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INVOLUTIONS ON THE SECOND DUALS OF GROUP ALGEBRAS AND A MULTIPLIER PROBLEM

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Abstract We show that if a locally compact group G is non-discrete or has an infinite amenable subgroup, then the second dual algebra $L^1(G)^{**}$ does not admit an involution extending the natural involution of $L^1(G)$. Thus, for the above classes of groups we answer in the negative a question raised by Duncan and Hosseiniun in 1979. We also find necessary and sufficient conditions for the dual of certain left-introverted subspaces of the space $C_b(G)$ (of bounded continuous functions on G) to admit involutions. We show that the involution problem is related to a multiplier problem. Finally, we show that certain non-trivial quotients of $L^1(G)^{**}$ admit involutions.

Keywords: amenable group; Arens product; involution; multiplier; left uniformly continuous function; weakly almost periodic function

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1. Introduction and preliminaries

Let G be a locally compact group and $L^1(G)$ be its group algebra. On $L^1(G)$ there is a *natural involution* $\sim$ defined by

$$\tilde{f}(x) = \Delta(x^{-1})\overline{f(x^{-1})}, \qquad f \in L^1(G),\ x \in G,$$

where Δ is the modular function of the group G and the bar denotes complex conjugation. Suppose that the second dual space $L^1(G)^{**}$ is given the first Arens product (see the definition below). In [**4**, p. 323] Duncan and Hosseiniun ask whether there is an involution on $L^1(G)^{**}$ extending the natural involution of $L^1(G)$. In this paper we show that if G is an infinite non-discrete or an infinite amenable group, the answer to the above question is negative. For an amenable group G we show that the dual $\mathrm{LUC}(G)^*$ of the space of left uniformly continuous functions on G admits an involution extending the natural involution of $L^1(G)$ if and only if G is compact, whereas for any G the dual $\mathrm{WAP}(G)^*$ of the space of weakly almost periodic functions on G admits an involution extending the natural involution of $L^1(G)$. We show that, for the remaining class of groups, the above

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question can be answered in the affirmative if a question about right multipliers (right $L^1(G)$-module morphisms) from $L^\infty(G)^{**}$ into $L^\infty(G)^{**}$ has a positive answer. Finally, we show that certain non-trivial quotients of $L^1(G)^{**}$ always admit involutions.

We recall the definition of the first (left) Arens product on the second dual A^{**} of a Banach algebra A. For $f \in A^*$, $a \in A$, let $f \cdot a \in A^*$ be defined by

$$\langle f \cdot a, b\rangle = \langle f, ab\rangle, \quad b \in A.$$

Now for $G \in A^{**}$ and $f \in A^*$, let $G \cdot f \in A^*$ be defined by

$$\langle G \cdot f, a\rangle = \langle G, f \cdot a\rangle, \quad a \in A.$$

Finally, for $F, G \in A^{**}$ let $F \,\square\, G \in A^{**}$ be defined by

$$\langle F \,\square\, G, f\rangle = \langle F, G \cdot f\rangle, \quad f \in A^*.$$

We will also need the following definition of the second (right) Arens product. With the same notation as above we define $a \cdot f \in A^*$, $f \cdot F$ and $F \diamond G$ respectively by

$$\begin{aligned} \langle a \cdot f, b\rangle &= \langle f, ba\rangle, & b \in A,\\ \langle f \cdot F, a\rangle &= \langle F, a \cdot f\rangle, & a \in A,\\ \langle F \diamond G, f\rangle &= \langle G, f \cdot F\rangle, & f \in A^*. \end{aligned}$$

The Banach algebra A is said to be Arens regular if $F \,\square\, G = F \diamond G$, for all $F, G \in A^{**}$, i.e. whenever the two products coincide. For a Banach space X and any element $x \in X$, the image of x in X^{**} under the canonical mapping will be denoted by $\hat{x}$. Let $F, G \in A^{**}$, and let (f_i), (g_i) be nets in A^{**} such that $\hat{f}_i \xrightarrow{w^*} f$ and $\hat{g}_i \xrightarrow{w^*} g$ (in the weak-$*$ topology of A^{**}). Then

$$F \,\square\, G = w^* - \lim_i \lim_j \widehat{(f_i g_j)},$$

$$F \diamond G = w^* - \lim_j \lim_i \widehat{(f_i g_j)}.$$

For a locally compact group G we will denote the Banach space of all continuous bounded complex-valued functions on G by $C_b(G)$, where the norm is taken to be the sup-norm. A norm closed subspace X of $C_b(G)$ is left introverted if, for every $n \in X^*$, $f \in X$ and $t \in G$, we have both $n \cdot f \in X$ and $l_t f \in X$, where $n \cdot f$ and $l_x f$ are defined respectively by

$$(n \cdot f)(t) = \langle n, l_t f\rangle,$$

$$(l_t f)(s) = f(ts), \quad s \in G.$$

Two examples of left-introverted subspaces of $C_b(G)$, with which we will be concerned in the following, are the space $\mathrm{LUC}(G)$ (of left uniformly continuous functions on G) and the space $\mathrm{WAP}(G)$ (of weakly almost periodic functions on G); the first one consists of

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all functions $f \in C_b(G)$ for which the map $x \mapsto l_x f$ from G into $C_b(G)$ is continuous, while the second space consists of all $f \in C_b(G)$ such that $\{l_x f : x \in G\}$ has weakly compact closure in $C_b(G)$. When X is a left-introverted subspace of $C_b(G)$, X^* can be made into a Banach algebra if we define the product mn of elements $m, n \in X^*$ by

$$\langle mn, f\rangle = \langle m, n \cdot f\rangle, \quad f \in X,$$

where $n \cdot f$ is as defined earlier.

2. Some lemmas

Lemma 2.1. *Suppose A is a Banach algebra with a continuous involution $\sim$. Then $\sim$ has an extension to a continuous conjugate linear mapping on A^{**}, denoted by the same symbol $\sim$, such that $(m \,\square\, n)^{\sim} = \tilde{n} \diamond \tilde{m}$ for all $m, n \in A^{**}$. Furthermore, if the original involution is isometric, then so is the extension.*

Proof. We can define a new scalar product $\lambda \cdot a$, by $\lambda \cdot a = \bar{\lambda} a$, $\lambda \in \mathbb{C}$, $a \in A$. With the new scalar product, $\sim$ can be identified with a bounded linear operator T. Let T^{**} be the second conjugate of T. Then T^{**} is the required extension; in fact let $m, n \in A^{**}$ and let (a_i) and (b_j) be bounded nets in A such that $\hat{a}_i \xrightarrow{w^*} m$ and $\hat{b}_j \xrightarrow{w^*} n$ in A^{**}. Then

$$\begin{aligned} T^{**}(m \,\square\, n) &= w^* - \lim_i \lim_j T^{**}((a_i b_j)\hat{\ }) \\ &= w^* - \lim_i \lim_j (T(b_j) T(a_i))\hat{\ } \\ &= T^{**}(n) \diamond T^{**}(m). \end{aligned}$$

Obviously, T^{**} is isometric when T is isometric. □

Corollary 2.2. *Let $\sim$ be a continuous involution on the Banach algebra A, and let $\sim$ be extended to A^{**} as in Lemma 2.1. Then the extension is an involution if and only if A is Arens regular.*

Remark 2.3. From Corollary 2.2 we see that if G is an infinite group, then the second adjoint of the natural involution on $L^1(G)$ is not an involution on $L^1(G)^{**}$, since $L^1(G)$ is not Arens regular [**12**].

The proof of the next lemma is straightforward.

Lemma 2.4. *Let X be a left-introverted subspace of $C_b(G)$. Then the map $\pi : L^\infty(G)^* \to X^*$, induced by restriction, is an algebra homomorphism.*

Lemma 2.5. *$L^1(G)^{**}$ with the first (or second) Arens product has an identity if and only if G is discrete.*

Proof. If G is discrete, then $L^1(G)$ has an identity and so $L^1(G)^{**}$ has an identity. Now suppose that $L^1(G)^{**}$ with the first Arens product has an identity E. We first show that $\mathrm{LUC}(G) = L^\infty(G)$. Suppose to the contrary that $\mathrm{LUC}(G) \neq L^\infty(G)$. Then we can find a non-zero $m \in L^\infty(G)^*$ annihilating $\mathrm{LUC}(G)$. By using the factorization

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$\mathrm{LUC}(G) = L^\infty(G) \cdot L^1(G)$ [**6**,**7**] together with the weak star continuity of the first Arens product in the first variable and weak star denseness of $L^1(G)$ in $L^1(G)^{**}$, we find that m is a right annihilator of $L^1(G)^{**}$. In particular, $m = Em = 0$, which is a contradiction. So $\mathrm{LUC}(G) = L^\infty(G)$, whenever $L^1(G)^{**}$ has an identity. Thus, G is discrete in this case. □

3. Consequences of existence of involutions

Proposition 3.1. *Suppose that there is an involution on $L^1(G)^{**}$. Then G is discrete.*

Proof. Let † be an involution on $L^1(G)^{**}$. Since $L^1(G)$ has a bounded approximate identity, $L^1(G)^{**}$ has a right identity, say E [**1**, p. 146]. For any $m \in L^1(G)^{**}$ we have

$$E^\dagger \mathbin{\square} m = (m^\dagger \mathbin{\square} E)^\dagger = (m^\dagger)^\dagger = m.$$

Hence, $L^1(G)^{**}$ has an identity and we have the result from Lemma 2.5. □

Before proving the next theorem, we need to explain some notation and facts: for $f \in C_0(G)$, define $\check{f} \in C_0(G)$ by $\check{f}(x) = \overline{f(x^{-1})}$. Then for $\mu \in M(G)$ (the measure algebra of G) define $\tilde{\mu} \in M(G)$ by

$$\langle \tilde{\mu}, f \rangle = \overline{\langle \mu, \check{f} \rangle}, \quad f \in C_0(G).$$

Then one can easily see that $\sim$ on $M(G)$ extends the natural involution of $L^1(G)$, so especially for $\varphi \in L^1(G)$ we have $\langle \tilde{\varphi}, f \rangle = \overline{\langle \varphi, \check{f} \rangle}$.

We had originally proved part (a) of the following theorem for amenable G. We thank the referee for pointing out that the result holds only if one assumes that G has an infinite amenable subgroup.

Theorem 3.2.

(a) *Suppose that the locally compact group G has an infinite amenable subgroup. Then $L^1(G)^{**}$ does not admit an involution extending the natural involution of $L^1(G)$.*

(b) *Suppose that G is amenable. Then $\mathrm{LUC}(G)^*$ has an involution extending the natural involution of $L^1(G)$ if and only if G is compact.*

Proof. (a) Let $*$ be an involution on $L^1(G)^{**}$ extending the natural involution of $L^1(G)$. From Proposition 3.1 we can assume that G is discrete and, hence, for every $x \in G$ the point mass δ_x belongs to $L^1(G)$ $(= l^1(G))$. Let H be an infinite amenable subgroup of G and let m be a two-sided invariant mean on $l^\infty(H)$. For $x \in H$ we have

$$\delta_x m^* = (m\delta_{x^{-1}})^* = m^*, \tag{3.1}$$

and so m^* is left translation invariant.

From (2.1) it follows that

$$mm^* = m, \tag{3.2}$$

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since $l^1(H)$ is weak* dense in $l^1(H)^{**}$ and the embedding of $l^1(H)^{**}$ into $l^1(G)^{**}$ is weakly* continuous. From (2.4) it follows that $m = m^*$. Now let m_1 and m_2 be any pair of two-sided invariant means on $l^\infty(H)$. Then $m_2 = m_1 m_2 = (m_1 m_2)^* = m_2 m_1 = m_1$, contradicting [**2**].

(b) If G is compact, then $\mathrm{LUC}(G)^* = M(G)$ and the result is clear.

For the converse, let $\dagger$ be an involution on $\mathrm{LUC}(G)^*$ extending the natural involution $\sim$ of $L^1(G)$. We have $\mathrm{LUC}(G)^* = M(G) \oplus C_0(G)^\perp$ [**5**, Lemma 1.1]). Let $\mu \in M(G)$ and $f \in \mathrm{LUC}(G)$. Using the factorizations $\mathrm{LUC}(G) = L^\infty(G) \cdot L^1(G)$ and $L^1(G) = L^1(G) * L^1(G)$ [**6**, **7**], one has the factorization $\mathrm{LUC}(G) = \mathrm{LUC}(G) \cdot L^1(G)$, so that $f = f_1 \cdot \varphi$ for some $f_1 \in \mathrm{LUC}(G)$ and $\varphi \in L^1(G)$. Hence,

$$\begin{aligned}\langle \mu^\dagger, f\rangle &= \langle \mu^\dagger, f_1 \cdot \varphi\rangle = \langle \varphi\mu^\dagger, f_1\rangle = \langle (\mu\varphi^\dagger)^\dagger, f_1\rangle \\ &= \langle (\mu\tilde{\varphi})^\dagger, f_1\rangle = \langle (\mu * \tilde{\varphi})^\sim, f_1\rangle = \langle \varphi * \tilde{\mu}, f_1\rangle \\ &= \langle \tilde{\mu}, f_1 \cdot \varphi\rangle.\end{aligned}$$

Hence, $\mu^\dagger = \tilde{\mu}$. In particular, $(\delta_x)^\dagger = \delta_{x^{-1}}$. Now one can repeat the argument of part (a) word for word, except that, concerning the cardinality of invariant means, one needs to use the more general result of [**8**]. □

The result of the preceding theorem may suggest that G is compact whenever there is a left-introverted subspace X of $C_b(G)$ containing $C_0(G)$ such that X has a left invariant mean and such that X^* has an involution extending the natural involution of $L^1(G)$. We will show that this is not the case for $X = \mathrm{WAP}(G)$.

Definition 3.3. Let A be a Banach algebra. A linear functional $f \in A^*$ is called *weakly almost periodic* on A if the operator $a \mapsto f \cdot a$ from A into A^* is weakly compact (equivalently, $a \mapsto a \cdot f$ is weakly compact, by the Grothendieck double limit theorem).

We denote the space of all weakly almost periodic functionals on A by $\mathrm{WAP}(A)$. It is easily verified that $\mathrm{WAP}(A)$ is a Banach submodule of the dual Banach A-module A^*. The space $\mathrm{WAP}(A)^*$ can be made into a Banach algebra by an Arens-type product: for $n \in \mathrm{WAP}(A)^*$ and $f \in \mathrm{WAP}(A)$, let $n \cdot f \in A^*$ such that

$$\langle n \cdot f, a\rangle = \langle n, f \cdot a\rangle, \quad a \in A,$$

and for $m, n \in \mathrm{WAP}(A)^*$, let $mn \in \mathrm{WAP}(G)^*$ such that

$$\langle mn, f\rangle = \langle m, n \cdot f\rangle, \quad f \in \mathrm{WAP}(A).$$

It is easy to prove that with this product $\mathrm{WAP}(A)^*$ is a Banach algebra, and the map $a \mapsto \hat{a}$, where $\hat{a}(f) = f(a)$, is a continuous algebra homomorphism from A into $\mathrm{WAP}(A)^*$.

The following result is due to Pym [**9**].

Proposition 3.4. *Let A be a Banach algebra and $f \in A^*$. Then f is weakly almost periodic on A if and only if $\langle m \,\Box\, n, f\rangle = \langle m \diamond n, f\rangle$, for all $m, n \in A^{**}$.*

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Theorem 3.5. *Suppose that the Banach algebra A has a continuous involution $\sim$, and further, suppose that the algebra homomorphism $a \mapsto \hat{a}$ from A into $\mathrm{WAP}(A)^*$ is an embedding. Then $\sim$ extends to a continuous involution on $\mathrm{WAP}(A)^*$.*

Proof. Let T be the bounded operator on A associated with $\sim$, as in the proof of Lemma 2.1. We note that $T^*(\mathrm{WAP}(A)) \subset \mathrm{WAP}(A)$. In fact, if $f \in \mathrm{WAP}(A)$ and $a \in A$, then direct calculations show that $a \cdot T^*(f) = T^*(f \cdot T(a))$, so that $a \mapsto a \cdot T^*(f)$ is weakly compact. Now we define $\dagger$ on $\mathrm{WAP}(A)^*$ by $\langle n^\dagger, f\rangle = \langle T^{**}(n'), f\rangle$, where $n \in \mathrm{WAP}(A)^*$, $f \in \mathrm{WAP}(A)$ and n' is any Hahn–Banach extension of n. From the above observations we see that $n^\dagger$ is well defined. Now let n_1, $n_2 \in \mathrm{WAP}(A)^*$, and let n_i', $i = 1, 2$, be an extension of n_i. Then $n_1' \mathbin{\square} n_2'$ is an extension of $n_1 n_2$, and so for $f \in \mathrm{WAP}(A)$ we have

$$\begin{aligned} \langle (n_1 n_2)^\dagger, f\rangle &= \langle T^{**}(n_1' \mathbin{\square} n_2'), f\rangle = \langle T^{**}(n_2') \diamond T^{**}(n_1'), f\rangle \\ &= \langle T^{**}(n_2') \mathbin{\square} T^{**}(n_1'), f\rangle = \langle T^{**}(n_2'), T^{**}(n_1') f\rangle \\ &= \langle n_1^\dagger, T^{**}(n_1') f\rangle. \end{aligned} \tag{3.3}$$

Now we calculate $T^{**}(n_1')f$. In fact if $a \in A$, then

$$\langle T^{**}(n_1')f, a\rangle = \langle T^{**}(n_1'), f \cdot a\rangle = \langle n_1^\dagger, f \cdot a\rangle$$

and so $T^{**}(n_1')f = n_1^\dagger f$. Together with (3.3), this shows that $(n_1 n_2)^\dagger = n_2^\dagger n_1^\dagger$. The proof is complete. □

Corollary 3.6. *For a locally compact group G, the Banach algebra $\mathrm{WAP}(G)^*$ has an isometric involution that extends the natural involution of $L^1(G)$.*

Proof. First we note that a bounded continuous function f on G is weakly almost periodic if and only if the functional

$$\Lambda_f : g \mapsto \int_G f(x) g(x) \, \mathrm{d}x$$

is a weakly almost periodic functional on $L^1(G)$ [**11**]. Thus, from Theorem 3.5, $\mathrm{WAP}(G)$ admits an isometric involution which extends the natural involution of $L^1(G)$. □

4. A multiplier problem related to involutions

Suppose that $T : L^\infty(G) \to L^\infty(G)$ is a continuous right $L^1(G)$-module morphism, so that $T(f \cdot \varphi) = T(f) \cdot \varphi$, $f \in L^\infty(G)$, $\varphi \in L^1(G)$. We can give a concrete description of T. In fact, if T^* is the adjoint of T, then T^* is a left $L^1(G)$ module morphism, $T^*(\varphi \cdot n) = \varphi \cdot T^*(n)$. Hence, by weak* continuity of the first Arens product with respect to the first variable, we have $T^*(m \mathbin{\square} n) = m \mathbin{\square} T^*(n)$, for all $m, n \in L^1(G)^{**}$. By letting $n = E$, a right identity for $L^1(G)^{**}$, we find $T^*(m) = m \mathbin{\square} T^*(E)$, for any $m \in L^1(G)^{**}$. Let $N = T^*(E)$. Then, for $m \in L^1(G)^{**}$ and $f \in L^\infty(G)$,

$$\langle m, T(f)\rangle = \langle T^*(m), f\rangle = \langle m \mathbin{\square} N, f\rangle = \langle m, N \cdot f\rangle$$

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and so $T(f) = N \cdot f$, $f \in L^\infty(G)$. Conversely, for every $N \in L^1(G)^{**}$, the map $T_N : f \mapsto N \cdot f$ defines a continuous right $L^1(G)$-module morphism. Similarly, by using the second Arens product, it can be shown that a bounded operator $T : L^\infty(G) \to L^\infty(G)$ is a left $L^1(G)$-module morphism if and only if there is $N \in L^1(G)^{**}$ such that $T(f) = f \cdot N$. Now let $N \in L^1(G)^{****}$. Then it is easy to verify that

$$R_N f \mapsto N \cdot f, \quad f \in L^\infty(G)^{**},$$

is a right $L^1(G)$-module morphism from $L^\infty(G)^{**}$ into $L^\infty(G)^{**}$.

Problem 4.1. Is every weak-$*$ continuous surjective right $L^1(G)$-module morphism $R : L^\infty(G)^{**} \to L^\infty(G)^{**}$ equal to R_N for some $N \in L^1(G)^{****}$?

Theorem 4.2. *Suppose that problem 4.1 is answered affirmatively. Then $L^1(G)^{**}$ has a continuous involution extending the natural involution of $L^1(G)$ if and only if G is finite.*

Proof. Let θ be an involution on $L^1(G)^{**}$ extending the natural involution of $L^1(G)$. Then from Proposition 3.1, G is discrete. Let T be the bounded operator associated with the natural involution on $l^1(G)$, as in Lemma 2.1, and let T^{**} be the second adjoint of T. We have $T(\delta_x) = \delta_{x^{-1}}$, $x \in G$, and

$$T^{**}(m \,\Box\, n) = T^{**}(n) \diamond T^{**}(m), \quad m, n \in l^1(G)^{**}.$$

Since $\theta(\delta_x n) = \theta(n)\delta_{x^{-1}}$, $x \in G$, we have $T^{**}\theta(\delta_x n) = \delta_x T^{**}\theta(n)$. Hence, by linearity and continuity we have

$$(T^{**}\theta)(fn) = f(T^{**}\theta)(n), \quad f \in l^1(G),\ n \in l^1(G)^{**}.$$

Let $S = T^{**}\theta$. Then from the hypothesis, there exists $N \in l^1(G)^{****}$ such that $S^*(f) = N \cdot f$, $f \in l^\infty(G)^{**}$. Then, for $m \in l^1(G)^{**}$ and $f \in l^\infty(G)^{**}$,

$$\begin{aligned} \langle S(m), f\rangle &= \langle S^*(f), m\rangle = \langle N \cdot f, m\rangle \\ &= \langle \hat{m}, N \cdot f\rangle = \langle \hat{m} \,\Box\, N, f\rangle. \end{aligned} \tag{4.1}$$

For $m = \delta_e$, the above equation yields $\langle \delta_e, f\rangle = \langle N, f\rangle$, $f \in l^\infty(G)^{**}$, and so $N = \delta_e$. Hence, $S = I$, the identity operator on $l^1(G)^{**}$. Since $(T^{**})^2 = T^{**}$, we obtain $\theta = T^{**}$. But T^{**} is an involution if and only if $l^1(G)$ is Arens regular and this can happen if and only if G is finite [**12**]. □

5. Some non-trivial quotients of $L^1(G)^{**}$ admitting involutions

Although we have shown that when an infinite group G is non-discrete or amenable then $L^1(G)^{**}$ does not admit any involution extending the natural involution of $L^1(G)$, in this section we show that, for certain non-trivial quotients of $L^1(G)^{**}$, admission can happen.

We recall that the *left regular representation* of $L^1(G)$ onto $L^2(G)$ is the mapping $\lambda : L^1(G) \to B(L^2(G))$ such that

$$\lambda(\varphi)(g) = \varphi * g, \quad \varphi \in L^1(G),\ g \in L^2(G).$$

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The mapping λ is a norm-decreasing $*$-homomorphism, when $L^1(G)$ and $B(L^2(G))$ have their natural involutions [**7**].

Proposition 5.1. *Let $\lambda : L^1(G) \to B(L^2(G))$ be the left regular representation. Then*

(a) *the image of λ^{**} is a self-adjoint subalgebra of the C^*-algebra $B(L^2(G))^{**}$,*

(b) *the kernel of λ^{**} contains the radical of $L^1(G)^{**}$.*

Proof. (a) Let $m \in L^1(G)^{**}$ and (φ_α) be a net in $L^1(G)$ such that $\|\varphi_\alpha\| \leqslant \|m\|$ and $\hat{\varphi}_\alpha \xrightarrow{w^*} m$. Let $\sim$ be the natural involution on $L^1(G)$. Then $\|\tilde{\varphi}_\alpha\| \leqslant \|m\|$, so that a subnet of $(\tilde{\varphi}_\alpha)\hat{}$ converges to some $n \in L^1(G)^{**}$. We can assume that $(\tilde{\varphi}_\alpha)\hat{} \xrightarrow{w^*} n$. By the weak-$*$ continuity of λ^{**} we have

$$\lim_\alpha \lambda^{**}(\hat{\varphi}_\alpha) = \lambda^{**}(m)$$

and

$$\lim_\alpha \lambda^{**}((\tilde{\varphi}_\alpha)\hat{}) = \lambda^{**}(n).$$

On the other hand, from [**10**, Theorem 1.7.8] the involution of a von Neumann algebra is continuous in the weak-$*$ topology, so that in the weak-$*$ topology we have

$$\begin{aligned}\lambda^{**}(n) &= \lim_\alpha \lambda^{**}((\tilde{\varphi}_\alpha)\hat{}) = \lim_\alpha (\lambda(\tilde{\varphi}_\alpha))\hat{} \\ &= \lim_\alpha (\lambda(\varphi_\alpha)^*)\hat{} = \lim_\alpha (\lambda^{**}(\hat{\varphi}_\alpha))^* = (\lambda^{**}(m))^*,\end{aligned}$$

indicating that $(\lambda^{**}(m))^* = \lambda^{**}(n)$. Hence, $\lambda^{**}(L^1(G)^{**})$ is self-adjoint. Furthermore, $\lambda^{**}(L^1(G)^{**})$ is an algebra since λ^{**} is an algebra homomorphism [**3**].

(b) Let $m \in \mathrm{Rad}(L^1(G)^{**})$ (the radical) and let $n \in L^1(G)^{**}$ be as obtained in part (a). Then the spectral radius $r(mn) = 0$. Since an algebra homomorphism is always spectral radius reducing, we have

$$r(\lambda^{**}(m)(\lambda^{**}(m))^*) = r(\lambda^{**}(m)\lambda^{**}(n)) \leqslant r(mn) = 0.$$

Since $\lambda^{**}(m)\lambda^{**}(m)^*$ is self-adjoint, we have $\lambda^{**}(m) = 0$, and the proof is complete. □

Corollary 5.2. *Let $I = \ker \lambda^{**}$. Then $L^1(G)^{**}/I$ admits an involution.*

Proof. This is clear from Proposition 5.1 (a). □

Concerning the radical of $L^1(G)^{**}$, there is an old open question asking whether, for all groups G, $\mathrm{Rad}(L^1(G)^{**}) \neq \{0\}$. When G is non-discrete or amenable the answer is affirmative; in fact, if G is non-discrete then $\mathrm{LUC}(G) \neq L^\infty(G)$ and, by using the factorization $\mathrm{LUC}(G) = L^\infty(G) \cdot L^1(G)$, one can easily show that any element of $\mathrm{LUC}(G)^\perp$ is a right annihilator of $L^1(G)^{**}$. If G is infinite, discrete and amenable, we let m and n be distinct left invariant means on $l^\infty(G)$. Then $(m-n)^2 = 0$, and so the spectral radius of $m-n$ is zero. Now if $p \in l^\infty(G)^*$, then $p \,\Box\, (m-n) = \langle p, \mathbf{1}\rangle(m-n)$. Hence, the spectral radius of $p \,\Box\, (m-n)$ is 0, showing that $(m-n) \in \mathrm{Rad}(l^1(G)^{**})$.

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Problem 5.3. Let G be discrete and suppose that $l^1(G)^{**}$ admits an involution that extends the natural involution of $l^1(G)$. Does it then follow that $\mathrm{Rad}(l^1(G)^{**}) = \{0\}$?

Problem 5.4. Is $\mathrm{Ker}(\lambda^{**})$ larger than $\mathrm{Rad}(L^1(G)^{**})$?

Corollary 5.5. *Let* $\rho : L^1(G)^{**} \to \mathrm{WAP}(G)^*$ *be the map restricting functionals on* $L^\infty(G)$ *to* $\mathrm{WAP}(G)$*, and let* $I = \mathrm{Ker}(\rho)$*. Then* $L^1(G)^{**}/I$ *admits an involution.*

Proof. This is immediate from Lemma 2.4 and Corollary 3.6. □

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References

1. F. F. Bonsall and J. Duncan, *Complete normed algebras* (Springer, 1973).
2. C. Chou, The exact cardinality of the set of invariant means on a group, *Proc. Am. Math. Soc.* **55** (1976), 103–106.
3. P. Civin and B. Yood, The second conjugate space of a Banach algebra as an algebra, *Pac. J. Math.* **11** (1961), 847–870.
4. J. Duncan and S. A. R. Hosseiniun, The second dual of a Banach algebra, *Proc. R. Soc. Edinb.* A **84** (1979), 309–325.
5. F. Ghahramani, A. T. Lau and V. Losert, Isometric isomorphism between Banach algebras related to locally compact groups, *Trans. Am. Math. Soc.* **321** (1990), 273–283.
6. E. Hewitt and K. A. Ross, *Abstract harmonic analysis*, Volume 2 (Springer, 1970).
7. E. Hewitt and K. A. Ross, *Abstract harmonic analysis*, Volume 1, 2nd edn (Springer, 1979).
8. A. T. Lau and A. L. T. Paterson, The exact cardinality of the set of topological left invariant means on an amenable locally compact group, *Proc. Am. Math. Soc.* **98** (1986), 75–80.
9. J. S. Pym, The convolution of functionals on spaces of bounded functions, *Proc. Lond. Math. Soc.* **15** (1965), 84–104.
10. S. Sakai, *C^*-algebras and W^*-algebras* (Springer, 1971).
11. A. Ülger, Continuity of weakly periodic functional on $L^1(G)$, *Q. J. Math.* **37** (1986), 495–497.
12. N. J. Young, The irregularity of multiplication in group algebras, *Q. J. Math.* **24** (1973), 59–62.

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