y) are subshifts and $\pi : X \to Y$ is a one-block factor map, then for $f \in C(X)$,

$$P(\sigma_X, \pi, f)(y) = \limsup_{n\to\infty} \frac{1}{n} \log g_n(y) \tag{33}$$

μ-almost everywhere for every invariant Borel probability measure μ on Y. Equation (33) was shown by Petersen and Shin [19] for the case when X is an irreducible shift of finite type. The result for general subshifts is obtained by combining [12, Proposition 3.7(i)] and the fact that $P(\sigma_X, \pi, f)(y) \le \limsup_{n\to\infty}(1/n) \log g_n(y)$ for all $y \in Y$. Note that the function $P(\sigma_X, \pi, f)$ is bounded on Y.

LEMMA 6.4. *Let (X, σ_X) be a subshift with the weak specification property, (Y, σ_Y) be a subshift and $\pi : X \to Y$ be a one-block factor map. If $f \in C(X)$, then the sequence $\mathcal{G} = \{\log g_n\}_{n=1}^{\infty}$ on Y satisfies (C1) and (D2) with bounded variation.*

Remark 6.5. In particular, the sequence $\{\log g_n\}_{n=1}^{\infty}$ on Y satisfies (C1) and (C2) with bounded variation if $f \in C(X)$ is in the Bowen class (see [**12**, **14**, **32**]).

Proof. First we show that $\mathcal{G} = \{\log g_n\}_{n=1}^{\infty}$ satisfies (C1). Let $y = (y_1, \ldots, y_n, \ldots, y_{n+m}, \ldots) \in Y$. For each $x \in E_{n+m}(y)$, define S_x by $S_x := \{x' \in E_{n+m}(y) : x'_i = x_i, 1 \leq i \leq n\}$. Take a point $x^* \in S_x$ such that $e^{(S_n f)(x^*)} = \max\{e^{(S_n f)(z)} : z \in S_x\}$. Then we can construct a set $E_n(y)$ such that $x^* \in E_n(y)$. In a similar manner, for each $x \in E_{n+m}(y)$, define $S_{\sigma^n x}$ by $S_{\sigma^n x} := \{x' \in E_{n+m}(y) : x'_i = x_i, n+1 \leq i \leq m+n\}$ and take a point $x^{**} \in S_{\sigma^n x}$ such that $e^{(S_m f)(\sigma^n x^{**})} = \max\{e^{(S_m f)(z)} : z \in S_{\sigma^n x}\}$. Then we can construct a set $E_m(\sigma^n y)$ such that $\sigma^n x^{**} \in E_m(\sigma^n y)$. Hence, we obtain $g_{n+m}(y) \leq g_n(y) g_m(\sigma^n y)$. Next we show that $\mathcal{G} = \{\log g_n\}_{n=1}^{\infty}$ satisfies (D2). We modify slightly the arguments found in [**14**] (see also [**12**]) by taking account of the tempered variation of *f*, and we write a proof for completeness. Given $u \in B_n(Y)$ and $v \in B_m(Y)$, let $x_1 \ldots x_n \in B_n(X)$ such that $\pi(x_1 \ldots x_n) = u$ and let $z_1 \ldots z_m \in B_m(X)$ such that $\pi(z_1 \ldots z_m) = v$. Let p be a weak specification number of *X*. Then there exists $\tilde{w} \in B_k(X)$, $0 \leq k \leq p$, such that $x_1 \ldots x_n \tilde{w} z_1 \ldots z_m \in B_{n+m+k}(X)$. Hence, if $x \in [x_1 \ldots x_n \tilde{w} z_1 \ldots z_m]$, by letting $\overline{m} = \min_{0 \leq k \leq p}\{e^{(S_k f)(x)} : x \in X\}$, where $e^{(S_0 f)(x)} := 1$ for all $x \in X$, we obtain

$$e^{(S_{n+k+m} f)(x)} \geq \overline{m} e^{(S_n f)(x)} e^{(S_m f)(\sigma^{n+k} x)}. \tag{34}$$

For $n \in \mathbb{N}$, let $M_n := \sup\{e^{(S_n f)(x)}/e^{(S_n f)(x')} : x_i = x'_i, 1 \leq i \leq n\}$. Since *X* has the weak specification, *Y* also satisfies the weak specification property with specification number *p*. Define *S* by $S = \{w \in B_k(Y) : 0 \leq k \leq p, uwv \in B(Y)\}$ and let y_w be a point from the cylinder set $[uwv]$. Then

$$\begin{aligned}
\sum_{w \in S} \sum_{x \in E_{n+m+|w|}(y_w)} e^{(S_{n+m+|w|} f)(x)} &\geq \sum_{\substack{x \in [x_1 \ldots x_n \tilde{w} z_1 \ldots z_m], \\ \pi(x_1 \ldots x_n \tilde{w} z_1 \ldots z_m) \in [uwv]}} \overline{m} e^{(S_n f)(x)} e^{(S_m f)(\sigma^{n+k} x)} \\
&\geq \frac{\overline{m}}{M_n M_m} \bigg(\sum_{\pi(x_1 \ldots x_n) = u} \sup_{x \in [x_1 \ldots x_n]} e^{(S_n f)(x)} \bigg) \bigg(\sum_{\pi(z_1 \ldots z_m) = v} \sup_{z \in [z_1 \ldots z_m]} e^{(S_m f)(z)} \bigg) \\
&\geq \frac{\overline{m}}{M_n M_m} \sup\{g_n(y) : y \in [u]\} \sup\{g_m(y) : y \in [v]\}.
\end{aligned}$$

Hence,

$$\sum_{w \in S} g_{n+m+|w|}(y_w) \geq \frac{\overline{m}}{M_n M_m} \sup\{g_n(y) : y \in [u]\} \sup\{g_m(y) : y \in [v]\}.$$

Hence, there exits $\bar{w} \in S$ such that

$$g_{n+m+|\bar{w}|}(y_{\bar{w}}) \geq \frac{\overline{m}}{M_n M_m |S|} \sup\{g_n(y) : y \in [u]\} \sup\{g_n(y) : y \in [v]\}.$$

If *Y* is a subshift on *l* symbols, then $|S| \leq l^p$. Hence, $\mathcal{G}$ satisfies (D2) by setting $D_{n,m} = \overline{m}/(l^p M_n M_m)$. By the definition of $\mathcal{G}$, clearly $\mathcal{G}$ has bounded variation. □

LEMMA 6.6. [**28**] *Let (X, σ_X) and (Y, σ_Y) be subshifts and $\pi : X \to Y$ be a one-block factor map. Given $f \in C(X)$, the following statements are equivalent for $h \in C(Y)$.*

(i) $f - h \circ \pi$ *is a compensation function for* π.
(ii) $\int P(\sigma_X, \pi, f - h \circ \pi)\, dm = 0$ *for each* $m \in M(Y, \sigma_Y)$.
(iii) $m(\{y \in Y : P(\sigma_X, \pi, f - h \circ \pi)(y) = 0\}) = 1$ *for each* $m \in M(Y, \sigma_Y)$.

LEMMA 6.7. *Let* (X, σ_X) *and* (Y, σ_Y) *be subshifts and* $\pi : X \to Y$ *be a one-block factor map. Given* $f \in C(X)$*, the following statement for* $h \in C(Y)$ *is equivalent to the equivalent statements in Lemma 6.6:*

$$\int P(\sigma_X, \pi, f)\, dm = \int h\, dm \text{ for each } m \in M(Y, \sigma_Y).$$

Proof. Suppose that the equation in Lemma 6.7 holds for every $m \in M(Y, \sigma_Y)$. Then (31) implies that $f - h \circ \pi$ is a compensation function for π. Suppose that Lemma 6.6(iii) holds. Then (33) implies that m-almost everywhere,

$$P(\sigma_X, \pi, f - h \circ \pi)(y) = \limsup_{n\to\infty} \frac{1}{n} \log \left(\frac{g_n(y)}{e^{(S_n h)(y)}} \right),$$

where $g_n(y)$ is defined as in (32). Since $\{\log g_n\}_{n=1}^{\infty}$ is subadditive, $\{\log(g_n/e^{S_n h})\}_{n=1}^{\infty}$ is subadditive. Applying the subadditive ergodic theorem, we obtain, for each $m \in M(Y, \sigma_Y)$,

$$\begin{aligned}\int P(\sigma_X, \pi, f - h \circ \pi)\, dm &= \int \limsup_{n\to\infty} \frac{1}{n} \log \left(\frac{g_n}{e^{S_n h}} \right) dm \\ &= \lim_{n\to\infty} \frac{1}{n} \int \log \left(\frac{g_n}{e^{S_n h}} \right) dm \\ &= \lim_{n\to\infty} \frac{1}{n} \int \log g_n\, dm - \int h\, dm = 0.\end{aligned}$$

Hence, we obtain Lemma 6.7. □

LEMMA 6.8. *Let* $m \in M(Y, \sigma_Y)$. *Then*

$$P(\sigma_X, \pi, f - h \circ \pi)(y) = \lim_{n\to\infty} \frac{1}{n} \log \left(\frac{g_n(y)}{e^{(S_n h)(y)}} \right)$$

m-almost everywhere on Y.

Proof. The result follows by the subadditive ergodic theorem. □

The main result of this section is the next theorem which relates the existence of a continuous compensation function for a factor map with the asymptotically additive property of the sequences $\mathcal{G} = \{\log g_n\}_{n=1}^{\infty}$. Given $f \in C(X)$, we continue to use g_n as defined in equation (32).

THEOREM 6.9. *Let* (X, σ_X) *be an irreducible shift of finite type and* (Y, σ_Y) *be a subshift. Let* $\pi : X \to Y$ *be a one-block factor map and* $f \in C(X)$. *Then the following statements are equivalent for* $h \in C(Y)$.

(i) $P(\sigma_X, \pi, f - h \circ \pi)(y) = 0$ *for every periodic point* $y \in Y$*; equivalently,*

$$\lim_{n\to\infty} \frac{1}{n} \log \left(\frac{g_n(y)}{e^{(S_n h)(y)}} \right) = 0$$

for every periodic point $y \in Y$.

https://doi.org/10.1017/etds.2022.30 Published online by Cambridge University Press

(ii) *The function $f - h \circ \pi$ is a compensation function for π.*

(iii)

$$\lim_{n\to\infty} \frac{1}{n} \log \left(\frac{g_n(y)}{e^{(S_n h)(y)}} \right) = 0$$

for every $y \in Y$.

(iv) *The sequence $\mathcal{G} = \{\log g_n\}_{n=1}^{\infty}$ is asymptotically additive on Y satisfying*

$$\lim_{n\to\infty} \frac{1}{n} \left\| \log \left(\frac{g_n}{e^{(S_n h)}} \right) \right\|_{\infty} = 0.$$

(v) *$\int P(\sigma_X, \pi, f)\, dm = \int h\, dm$ for all $m \in M(Y, \sigma_Y)$.*

Remark 6.10.

(1) Theorem 6.9(i) with $f = 0$ is equivalent to the condition found by Shin [**26**, Theorem 3.5] for the existence of a saturated compensation function between two-sided irreducible shifts of finite type (see §7). Hence, by [**26**, Theorem 3.5], Theorem 6.9(i), (ii) and (v) are equivalent when $f = 0$ for a factor map between two-sided irreducible shifts of finite type. By the result of Cuneo [**9**, Theorem 2.1], if $\mathcal{G}$ is asymptotically additive then (v) holds for some $h \in C(Y)$.

(2) See §7 for some examples and properties of h.

Proof. It is clear that (iv) implies (iii). By Lemma 6.6, (iii) implies (ii) and (ii) implies (i). Now we show that (i) implies (iv). Suppose that (i) holds. It is enough to show that Lemma 4.1(ii) holds. Let X be an irreducible shift of finite type on a set S of finitely many symbols and k be a weak specification number of X. Let L be the cardinality of the set S. Let $y = (y_1, y_2, \ldots, y_n, \ldots) \in Y$. For a fixed $n \geq 3$, let $y_1 = a$, $y_n = b$. Then $\pi^{-1}(y_1) = \{a_1, \ldots, a_{L_1}\}$, where $a_i \in S$ for $1 \leq i \leq L_1$, for some $L_1 \leq L$, and $\pi^{-1}(y_n) = \{b_1, \ldots, b_{L_2}\}$ where $b_j \in S$ for $1 \leq j \leq L_2$, for some $L_2 \leq L$. Define $W_{ij} := \{a_i x_2 \ldots x_{n-1} b_j \in B_n(X) : \pi(a_i x_2 \ldots x_{n-1} b_j) = y_1 \ldots y_n\}$. Let $E_n^{i,j}(y)$ be a set consisting of exactly one point from each cylinder set $[u]$ of length n of X, where $u \in W_{ij}$. Define $C_{i,j} := \sum_{x\in E_n^{i,j}(y)} e^{(S_n f)(x)}$ and $M_n := \sup\{e^{(S_n f)(x)}/e^{(S_n f)(y)} : x_i = y_i, 1 \leq i \leq n\}$. If $W_{i,j} = \emptyset$, then define $C_{i,j} := 0$. Then

$$g_n(y) \geq \sum_{1\leq i\leq L_1, 1\leq j\leq L_2} C_{i,j} \geq \frac{1}{M_n} g_n(y),$$

where in the second equality we use the fact that, for any $E_n(y)$,

$$\frac{g_n(y)}{M_n} \leq \sum_{x\in E_n(y)} e^{(S_n f)(x)}.$$

Hence, there exist i_0, j_0 such that

$$C_{i_0,j_0} \geq \frac{1}{L_1 L_2 M_n} g_n(y) \geq \frac{1}{L^2 M_n} g_n(y). \tag{35}$$

Note that (i_0, j_0) depends on n. There exists an allowable word $w = w_1 \ldots w_q$ of length q in X, $0 \leq q \leq k$, such that $b_{j_0} w a_{i_0}$ is an allowable word of X. Take an allowable word

$a_{i_0}x_2 \ldots x_{n-1}b_{j_0} \in W_{i_0,j_0}$. Since X is an irreducible shift of finite type, we obtain a periodic point $\tilde{x} := (a_{i_0}, x_2, \ldots, x_{n-1}, b_{j_0}, w_1, \ldots, w_q)^\infty \in X$. Let $\pi(w_i) = d_i$ for each $i = 1, \ldots, q$. Let $y^* := \pi(\tilde{x})$. Then $y^* = (y_1, \ldots, y_n, d_1, \ldots d_q)^\infty$ is a periodic point of σ_Y.

For a fixed $n \geq 3$, define $P_0 := E_n^{i_0,j_0}(y)$. Define P_1 by

$$P_1 = \{z = (z_i)_{i=1}^\infty \in X : z_1 \ldots z_n \in W_{i_0,j_0}, z_{n+1} \ldots z_{n+q} = w, \sigma^{n+q} z = z\}.$$

Observe that if $z \in P_1$, then $\pi(z) = y^*$ and P_1 is a set consisting of exactly one point from each cylinder $[u]$ of length $(n+q)$ of X such that $\pi(u) = y_1 \ldots y_n d_1 \ldots d_q$ satisfying $u_1 \ldots u_n \in W_{i_0,j_0}$ and $u_{n+1} \ldots u_{n+q} = w$. Then

$$g_{n+q}(y^*) = \sup_{E_{n+q}(y^*)} \left\{ \sum_{x \in E_{n+q}(y^*)} e^{(S_{n+q}f)(x)} \right\} \geq \sum_{x \in P_1} e^{(S_{n+q}f)(x)} \geq \frac{e^m}{M_n} \left(\sum_{x \in P_0} e^{(S_n f)(x)} \right),$$

where $m := \min_{0 \leq i \leq k}\{e^{(S_i f)(x)} : x \in X\}$, $(S_0 f)(x) := 1$ for every $x \in X$. Next define P_2 by

$$P_2 = \{z = (z_i)_{i=1}^\infty \in X : \text{ for each } j = 0, 1, z_{j(n+q)+1} \ldots z_{n(j+1)+jq} \in W_{i_0,j_0},$$
$$z_{(j+1)n+jq+1} \ldots z_{(j+1)(n+q)} = w, \sigma^{2(n+q)} z = z\}.$$

Observe that if $z \in P_2$, then $\pi(z) = y^*$ and P_2 is a set consisting of one point from each cylinder $[u]$ of length $(2n+2q)$ of X such that $\pi(u) = y_1 \ldots y_n d_1 \ldots d_q y_1 \ldots y_n d_1 \ldots d_q$ satisfying $u_1 \ldots u_n, u_{n+q+1} \ldots u_{2n+q} \in W_{i_0,j_0}$ and $u_{n+1} \ldots u_{n+q} = u_{2n+q+1} \ldots u_{2n+2q} = w$. Hence,

$$\begin{aligned} g_{2(n+q)}(y^*) &= \sup_{E_{2(n+q)}(y^*)} \left\{ \sum_{x \in E_{2(n+q)}(y^*)} e^{(S_{2(n+q)}f)(x)} \right\} \\ &\geq \sum_{x \in P_2} e^{(S_{2(n+q)}f)(x)} \geq \frac{e^{2m}}{M_n^2} \left(\sum_{x \in P_0} e^{(S_n f)(x)} \right)^2. \end{aligned}$$

Applying (35), we obtain

$$g_{2(n+q)}(y^*) \geq \frac{e^{2m}}{M_n^2} \left(\sum_{x \in P_0} e^{(S_n f)(x)} \right)^2 \geq \frac{e^{2m} g_n^2(y^*)}{L^4 M_n^4}.$$

Similarly, for $j \geq 3$, define the set P_j of periodic points by

$$P_j = \{z = (z_i)_{i=1}^\infty \in X : \quad \text{for each } 0 \leq l \leq j-1, z_{l(n+q)+1} \ldots z_{(l+1)n+lq} \in W_{i_0,j_0},$$
$$z_{(l+1)n+lq+1} \ldots z_{(l+1)(n+q)} = w, \sigma^{j(n+q)} z = z\}.$$

If $z \in P_j$, then $\pi(z) = y^*$ and P_j is a set consisting of one point from each cylinder $[u]$ of length $j(n+q)$ such that $\pi(u) = (y_1 \ldots y_n d_1 \ldots d_q)^j$ satisfying $u_{l(n+q)+1} = a_{i_0}$, $u_{(l+1)n+lq} = b_{j_0}$ and $u_{(l+1)n+lq+1} \ldots u_{(l+1)(n+q)} = w$ for each $0 \leq l \leq j-1$.

https://doi.org/10.1017/etds.2022.30 Published online by Cambridge University Press

Then we obtain

$$g_{j(n+q)}(y^*) = \sup_{E_{j(n+q)}(y^*)} \left\{ \sum_{x \in E_{j(n+q)}(y^*)} e^{(S_{j(n+q)} f)(x)} \right\}$$
$$\geq \sum_{x \in P_j} e^{(S_{j(n+q)} f)(x)} \geq \frac{e^{jm}}{M_n^j} \left(\sum_{x \in P_0} e^{(S_n f)(x)} \right)^j.$$

Applying (35), we obtain

$$g_{j(n+q)}(y^*) \geq \frac{e^{jm}}{M_n^j} \left(\sum_{x \in P_0} e^{(S_n f)(x)} \right)^j \geq \left(\frac{e^m}{L^2 M_n^2} \right)^j g_n^j(y^*).$$

Since the function g_n is locally constant, for $n \geq 3$,

$$g_{j(n+q)}(y^*) \geq \left(\frac{e^m}{L^2 M_n^2} \right)^j \sup\{g_n(z) : z \in [y_1 \ldots y_n]\}^j$$

for every $j \in \mathbb{N}$. Hence, condition (ii) in Lemma 4.1 holds. Applying Lemma 4.1, we obtain (iv). Finally, (iv) $\Longrightarrow$ (v) is immediate and (v) implies (ii) by Lemma 6.7. □

Recall that if $f \in C(X)$ is in the Bowen class, then $\mathcal{G} = \{\log g_n\}_{n=1}^{\infty}$ satisfies (C1) and (C2) and $\mathcal{G}$ has the unique Gibbs equilibrium state.

COROLLARY 6.11. *Under the assumptions of Theorem 6.9, assume also that $f \in C(X)$ is a function in the Bowen class and let m be the unique Gibbs equilibrium state for $\mathcal{G} = \{\log g_n\}_{n=1}^{\infty}$. Suppose that one of the equivalent statements in Theorem 6.9 holds. Then:*

(i) *m is an invariant weak Gibbs measure for h;*
(ii) *equation (28) holds by replacing f_n by g_n;*
(iii) *if m is a Gibbs measure for a continuous function, then $\mathcal{G}$ is almost additive.*

Hence, if there is no sequence $\{C_{n,m}\}_{n,m \in \mathbb{N}}$ satisfying (28) by replacing f_n by g_n, then there does not exist a continuous function h on Y such that

$$\int P(\sigma_X, \pi, f)\, d\mu = \int h\, d\mu$$

for every $\mu \in M(Y, \sigma_Y)$.

Proof. Since $\lim_{n\to\infty}(1/n)|| \log(g_n/e^{S_n h})||_\infty = 0$, applying the first part of the proof of Lemma 5.5, m is an invariant weak Gibbs measure for h. To show the second statement, we use similar arguments to the proof of Theorem 5.6. To show the third statement, we apply the proof of Theorem 5.6. The last statement is obvious by Theorem 6.9. □

Remark 6.12. Applying Theorem 4.3, we can study Theorem 6.9 under a more general setting. Let (X, σ_X), (Y, σ_Y) be subshifts and $\pi : X \to Y$ be a one-block factor map. Given a function $f \in C(X)$, suppose that $\mathcal{G} = \{\log g_n\}_{n=1}^{\infty}$ satisfies Theorem 4.4(ii). Then Theorem 6.9 holds. It would be interesting to study the conditions on factor maps π satisfying Theorem 4.4(ii) for $\mathcal{G}$.

7. *Applications*

In this section, we give some examples and applications. Applying the results from the previous sections, we study the existence of a saturated compensation function for a factor map between subshifts and factors of weak Gibbs measures for continuous functions.

7.1. *Existence of continuous saturated compensation functions.* Let (X, σ_X) be an irreducible shift of finite type, Y be a subshift and $\pi : X \to Y$ be a one-block factor map. For $n \in \mathbb{N}$, let ϕ_n be the continuous function on Y obtained by setting $f = 0$ in equation (32). Set $\Phi = \{\log \phi_n\}_{n=1}^{\infty}$.

For a factor map π between subshifts, there always exists a Borel measurable saturated compensation function $-P(\sigma_X, \pi, 0) \circ \pi$ given by a superadditive sequence $-\Phi \circ \pi$; however, a continuous saturated compensation function does not always exist. Shin [**26**] considered a one-block factor map $\pi : X \to Y$ between two-sided irreducible shifts of finite type and gave an equivalent condition for the existence of a saturated compensation function (see [**26**, Theorem 3.5] for details). Note that the condition is equivalent to Theorem 6.9(i) with $f = 0$.

Here we characterize the existence of a saturated compensation function in terms of the type of the sequence Φ by applying Theorem 6.9.

COROLLARY 7.1. *Let (X, σ_X) be an irreducible shift of finite type, Y be a subshift and $\pi : X \to Y$ be a one-block factor map. Then $-h \circ \pi$, $h \in C(Y)$ is a saturated compensation function if and only if one of the equivalent statements in Theorem 6.9 holds with $f = 0$. In particular, a saturated compensation function exists if and only if Φ is asymptotically additive on Y. If $-h \circ \pi$ is a compensation function, then h has the unique equilibrium state and it is a weak Gibbs measure for h. If there does not exist $\{C_{n,m}\}_{(n,m)\in\mathbb{N}\times\mathbb{N}}$ satisfying equation (28) for Φ, then there exists no continuous saturated compensation function for π.*

Proof. The result follows by setting $f = 0$ in Theorem 6.9 and Corollary 6.11. □

Example 7.2. (A sequence satisfying (C1) and (C2) which is not asymptotically additive [**26**]) Shin [**26**, Example 3.1] gave an example of a factor map $\pi : X \to Y$ between two-sided irreducible shifts of finite type X, Y without a saturated compensation function. We note that the same results hold for one-sided subshifts. The sequence $\Phi = \{\log \phi_n\}_{n=1}^{\infty}$ is a subadditive sequence satisfying (C1) and (C2) with bounded variation and there exists a unique Gibbs equilibrium state ν for Φ. Since there is no saturated compensation function, there does not exist a continuous function $h \in C(Y)$ such that

$$\lim_{n\to\infty} \frac{1}{n} \int \log \phi_n \, dm = \int h \, dm$$

for every $m \in M(Y, \sigma_Y)$. Hence, Φ is not an asymptotically additive sequence and there does not exist a continuous function on Y for which ν is an invariant weak Gibbs measure (see Theorem 7.8). Alternatively, a simple calculation shows that for any $x \in [12^m 1]$ where

https://doi.org/10.1017/etds.2022.30 Published online by Cambridge University Press

$m \geq 3$ is odd,

$$\frac{\phi_{2+m}(x)}{\phi_2(x)\phi_m(\sigma^2 x)} = \frac{|\pi^{-1}[12^m 1]|}{|\pi^{-1}[12]||\pi^{-1}[2^{m-1}1]|} = \frac{1}{2^{(m-1)/2}+2}$$

(see [**26**]). Hence, for any sequence $\{C_{n,m}\}_{(n,m)\in\mathbb{N}\times\mathbb{N}}$ satisfying equation (28) for Φ, we obtain that $C_{2,m} \geq 2^{(m-1)/2}+2$. By Corollary 7.1, there does not exist a continuous saturated compensation function.

Remark 7.3.

(1) Pfister and Sullivan [**20**] studied a class of continuous functions satisfying bounded total oscillations on two-sided subshifts and showed that if a continuous function f belongs to the class under a certain condition then an equilibrium state for f is a weak Gibbs measure for some continuous function. Shin [**24**, Proposition 3.5] gave an example of a saturated compensation function $G \circ \pi$ for a factor map $\pi : X \to Y$ between two-sided irreducible shifts of finite type X, Y where $-G$ does not have bounded total oscillations. Let (X^+, σ_X^+) and (Y^+, σ_Y^+) be the corresponding one-sided shifts of finite type and consider the factor map $\pi^+ : X^+ \to Y^+$. Then the corresponding saturated compensation function $G^+ \circ \pi$ for π^+, $G^+ \in C(Y^+)$, is obtained. Applying Theorem 6.9 and Corollary 6.11, $-G^+$ has a unique equilibrium state and it is a weak Gibbs measure for $-G^+$.

(2) See §2 in [**3**] for examples of measures which are not weak Gibbs studied in quantum physics.

Example 7.4. (A sequence satisfying (C1) and (C2) which is also asymptotically additive) In [**30**], saturated compensation functions were studied to find the Hausdorff dimensions of some compact invariant sets of expanding maps of the torus. In [**30**, Example 5.1], given a factor map π between topologically mixing shifts of finite type X and Y, a saturated compensation function $G \circ \pi$, $G \in C(Y)$, was found and $-G$ has a unique equilibrium state ν which is not Gibbs. Applying Theorem 6.9 and Corollary 6.11, ν is an invariant weak Gibbs measure for $-G$.

Remark 7.5.

(1) In [**12**, **31**], the ergodic measures of full Hausdorff dimension for some compact invariant sets of certain expanding maps of the torus were identified with equilibrium states for sequences of continuous functions. If a saturated compensation function exists, then they are the equilibrium states of a constant multiple of a saturated compensation [**30**].

(2) In Example 7.4 [**30**, Example 5.1], X and Y are one-sided shifts of finite type. Considering the corresponding two-sided shifts of finite type $\hat{X}, \hat{Y}$ and the factor map $\hat{\pi}$ between them, a saturated compensation function $\hat{G} \circ \pi$ for $\hat{\pi}$, $\hat{G} \in C(\hat{Y})$, is obtained in the same manner as G is obtained. The function $-\hat{G}$ on $\hat{Y}$ does not have bounded total oscillations (see Remark 7.3(1)).

7.2. *Factors of invariant weak Gibbs measures.* Factors of invariant Gibbs measures for continuous functions and related topics have been widely studied (see, for example, [**7**, **8**, **12**, **16**, **21**–**23**, **27**, **31**–**33**]). For a survey of the study of factors of Gibbs measures, see the

https://doi.org/10.1017/etds.2022.30 Published online by Cambridge University Press

paper by Boyle and Petersen [4]. In this section, more generally, we study the properties of factors of invariant weak Gibbs measures. Given a one-block factor map $\pi : X \to Y$, and $f \in C(X)$, define g_n for each $n \in \mathbb{N}$ as in (32) and $\mathcal{G} = \{\log g_n\}_{n=1}^{\infty}$ on Y.

THEOREM 7.6. *Let (X, σ_X) be an irreducible shift of finite type, Y be a subshift and $\pi : X \to Y$ be a one-block factor map. Suppose there exists μ such that μ is an invariant weak Gibbs measure for $f \in C(X)$. Then $\pi\mu$ is an invariant weak Gibbs measure for $\mathcal{G} = \{\log g_n\}_{n=1}^{\infty}$ on Y. There exists $h \in C(Y)$ such that $\lim_{n\to\infty}(1/n) \int \log g_n \, dm = \int h \, dm$ for all $m \in M(Y, \sigma_Y)$ if and only if one of the equivalent statements in Theorem 6.9(i)–(iv) holds. Moreover, such a function h exists if and only if the invariant measure $\pi\mu$ is a weak Gibbs measure for a continuous function on Y.*

Remark 7.7.

(1) If f is in the Bowen class, then there is a unique Gibbs equilibrium state for $\mathcal{G}$ and Corollary 6.11 also applies.

(2) If there exists μ such that μ is an invariant weak Gibbs measure for $f \in C(X)$, then $\pi\mu$ is an equilibrium state for $\mathcal{G}$.

Proof. To prove the first statement, we apply similar arguments to the proof of [32, Theorem 3.7] and we outline the proof. Suppose that $f \in C(X)$ has an invariant weak Gibbs measure μ. Then there exists $C_n > 0$ such that

$$\frac{1}{C_n} \le \frac{\mu[x_1 \ldots x_n]}{e^{-nP(f)+(S_n f)(x)}} \le C_n$$

for each $x \in [x_1 \ldots x_n]$, where $\lim_{n\to\infty}(1/n) \log C_n = 0$. Since f has tempered variation, if we let

$$M_n = \sup\left\{\frac{e^{(S_n f)(x)}}{e^{(S_n f)(y)}} : x, y \in X, x_i = y_i \text{ for } 1 \le i \le n\right\},$$

then $\lim_{n\to\infty}(1/n) \log M_n = 0$. Using the definition of the topological pressure and after some calculations, we obtain that $P(f) = P(\mathcal{G})$. Since

$$\pi\mu[y_1 \ldots y_n] = \sum_{\substack{x_1\ldots x_n \in B_n(X) \\ \pi(x_1\ldots x_n) = y_1\ldots y_n}} \mu[x_1 \ldots x_n],$$

using similar arguments to the proof of [32, Theorem 3.7], we obtain

$$\frac{1}{C_n M_n} \le \frac{\pi\mu[y_1 \ldots y_n]}{e^{-nP(\mathcal{G})} g_n(y)} \le C_n M_n.$$

Hence, $\pi\mu$ is an invariant weak Gibbs measure for $\mathcal{G}$. The second statement holds by Theorem 6.9. Now we show the last statement. Suppose such h exists. Modifying slightly the proof of Corollary 6.11(i), taking into account the fact that $\pi\mu$ is a weak Gibbs measure for $\mathcal{G}$, we obtain that $\pi\mu$ is a weak Gibbs measure for h. To see the reverse implication, suppose $\pi\mu$ is weak Gibbs for some $\tilde{h}$. Then there exists $A_n > 0$ such that

$$\frac{1}{A_n} \le \frac{\pi\mu[y_1 \ldots y_n]}{e^{-nP(\tilde{h})+(S_n\tilde{h})(y)}} \le A_n \tag{36}$$

https://doi.org/10.1017/etds.2022.30 Published online by Cambridge University Press

for each $y \in [y_1 \dots y_n]$, where $\lim_{n\to\infty}(1/n)\log A_n = 0$. If we let $K_n = C_n M_n$, then the similar arguments to the latter part of the proof of Lemma 5.5 show that

$$\frac{1}{K_n A_n} \leq \frac{g_n(y)}{e^{(S_n(\tilde{h}-P(\tilde{h})+P(\mathcal{G})))(y)}} \leq K_n A_n$$

for each $y \in [y_1 \dots y_n]$. Hence, $\mathcal{G}$ is asymptotically additive. Set $h = \tilde{h} - P(\tilde{h}) + P(\mathcal{G})$. □

The proof of Theorem 7.6 gives us the following result.

THEOREM 7.8. *Under the assumptions of Theorem 6.9, suppose there exists μ such that μ is an invariant weak Gibbs measure for $f \in C(X)$. Then there exists $h \in C(Y)$ satisfying the equivalent statements in Theorem 6.9 if and only if there exists a continuous function on Y for which $\pi\mu$ is an invariant weak Gibbs measure on Y.*

COROLLARY 7.9. *Under the assumptions of Theorem 7.6, if there is no sequence $\{C_{n,m}\}_{n,m\in\mathbb{N}}$ satisfying equation (28) by replacing f_n by g_n, then there does not exist a continuous function h on Y such that $\lim_{n\to\infty}(1/n)\int \log g_n\, dm = \int h\, dm$ for every $m \in M(Y, \sigma_Y)$. Hence, there exists no continuous function on Y for which $\pi\mu$ is an invariant weak Gibbs measure on Y.*

Proof. Suppose there exists $h \in C(Y)$ such that $\lim_{n\to\infty}(1/n)\int \log g_n\, dm = \int h\, dm$ for every $m \in M(Y, \sigma_Y)$. By Theorem 7.6, $\mathcal{G}$ is asymptotically additive and $\pi\mu$ is an invariant weak Gibbs measure for h. Hence, there exists $A_n > 0$ such that (36) holds for h for each $y \in [y_1 \dots y_n]$, where $\lim_{n\to\infty}(1/n)\log A_n = 0$. Let K_n be defined as in the proof of Theorem 7.6. Using $P(h) = P(\mathcal{G})$ and additivity of $\{S_n h\}_{n=1}^{\infty}$, we obtain

$$\frac{1}{K_{n+m}A_{n+m}K_n A_n K_m A_m} \leq \frac{g_{n+m}(y)}{g_n(y)g_m(\sigma_Y^n y)} \leq K_{n+m}A_{n+m}K_n A_n K_m A_m.$$

Define $C_{n,m} := K_{n+m}A_{n+m}K_n A_n K_m A_m$ for each $n, m \in \mathbb{N}$. Then $\lim_{n\to\infty}(1/n)\log C_{n,m} = \lim_{m\to\infty}(1/m)\log C_{n,m} = 0$. Hence, the result follows from Theorem 7.6. □

Acknowledgements. The author was partly supported by CONICYT PIA ACT172001 and by 196108 GI/C at the Universidad del Bío-Bío. The author thanks the referee for valuable comments.

REFERENCES

[1] J. Antonioli. Compensation functions for factors of shifts of finite type. *Ergod. Th. & Dynam. Sys.* **36**(2) (2016), 375–389.

[2] L. Barreira. Nonadditive thermodynamic formalism: equilibrium and Gibbs measures. *Discrete Contin. Dyn. Syst.* **16**(2) (2006), 279–305.

[3] T. Benoist, N. Cuneo, V. Jakšić and C.-A. Pillet. On entropy production of repeated quantum measurements II. Examples. *J. Stat. Phys.* **182**(3) (2021), Paper no. 44.

[4] M. Boyle and K. Petersen. Hidden Markov processes in the context of symbolic dynamics. *Entropy of Hidden Markov Processes and Connections to Dynamical Systems (London Mathematical Society Lecture Note Series, 385)*. Eds. B. Marcus, K. Petersen and T. Weissman. Cambridge University Press, Cambridge, 2011, pp. 5–71.

[5] M. Boyle and S. Tuncel. Infinite-to-one codes and Markov measures. *Trans. Amer. Math. Soc.* **285**(2) (1984), 657–684.
[6] Y. L. Cao, D. J. Feng and W. Huang. The thermodynamic formalism for sub-additive potentials. *Discrete Contin. Dyn. Syst.* **20**(3) (2008), 639–657.
[7] J. R. Chazottes and E. Ugalde. Projection of Markov measures may be Gibbsian. *J. Stat. Phys.* **111**(5–6) (2003), 1245–1272.
[8] J. R. Chazottes and E. Ugalde. On the preservation of Gibbsianness under symbol amalgamation. *Entropy of Hidden Markov Processes and Connections to Dynamical Systems (London Mathematical Society Lecture Note Series, 385)*. Eds. B. Marcus, K. Petersen and T. Weissman. Cambridge University Press, Cambridge, 2011, pp. 72–97.
[9] N. Cuneo. Additive, almost additive and asymptotically additive potential sequences are equivalent. *Comm. Math. Phys.* **37**(3) (2020), 2579–2595.
[10] K. J. Falconer. A subadditive thermodynamic formalism for mixing repellers. *J. Phys. A* **21**(14) (1988), 737–742.
[11] D. J. Feng. Lyapunov exponents for products of matrices and multifractal analysis. II. General matrices. *Israel J. Math.* **170** (2009), 355–394.
[12] D. J. Feng. Equilibrium states for factor maps between subshifts. *Adv. Math.* **226**(3) (2011), 2470–2502.
[13] D. J. Feng and W. Huang. Lyapunov spectrum of asymptotically sub-additive potentials. *Comm. Math. Phys.* **297**(1) (2010), 1–43.
[14] G. Iommi, C. Lacalle and Y. Yayama. Hidden Gibbs measures on shift spaces over countable alphabets. *Stoch. Dyn.* **20**(4) (2020), 2050028.
[15] A. Käenmäki and H. Reeve. Multifractal analysis of Birkhoff averages for typical infinitely generated self-affine sets. *J. Fractal Geom.* **1**(1) (2014), 83–152.
[16] T. Kempton. Factors of Gibbs measures for subshifts of finite type. *Bull. Lond. Math. Soc.* **43**(4) (2011), 751–764.
[17] F. Ledrappier and P. Walters. A relativised variational principle for continuous transformations. *J. Lond. Math. Soc. (2)* **16**(3) (1977), 568–576.
[18] A. Mummert. The thermodynamic formalism for almost-additive sequences. *Discrete Contin. Dyn. Syst.* **16**(2) (2006), 435–454.
[19] K. Petersen and S. Shin. On the definition of relative pressure for factor maps on shifts of finite type. *Bull. Lond. Math. Soc.* **37**(4) (2005), 601–612.
[20] C.-E. Pfister and W. G. Sullivan. Asymptotic decoupling and weak Gibbs measures for finite alphabet shift spaces. *Nonlinearity* **33**(9) (2020), 4799–4817.
[21] M. Piraino. Projections of Gibbs states for Hölder potentials. *J. Stat. Phys.* **170**(5) (2018), 952–961.
[22] M. Piraino. Single site factors of Gibbs measures. *Nonlinearity* **33**(2) (2020), 742–761.
[23] M. Pollicott and T. Kempton. Factors of Gibbs measures for full shifts. *Entropy of Hidden Markov Processes and Connections to Dynamical Systems (London Mathematical Society Lecture Note Series, 385)*. Eds. B. Marcus, K. Petersen and T. Weissman. Cambridge University Press, Cambridge, 2011, pp. 246–257.
[24] S. Shin. Measures that maximize weighted entropy for factor maps between subshifts of finite type. *Ergod. Th. & Dynam. Sys.* **21**(4) (2001), 1249–1272.
[25] S. Shin. An example of a factor map without a saturated compensation function. *Ergod. Th. & Dynam. Sys.* **21**(6) (2001), 1855–1866.
[26] S. Shin. Relative entropy functions for factor maps between subshifts. *Trans. Amer. Math. Soc.* **358**(6) (2006), 2205–2216.
[27] E. Verbitskiy. On factors of g-measures. *Indag. Math. (N.S.)* **22**(3–4) (2011), 315–329.
[28] P. Walters. Relative pressure, relative equilibrium states, compensation functions and many-to-one codes between subshifts. *Trans. Amer. Math. Soc.* **296**(1) (1986), 1–31.
[29] P. Walters. Regularity conditions and Bernoulli properties of equilibrium states and g-measures. *J. Lond. Math. Soc. (2)* **71**(2) (2005), 379–396.
[30] Y. Yayama. Dimension of compact invariant sets of some expanding maps. *Ergod. Th. & Dynam. Sys.* **29**(1) (2009), 281–315.
[31] Y. Yayama. Existence of a measurable saturated compensation function between subshifts and its applications. *Ergod. Th. & Dynam. Sys.* **31**(5) (2011), 1563–1589.
[32] Y. Yayama. On factors of Gibbs measures for almost additive potentials. *Ergod. Th. & Dynam. Sys.* **36**(1) (2016), 276–309.
[33] J. Yoo. On factor maps that send Markov measures to Gibbs measure. *J. Stat. Phys.* **141**(6) (2010), 1055–1070.