J. Inst. Math. Jussieu (2023), **22**(1), 213–273 213
doi:10.1017/S1474748021000128

R-GROUP AND WHITTAKER SPACE OF SOME GENUINE REPRESENTATIONS

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(Received 16 December 2019; revised 28 January 2021; accepted 9 February 2021; first published online 8 March 2021)

Abstract For a unitary unramified genuine principal series representation of a covering group, we study the associated R-group. We prove a formula relating the R-group to the dimension of the Whittaker space for the irreducible constituents of such a principal series representation. Moreover, for certain saturated covers of a semisimple simply connected group, we also propose a simpler conjectural formula for such dimensions. This latter conjectural formula is verified in several cases, including covers of the symplectic groups.

Keywords: covering groups, R-groups, Whittaker functionals, scattering matrix, gamma factor, Plancherel measure

2020 *Mathematics Subject Classification*: Primary 11F70
Secondary 22E50

Contents

1. Introduction

Let F be a local field of characteristic 0. Let G be the F-rational points of a split connected reductive group over F. Assume that $F^\times$ contains the full group μ_n of nth roots of unity. In this paper we consider a central extension

$$\mu_n \hookrightarrow \overline{G} \twoheadrightarrow G$$

of G by μ_n arising from the Brylinski–Deligne framework [11]. A representation (π, V_π) of $\overline{G}$ is called genuine if μ_n acts on V_π by a fixed embedding $\mu_n \hookrightarrow \mathbf{C}^\times$.

In [18], we proposed and partially proved a conjectural formula for the dimension of a certain relative Whittaker space of the irreducible constituents of a *regular* unramified genuine principal series representation $I(\chi)$ of $\overline{G}$ over a non-archimedean field F. The formula in [18] relates certain Kazhdan–Lusztig representations of the Weyl group to the dimensions of such relative Whittaker spaces.

This paper, as a companion to [18], deals with the case where $I(\chi)$ is a *unitary* unramified genuine principal series – that is, χ is a unitary unramified genuine character of the centre $Z\left(\overline{T}\right)$ of the covering torus $\overline{T} \subset \overline{G}$. In this case, $I(\chi)$ is a semisimple $\overline{G}$-module, and has a decomposition

$$I(\chi) = \bigoplus_{\pi \in \Pi(\chi)} m_\pi \cdot \pi,$$

where π are the nonequivalent constituents of $I(\chi)$. These representations π thus should constitute an L-packet $\Pi(\chi)$. The reducibility of $I(\chi)$ and this decomposition is controlled by a certain Knapp–Stein R-group $R_\chi \subset W_\chi$, where $W_\chi \subset W$ is the stabiliser subgroup of χ inside the Weyl group W. In particular, there is a correspondence

$$\operatorname{Irr}(R_\chi) \longleftrightarrow \Pi(\chi), \qquad \sigma \leftrightarrow \pi_\sigma, \tag{1.1}$$

between the irreducible representations of R_χ and elements in $\Pi(\chi)$ such that $m_{\pi_\sigma} = \dim(\sigma)$. Since χ is unramified, one can show that R_χ is abelian and therefore $m_{\pi_\sigma} = 1$ for every σ.

Fix a Whittaker datum $\left(\overline{B} = \overline{T}U, \psi\right)$ for $\overline{G}$, where U is the unipotent radical of the Borel subgroup $\overline{B}$ and $\psi : U \to \mathbf{C}^\times$ is a nondegenerate character. It is well known that genuine representations of covering groups could have high-dimensional ψ-Whittaker space (i.e., the space of ψ-Whittaker functionals; see the introductions of [18, 19] for brief literature reviews). In particular, $\dim \mathrm{Wh}_\psi(I(\chi))$ increases as the degree of covering increases. In view of the correspondence in formula (1.1), it is natural to ask:

- How can $\dim \mathrm{Wh}_\psi(\pi_\sigma)$ be determined in terms of $\sigma \in \operatorname{Irr}(R_\chi)$?

Our goal is first to prove a formula for $\dim \mathrm{Wh}_\psi(\pi_\sigma)$ for general $\overline{G}$. Second, for certain saturated covers (see Definition 2.1) of a semisimple simply connected G, we propose a simpler conjectural formula for $\dim \mathrm{Wh}_\psi(\pi_\sigma)$ in terms of the character pairing of σ and a certain permutation representation $\sigma^{\mathscr{X}}$ of R_χ. We will also verify several cases of this conjectural formula in this paper.

https://doi.org/10.1017/S1474748021000128 Published online by Cambridge University Press

1.1. Background and motivation

We briefly recall some relevant works for linear algebraic groups which motivate our consideration in this paper.

For a linear algebraic group G and χ a character of T (not necessarily unramified), correspondence (1.1) arises from the theory of Harish-Chandra and Knapp and Stein for the commuting algebra $\mathrm{End}(I(\chi))$ of the principal series representation $I(\chi)$, especially from the algebra isomorphism

$$\mathbf{C}[R_\chi] \simeq \mathrm{End}(I(\chi)). \tag{1.2}$$

In fact, for general parabolic induction (i.e., parabolically induced representation) for G, the theory of R-groups was initiated in the work of Knapp and Stein for real groups [29]. Based on Harish-Chandra's work [46], Silberger worked out the formulation for p-adic groups [44, 45]; in particular, he showed that Harish-Chandra's commuting algebra theorem holds, which then gives an analogue of isomorphism (1.2) for general parabolic induction on linear algebraic groups. Here, for simplicity in this introduction, we ignore the subtleties of the two-cocycle twisting in the algebra of the R-group for general parabolic inductions (see [3] for details). For minimal parabolic induction of a Chevalley group, the group R_χ was computed explicitly by Keys [25]; the study in the unramified case was furthered in [26].

In the minimal parabolic case, the connection of the R-group with the Langlands correspondence was elucidated in [27, 28]. For example, let ϕ_χ be the L-parameter associated to the character χ; Keys showed that the component group $\mathcal{S}_{\phi_\chi}$ is isomorphic to R_χ, and thus elements inside the L-packet $\Pi(\chi)$ are also naturally parametrised by $\mathrm{Irr}\left(\mathcal{S}_{\phi_\chi}\right)$. Beyond the principal series case, it was conjectured by Arthur [2] that the R-group associated with the parabolic induction $I(\sigma)$ from a discrete series σ on the Levi subgroup is also isomorphic to the component group $\mathcal{S}_{\phi_\sigma}$, where ϕ_σ is the L-parameter associated to $I(\sigma)$. We refer to [7, 23, 6] and the references therein for works in this direction.

Such a relation between $\Pi(\chi)$ and ϕ_χ demonstrates a prototype of the general idea of the local Langlands parametrisation for admissible representations of a local reductive group G. Let $W_F' = W_F \times \mathrm{SL}_2(\mathbf{C})$ be the Weil–Deligne group of F, where W_F is the local Weil group. Let ${}^LG$ be the L-group of G. For each parameter

$$\phi : W_F' \longrightarrow {}^LG,$$

the local conjecture of Langlands asserts that there is an L-packet Π_ϕ consisting of irreducible admissible representations of G which satisfy certain desiderata (see [9]). Members inside the same packet Π_ϕ are equipped with the same L-function and ε-factor. Moreover, as already alluded to, if ϕ is a tempered parameter (i.e., the image of $\phi|_{W_F}$ in the dual group of G is relatively compact), then the component group $\mathcal{S}_\phi$ conjecturally parametrises elements in the L-packet Π_ϕ (see [4]) – that is, there is a bijection

$$\mathrm{Irr}\left(\mathcal{S}_\phi\right) \longleftrightarrow \Pi_\phi. \tag{1.3}$$

This bijection originally manifests in the theory of endoscopic transfer, and in particular in the character identity relations matching orbital integrals arising from the

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transfers (cf. [31, 42, 43, 1, 3]). More generally, in order to deal with non-tempered representation in the global L^2 spectral decomposition, Arthur postulated that one should consider a parameter φ of $W_F' \times \mathrm{SL}_2(\mathbf{C})$ valued in the Langlands L-group ${}^L G$; the component group $\mathcal{S}_\varphi$ should parametrise a certain Arthur-packet, which plays a crucial role in Arthur's work formulating his global multiplicity formula for the discrete spectrum of automorphic representations of a linear reductive group (see [2, 5]).

It should be noted that bijection (1.3) is not canonical, which depends from the geometric side on the normalisation of the Langlands–Shelstad transfer factors and from the representation-theoretic side on the normalisation of intertwining operators (see [28, 39, 40]). Indeed, one can always twist the bijection by a character of the component group $\mathcal{S}_\phi$. However, if ϕ is a tempered parameter, then it is conjectured in [40] that Π_ϕ always contains a unique generic element with respect to a fixed Whittaker datum of G. In particular, for a fixed Whittaker datum of G, there is a one-to-one correspondence between generic tempered representations and tempered L-packets. Granted with this tempered-packet conjecture, the correspondence in formula (1.3) can then be normalised such that the unique generic element (with respect to the fixed Whittaker datum) is parametrised by $\mathbb{1} \in \mathrm{Irr}(\mathcal{S}_\phi)$. We refer the reader to [41] and the references therein for works in this direction.

It is to this last problem on the genericity of representations inside Π_ϕ, where $\phi = \phi_\chi$ is the parameter for a unitary unramified genuine principal series $I(\chi)$ of a covering group, that our object in this paper pertains. The Whittaker space $\mathrm{Wh}_\psi(I(\chi))$ is understood as a consequence of Rodier's heredity and the fact that $\overline{T}$ has only trivial unipotent subgroup. However, if $I(\chi)$ is reducible, then for any constituent π it is a natural but delicate question to determine the dimension $\dim \mathrm{Wh}_\psi(\pi)$.

For a unitary unramified genuine χ, correspondence (1.1) continues to hold. The proof is essentially the same as in the linear algebraic case, and relies on the covering analogue of formula (1.2), which follows from recent work of W.-W. Li [33, 34] and C. Luo [35]. In fact, we will also show the isomorphism $R_\chi \simeq \mathcal{S}_{\phi_\chi}$. However, since it is possible to have $\dim \mathrm{Wh}_\psi(\pi) > 0$ for every constituent π of $I(\chi)$, there does not seem to be a preferred choice of $\pi \in \Pi(\chi)$ using the genericity criterion. Thus, we choose the normalisation for correspondence (1.1) such that the unique unramified constituent $\pi_\chi^{un} \subset I(\chi)$ corresponds to the trivial representation $\mathbb{1}$ of R_χ – that is,

$$\pi_{\mathbb{1}} = \pi_\chi^{un}.$$

For unramified principal series of a linear algebraic group, this is the natural choice, since π_χ^{un} is generic with respect to the fixed Whittaker datum, as a consequence of the Casselman–Shalika formula.

With this normalised correspondence $\sigma \leftrightarrow \pi_\sigma$, we want to determine $\dim \mathrm{Wh}_\psi(\pi_\sigma)$ in terms of σ for every $\sigma \in \mathrm{Irr}(R_\chi)$. We hope that the results in this paper will eventually find applications in the context of global automorphic representations for covering groups.

1.2. Main conjecture

Let χ be a unitary unramified genuine character of $Z(\overline{T})$. Associated to $\overline{T}$ is a finite abelian group

$$\mathscr{X}_{Q,n} := Y/Y_{Q,n},$$

which is the quotient of the cocharacter lattice Y of G by a certain sublattice $Y_{Q,n}$. Here $\mathscr{X}_{Q,n}$ is the 'moduli space' of $\mathrm{Wh}_\psi(I(\chi))$; in particular,

$$\dim \mathrm{Wh}_\psi(I(\chi)) = |\mathscr{X}_{Q,n}|.$$

The group $\mathscr{X}_{Q,n}$ is endowed with a natural twisted W-action which we denote by $\mathbb{w}[y]$. From this we have a permutation representation

$$\sigma^{\mathscr{X}} : W \longrightarrow \mathrm{Perm}(\mathscr{X}_{Q,n})$$

given by $\sigma^{\mathscr{X}}(w)(y) = \mathbb{w}[y]$. Let $\mathcal{O}_{\mathscr{X}}$ be the set of all W-orbits (with respect to this twisted action) in $\mathscr{X}_{Q,n}$. Clearly, for each W-orbit $\mathcal{O}_y \in \mathcal{O}_{\mathscr{X}}$ we have the permutation representation

$$\sigma^{\mathscr{X}}_{\mathcal{O}_y} : W \longrightarrow \mathrm{Perm}(\mathcal{O}_y);$$

moreover, $\sigma^{\mathscr{X}}$ decomposes as a sum of all $\sigma^{\mathscr{X}}_{\mathcal{O}_y}$ – that is,

$$\sigma^{\mathscr{X}} = \bigoplus_{\mathcal{O}_y \in \mathcal{O}_{\mathscr{X}}} \sigma^{\mathscr{X}}_{\mathcal{O}_y}.$$

By restriction, $\sigma^{\mathscr{X}}_{\mathcal{O}_y}$ could be viewed as a permutation representation of $R_\chi \subset W_\chi \subset W$.

For every W-orbit $\mathcal{O}_y \subset \mathscr{X}_{Q,n}$, there is also a natural subspace $\mathrm{Wh}_\psi(\pi_\sigma)_{\mathcal{O}_y} \subset \mathrm{Wh}_\psi(\pi_\sigma)$ (see definition (5.7)) such that

$$\mathrm{Wh}_\psi(\pi_\sigma) = \bigoplus_{\mathcal{O}_y \in \mathcal{O}_{\mathscr{X}}} \mathrm{Wh}_\psi(\pi_\sigma)_{\mathcal{O}_y}.$$

Conjecture 1.1 (Conjecture 5.3). *Let $\overline{G}$ be a saturated n-fold cover (see Definition 2.1) of a semisimple simply connected group G with $\overline{G}^\vee \simeq G^\vee$. In the normalised correspondence $Irr(R_\chi) \longleftrightarrow \Pi(\chi), \sigma \leftrightarrow \pi_\sigma$ such that $\pi_{\mathbb{1}} = \pi_\chi^{un}$, we have*

$$\dim \mathrm{Wh}_\psi(\pi_\sigma)_{\mathcal{O}_y} = \left\langle \sigma, \sigma^{\mathscr{X}}_{\mathcal{O}_y} \right\rangle_{R_\chi}$$

for every orbit $\mathcal{O}_y \in \mathcal{O}_{\mathscr{X}}$, where $\langle -,- \rangle_{R_\chi}$ denotes the pairing of two representations of R_χ. Consequently,

$$\dim \mathrm{Wh}_\psi(\pi_\sigma) = \langle \sigma, \sigma^{\mathscr{X}} \rangle_{R_\chi}$$

for every $\sigma \in \mathrm{Irr}(R_\chi)$; in particular, $\dim \mathrm{Wh}_\psi(\pi_\chi^{un})$ is equal to the number of R_χ-orbits in $\mathscr{X}_{Q,n}$.

1.3. Main results

Prior to the formulation of the main conjecture, a substantial part of this paper is devoted to analysing the group R_χ and proving for covers of general reductive groups an unconditional formula for $\dim \mathrm{Wh}_\psi(\pi_\sigma)_{\mathcal{O}_y}$ in terms of σ and a certain representation $\sigma^{\mathrm{Wh}}_{\mathcal{O}_y}$ of R_χ. We briefly outline the content of the paper and highlight some of our results.

After a brief introduction on covering groups in Section 2, we study in Section 3 the normalised intertwining operator between genuine principal series of $\overline{G}$. As in the case for linear algebraic groups, the normalisation is given by the Langlands L-functions, and one important property is the cocycle relation of the normalised intertwining operators, which does not depend on the length function of W.

In Section 4, we analyse the group R_χ based on the work of Keys [25, 27], W.-W. Li [33, 34] and C. Luo [35]. In particular, it follows from [35] that for a unitary unramified genuine principal series $I(\chi)$, there is an algebra isomorphism $\mathbf{C}[R_\chi] \simeq \mathrm{End}(I(\chi))$. We show how to compute R_χ by relating it to another group R_χ^{sc}, which is equal to the R-group of a certain unramified principal series of a simply connected Chevalley group H^{sc}. The group R_χ^{sc} is explicitly determined by Keys [25, 26] for principal series of simply connected Chevalley groups.

Moreover, by reducing to the linear algebraic case, we prove in Theorem 4.9 the isomorphism $R_\chi \simeq \mathcal{S}_{\phi_\chi}$, where

$$\phi_\chi : W_F \to {}^L\overline{G}$$

is the L-parameter of $I(\chi)$ valued in the L-group of $\overline{G}$ constructed by Weissman [53] and $\mathcal{S}_{\phi_\chi}$ is the connected component group of ϕ_χ (see definition (4.8)). We remark that the parameter ϕ_χ is associated to χ by the local Langlands correspondence for covering tori, and thus it is trivial on $\mathrm{SL}_2(\mathbf{C}) \subset W_F'$. Therefore, it suffices to consider the Weil group W_F alone.

Denote by $\overline{G}^\vee$ (resp., ${}^L\overline{G}$) the dual group (resp., L-group) for the covering group $\overline{G}$. The following is an amalgam of Proposition 4.4 and Theorems 4.6 and 4.9:

Theorem 1.2. *Let $\overline{G}$ be an n-fold cover of a linear algebraic group G. Let χ be a unitary unramified genuine character of $Z\left(\overline{T}\right)$. We have*

- $R_\chi \subseteq R_\chi^{sc}$, *with R_χ^{sc} being an abelian group, and if $\overline{G}$ is semisimple, then*

$$[R_\chi^{sc} : R_\chi] \le \left| Z\left(\overline{G}^\vee\right)\right|,$$

  where $Z\left(\overline{G}^\vee\right)$ is the centre of $\overline{G}^\vee$; and
- $R_\chi \simeq \mathcal{S}_{\phi_\chi}$.

Section 5 is devoted to stating and investigating several aspects of the main conjecture (Conjecture 5.3, which is Conjecture 1.1). First, the space $\mathrm{Wh}_\psi(I(\chi))$ affords a natural representation

$$\sigma^{\mathrm{Wh}} : R_\chi \longrightarrow \mathrm{GL}\left(\mathbf{C}^{|\mathscr{X}_{Q,n}|}\right)$$

(see definition (5.10)). Moreover, for $\mathcal{O}_y \in \mathcal{O}_{\mathscr{X}}$ there is a natural σ^{Wh}-stable subspace $\mathrm{Wh}_\psi(I(\chi))_{\mathcal{O}_y} \subset \mathrm{Wh}_\psi(I(\chi))$ of dimension $|\mathcal{O}_y|$ (see equation (5.6)); this gives a subrepresentation

$$\sigma^{\mathrm{Wh}}_{\mathcal{O}_y} : R_\chi \longrightarrow \mathrm{GL}\left(\mathbf{C}^{|\mathcal{O}_y|}\right),$$

and we have

$$\sigma^{\mathrm{Wh}} = \bigoplus_{\mathcal{O}_y \in \mathcal{O}_{\mathscr{X}}} \sigma^{\mathrm{Wh}}_{\mathcal{O}_y}.$$

Theorem 1.3 (Theorem 5.6)**.** *Let $\overline{G}$ be an n-fold cover of a connected reductive group G. For every orbit $\mathcal{O}_y \in \mathcal{O}_{\mathscr{X}}$, we have*

$$\dim \mathrm{Wh}_\psi(\pi_\sigma)_{\mathcal{O}_y} = \left\langle \sigma, \sigma^{\mathrm{Wh}}_{\mathcal{O}_y} \right\rangle_{R_\chi}.$$

Consequently, $\dim \mathrm{Wh}_\psi(\pi_\sigma) = \langle \sigma, \sigma^{\mathrm{Wh}} \rangle_{R_\chi}$.

In fact, $\sigma^{\mathrm{Wh}}_{\mathcal{O}_y}(\mathbbm{w})$ is represented by the matrix $\gamma(w,\chi) \cdot \mathcal{S}_{\mathfrak{R}}(w,i(\chi))_{\mathcal{O}_y}$, where $\gamma(w,\chi)$ is the γ-factor associated to $\mathbbm{w}$ and $\mathcal{S}_{\mathfrak{R}}(w,i(\chi))_{\mathcal{O}_y}$ is a so-called scattering matrix. As an application of this theorem, we show in Section 5.5 that a result of Szpruch [48] on the double cover of GSp_{2r} can be recovered from it (see Theorem 5.10). Here Theorem 1.3 also implies that Conjecture 1.1 is equivalent to the following (compare Conjecture 5.7):

Conjecture 1.4. *Let $\overline{G}$ be a saturated n-fold cover of a semisimple simply connected group G with $\overline{G}^\vee \simeq G^\vee$. Then for every orbit $\mathcal{O}_y \subset \mathscr{X}_{Q,n}$, we have $\sigma^{\mathrm{Wh}}_{\mathcal{O}_y} = \sigma^{\mathscr{X}}_{\mathcal{O}_y}$; or equivalently,*

$$\mathrm{Tr}\left(\mathcal{S}_{\mathfrak{R}}(w,i(\chi))_{\mathcal{O}_y}\right) = |(\mathcal{O}_y)^{\mathbbm{w}}| \cdot \gamma(w,\chi)^{-1}$$

for every $\mathbbm{w} \in R_\chi$, where the left-hand side denotes the trace of the matrix $\mathcal{S}_{\mathfrak{R}}(w,i(\chi))_{\mathcal{O}_y}$.

Using the formulation in this conjecture, we prove several results in Section 6:

- For a general reductive group G, we show that there is an exceptional set $\mathscr{X}^{\mathrm{exc}}_{Q,n} \subset (\mathscr{X}_{Q,n})^W$, which might be empty, such that $\sigma^{\mathrm{Wh}}_{\mathcal{O}_y} = \sigma^{\mathscr{X}}_{\mathcal{O}_y} = \mathbb{1}_{R_\chi}$ for $y \in \mathscr{X}^{\mathrm{exc}}_{Q,n}$. It follows that $\dim \mathrm{Wh}_\psi\left(\pi^{un}_\chi\right) \geq \left|\mathscr{X}^{\mathrm{exc}}_{Q,n}\right|$; this also implies that Conjecture 1.4 holds for such $\mathcal{O}_y$. This is the content of Theorem 6.1.
- In Section 6.2 we consider the Whittaker space $\mathrm{Wh}_\psi\left(\pi^{un}_\chi\right)$ from the perspective of unramified Whittaker functions. Using an analogue of the Casselman–Shalika formula proved in [19], we show in Theorem 6.5 a result on $\dim \mathrm{Wh}_\psi\left(\pi^{un}_\chi\right)$, which is compatible with Theorem 6.1.

In Section 7, we verify the following:

Theorem 1.5 (Theorem 7.1)**.** *Conjecture 1.4 (and thus Conjecture 1.1) holds for the n-fold covers $\overline{Sp}^{(n)}_{2r}$.*

We also prove in Section 7 that Conjecture 1.1 holds for the double cover of SL_3 by explicit computations.

https://doi.org/10.1017/S1474748021000128 Published online by Cambridge University Press

Lastly, in Section 8 we consider n-fold covers of SO_3 and the double cover of $\mathrm{Spin}_6 \simeq \mathrm{SL}_4$, and show that the naive analogue of Conjecture 1.4 fails for such covers. Thus, the constraints on G being simply connected and on $\overline{G}$ being saturated seem to be indispensable. For general reductive groups, a unified conjectural formula for $\mathrm{Wh}_\psi(\pi_\sigma)_{\mathcal{O}_y}$ in terms of $\sigma^{\mathscr{X}}_{\mathcal{O}_y}$ involves subtleties beyond the considerations of this paper, and its intimate relation with the R-group has yet to be unveiled in full generality.

2. Covering groups

Our exposition on covering groups is essentially the same as in [18, §2]. However, to ensure that this paper is self-contained, we will briefly recall some summarised results on $\overline{G}$.

Let F be a finite extension of $\mathbf{Q}_p$. Denote by $O \subset F$ the ring of integers of F and by $\varpi \in O$ a fixed uniformiser.

2.1. Covering groups

Let $\mathbf{G}$ be a split connected linear algebraic group over F with a maximal split torus $\mathbf{T}$. Let

$$\{X, \Delta, \Phi;\ Y, \Delta^\vee, \Phi^\vee\}$$

be the based root datum of $\mathbf{G}$. Here X (resp., Y) is the character lattice (resp., cocharacter lattice) for $(\mathbf{G}, \mathbf{T})$. Choose a set $\Delta \subseteq \Phi$ of simple roots from the set of roots Φ, and let $\Delta^\vee$ be the corresponding simple coroots from $\Phi^\vee$. This gives us a choice of positive roots Φ_+ and positive coroots $\Phi_+^\vee$. Write $Y^{sc} \subseteq Y$ for the sublattice generated by $\Phi^\vee$. Let $\mathbf{B} = \mathbf{T}\mathbf{U}$ be the Borel subgroup associated with Δ. Denote by $\mathbf{U}^- \subset \mathbf{G}$ the unipotent subgroup opposite $\mathbf{U}$.

Fix a Chevalley–Steinberg system of pinnings for $(\mathbf{G}, \mathbf{T})$. That is, we fix a set of compatible isomorphisms

$$\{e_\alpha : \mathbf{G}_\mathrm{a} \to \mathbf{U}_\alpha\}_{\alpha \in \Phi},$$

where $\mathbf{U}_\alpha \subseteq \mathbf{G}$ is the root subgroup associated with α. In particular, for each $\alpha \in \Phi$, there is a unique homomorphism $\varphi_\alpha : \mathbf{SL}_2 \to \mathbf{G}$ which restricts to $e_{\pm\alpha}$ on the upper and lower triangular subgroup, respectively, of unipotent matrices of $\mathbf{SL}_2$.

Denote by W the Weyl group of $(\mathbf{G}, \mathbf{T})$, which we identify with the Weyl group of the coroot system. In particular, W is generated by simple reflections $\{\mathbb{w}_\alpha : \alpha^\vee \in \Delta^\vee\}$ in $Y \otimes \mathbf{Q}$. Let $l : W \to \mathbf{N}$ be the length function. Let $\mathbb{w}_G$ be the longest element in W.

Consider the algebro-geometric $\mathbf{K}_2$-extension $\overline{\mathbf{G}}$ of $\mathbf{G}$ studied by Brylinski and Deligne [11], which is categorically equivalent to the pairs $\{(D, \eta)\}$ (see [14, §2.6]). Here $\eta : Y^{sc} \to F^\times$ is a homomorphism. On the other hand,

$$D : Y \times Y \to \mathbf{Z}$$

is a (not necessarily symmetric) bilinear form on Y such that

$$Q(y) := D(y, y)$$

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is a Weyl-invariant integer-valued quadratic form on Y. We call D a bisector. Let B_Q be the Weyl-invariant bilinear form associated to Q by

$$B_Q(y_1,y_2) = Q(y_1+y_2) - Q(y_1) - Q(y_2).$$

Clearly, $D(y_1,y_2)+D(y_2,y_1) = B_Q(y_1,y_2)$. Any $\overline{\mathbf{G}}$ is, up to isomorphism, incarnated by (i.e., categorically associated to) a pair (D,η) for a bisector D and η.

The couple (D,η) plays the following role for the structure of $\overline{\mathbf{G}}$:

- First, the group $\overline{\mathbf{G}}$ splits canonically and uniquely over any unipotent subgroup of $\mathbf{G}$. For $\alpha \in \Phi$ and $a \in \mathbf{G}_a$, denote by $\overline{e}_\alpha(a) \in \overline{\mathbf{G}}$ the canonical lifting of $e_\alpha(a) \in \mathbf{G}$. For $\alpha \in \Phi$ and $a \in \mathbf{G}_m$, define

$$w_\alpha(a) := e_\alpha(a)\cdot e_{-\alpha}\left(-a^{-1}\right)\cdot e_\alpha(a) \text{ and } \overline{w}_\alpha(a) := \overline{e}_\alpha(a)\cdot \overline{e}_{-\alpha}\left(-a^{-1}\right)\cdot \overline{e}_\alpha(a).$$

  This gives natural representatives $w_\alpha := w_\alpha(1)$ in $\mathbf{G}$, and also $\overline{w}_\alpha := \overline{w}_\alpha(1)$ in $\overline{\mathbf{G}}$, of the Weyl element $\mathsf{w}_\alpha \in W$. Moreover, for $h_\alpha(a) := \alpha^\vee(a) \in \mathbf{T}$, there is a natural lifting

$$\overline{h}_\alpha(a) := \overline{w}_\alpha(a)\cdot \overline{w}_\alpha(-1) \in \overline{\mathbf{T}},$$

  which depends only on the pinnings and the canonical unipotent splittings.
- Second, there is a section $\mathbf{s}$ of $\overline{\mathbf{T}}$ over $\mathbf{T}$ such that the group law on $\overline{\mathbf{T}}$ includes the relation

$$\mathbf{s}(y_1(a))\cdot \mathbf{s}(y_2(b)) = \{a,b\}^{D(y_1,y_2)}\cdot \mathbf{s}(y_1(a)\cdot y_2(b)) \tag{2.1}$$

  for any $a,b \in \mathbf{G}_m$. Moreover, for $\alpha \in \Delta$ and the natural lifting $\overline{h}_\alpha(a)$ of $h_\alpha(a)$, we have

$$\overline{h}_\alpha(a) = \{\eta(\alpha^\vee),a\}\cdot \mathbf{s}(h_\alpha(a)) \in \overline{\mathbf{T}}.$$

- Third, let $w_\alpha \in \mathbf{G}$ be the natural representative of $\mathsf{w}_\alpha \in W$. For every $\overline{y(a)} \in \overline{\mathbf{T}}$ with $y \in Y$ and $a \in \mathbf{G}_m$, we have

$$w_\alpha \cdot \overline{y(a)}\cdot w_\alpha^{-1} = \overline{y(a)}\cdot \overline{h}_\alpha\left(a^{-\langle y,\alpha\rangle}\right), \tag{2.2}$$

  where $\langle -,-\rangle$ is the pairing between Y and X.

If the derived subgroup of $\mathbf{G}$ is simply connected, then the isomorphism class of $\overline{\mathbf{G}}$ is determined by the Weyl-invariant quadratic form Q. In particular, for such $\mathbf{G}$, any extension $\overline{\mathbf{G}}$ is incarnated by $(D,\eta = \mathbb{1})$ for some bisector D, up to isomorphism. In this paper, we assume that the composite

$$\eta_n : Y^{sc} \to F^\times \twoheadrightarrow F^\times/(F^\times)^n$$

of η with the obvious quotient is trivial. For some consequences of this assumption, see Section 2.2 and the beginning of Section 3.

Set $n \in \mathbf{N}$. We assume that F contains the full group of nth roots of unity, denoted by μ_n. An n-fold cover of $\mathbf{G}$, in the sense of [53, Definition 1.2], is just a pair $\left(n,\overline{\mathbf{G}}\right)$. The $\mathbf{K}_2$-extension $\overline{\mathbf{G}}$ gives rise to an n-fold covering $\overline{G}$ as follows. Let

$$(-,-)_n : F\times F \to \mu_n$$

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be the nth Hilbert symbol. The cover $\overline{G}$ arises from the central extension

$$\mathbf{K}_2(F) \hookrightarrow \overline{\mathbf{G}}(F) \xrightarrow{\phi} \mathbf{G}(F)$$

by pushout via the natural map $\mathbf{K}_2(F) \to \mu_n$ given by $\{a,b\} \mapsto (a,b)_n$. This gives

$$\mu_n \hookrightarrow \overline{G} \xrightarrow{\phi} G.$$

We may write $\overline{G}^{(n)}$ for $\overline{G}$ to emphasise the degree of covering.

For any subset $H \subset G$, denote $\overline{H} := \phi^{-1}(H)$. The relations for $\overline{\mathbf{G}}$ already described give rise to the corresponding relations for $\overline{G}$. For example, inherited from equation (2.1) is the following relation on $\overline{T}$:

$$\mathbf{s}(y_1(a)) \cdot \mathbf{s}(y_2(b)) = (a,b)_n^{D(y_1,y_2)} \cdot \mathbf{s}(y_1(a) \cdot y_2(b)), \tag{2.3}$$

where $y_i \in Y$ and $a,b \in F^\times$. The commutator $[\bar{t}_1,\bar{t}_2] := \bar{t}_1\bar{t}_2\bar{t}_1^{-1}\bar{t}_2^{-1}$ of $\overline{T}$, which descends to a map $[-,-] : T \times T \to \mu_n$, is thus given by

$$[y_1(a), y_2(b)] = (a,b)_n^{B_Q(y_1,y_2)}.$$

A representation of $\overline{G}$ is called ϵ-genuine (or simply genuine) if μ_n acts by a fixed embedding $\epsilon : \mu_n \hookrightarrow \mathbf{C}^\times$. We consider only genuine representations of a covering group in this paper.

Let $W' \subset \overline{G}$ be the group generated by $\overline{w}_\alpha$ for all α. Then the map $\overline{w}_\alpha \mapsto \mathbb{w}_\alpha$ gives a surjective homomorphism

$$W' \twoheadrightarrow W$$

with kernel a finite group. For any $\mathbb{w} = \mathbb{w}_{\alpha_k} \cdots \mathbb{w}_{\alpha_2}\mathbb{w}_{\alpha_1} \in W$ in a minimal decomposition, we let

$$\overline{w} := \overline{w}_{\alpha_k} \cdots \overline{w}_{\alpha_2}\overline{w}_{\alpha_1} \in W'$$

be its representative in W', which is independent of the minimal decomposition (see [47, Lemma 83 (b)]). In particular, we denote by $\overline{w}_G \in \overline{G}$ this representative of the longest Weyl element $\mathbb{w}_G$. Note that we also have the natural representative

$$w := w_{\alpha_k} \cdots w_{\alpha_2} w_{\alpha_1} \in G$$

of $\mathbb{w}$. In particular, we have the representative $w_G \in G$ for $\mathbb{w}_G$, which is the image of $\overline{w}_G$ in G.

2.2. Dual groups and L-groups

For a cover $(n, \overline{\mathbf{G}})$ associated to (D,η), with Q and B_Q arising from D, we define

$$Y_{Q,n} := Y \cap nY^*, \tag{2.4}$$

where $Y^* \subset Y \otimes \mathbf{Q}$ is the dual lattice of Y with respect to B_Q; more explicitly,

$$Y_{Q,n} = \{y \in Y : B_Q(y,y') \in n\mathbf{Z} \text{ for all } y' \in Y\} \subset Y.$$

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For every $\alpha^\vee \in \Phi^\vee$, denote

$$n_\alpha := \frac{n}{\gcd(n, Q(\alpha^\vee))}$$

and

$$\alpha_{Q,n}^\vee = n_\alpha \alpha^\vee, \qquad \alpha_{Q,n} = \frac{\alpha}{n_\alpha}.$$

Let

$$Y_{Q,n}^{sc} \subset Y_{Q,n}$$

be the sublattice generated by $\Phi_{Q,n}^\vee = \{\alpha_{Q,n}^\vee : \alpha^\vee \in \Phi^\vee\}$. Denote $X_{Q,n} = \mathrm{Hom}_{\mathbf{Z}}(Y_{Q,n}, \mathbf{Z})$ and $\Phi_{Q,n} = \{\alpha_{Q,n} : \alpha \in \Phi\}$. We also write

$$\Delta_{Q,n}^\vee = \{\alpha_{Q,n}^\vee : \alpha^\vee \in \Delta^\vee\} \text{ and } \Delta_{Q,n} = \{\alpha_{Q,n} : \alpha \in \Delta\}.$$

Then

$$\left(Y_{Q,n}, \Phi_{Q,n}^\vee, \Delta_{Q,n}^\vee; X_{Q,n}, \Phi_{Q,n}^\vee, \Delta_{Q,n}\right)$$

forms a root datum with a choice of simple roots $\Delta_{Q,n}$. It gives a unique (up to unique isomorphism) pinned reductive group $\overline{\mathbf{G}}_{Q,n}^\vee$ over $\mathbf{Z}$, called the dual group of $(n, \overline{\mathbf{G}})$. In particular, $Y_{Q,n}$ is the character lattice for $\overline{G}_{Q,n}^\vee$ and $\Delta_{Q,n}^\vee$ the set of simple roots. Let

$$\overline{G}_{Q,n}^\vee := \overline{\mathbf{G}}_{Q,n}^\vee(\mathbf{C})$$

be the associated complex dual group. For simplicity, we may also write $\overline{G}^\vee$ for $\overline{G}_{Q,n}^\vee$, which is the Langlands dual group $G^\vee$ of G if $n = 1$. We have

$$Z\left(\overline{G}_{Q,n}^\vee\right) = \mathrm{Hom}\left(Y/Y_{Q,n}, \mathbf{C}\right).$$

In [52, 53], Weissman constructed the global L-group as well as the local L-group extension

$$\overline{G}_{Q,n}^\vee \hookrightarrow {}^L\overline{G} \twoheadrightarrow W_F,$$

which is compatible with the global L-group extension. (It may as well be an extension over the Weil–Deligne group. However, the Weil group W_F suffices in this paper, since we eventually only consider unitary principal series.) His construction of L-groups is functorial, and in particular it behaves well with respect to the restriction of $\overline{\mathbf{G}}$ to parabolic subgroups. More precisely, let $\mathbf{M} \subset \mathbf{G}$ be a Levi subgroup. By restriction, we have the n-cover $\overline{M}$ of M. The L-groups ${}^L\overline{M}$ and ${}^L\overline{G}$ are compatible – that is, there are natural homomorphisms of extensions:

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$$\begin{array}{ccccc} \overline{G}^\vee_{Q,n} & \hookrightarrow & {}^L\overline{G} & \twoheadrightarrow & W_F \\ \uparrow & & \uparrow & & \| \\ \overline{M}^\vee_{Q,n} & \hookrightarrow & {}^L\overline{M} & \twoheadrightarrow & W_F. \end{array}$$

This applies in particular to the case when $M = T$ is a torus.

The extension ${}^L\overline{G}$ does not split over W_F. However, if $\overline{G}^\vee_{Q,n}$ is of adjoint type, then we have a canonical isomorphism

$$ {}^L\overline{G} \simeq \overline{G}^\vee_{Q,n} \times W_F. $$

For general $\overline{G}$, under the assumption that $\eta_n = \mathbb{1}$, there exists a so-called distinguished genuine character $\chi_\psi : Z\left(\overline{T}\right) \to \mathbf{C}^\times$ (see [14, §6.4]), depending on a nontrivial additive character ψ of F, such that χ_ψ gives rise to a splitting of ${}^L\overline{G}$ over W_F, with respect to which there is an isomorphism

$$ {}^L\overline{G} \simeq_{\chi_\psi} \overline{G}^\vee_{Q,n} \times W_F. \tag{2.5}$$

For details on the construction and properties regarding the L-group, we refer the reader to [52, 53, 14].

2.3. Twisted Weyl action

For a group H acting on a set S, we denote by

$$ \mathcal{O}^H_S $$

the set of all H-orbits in S. For every $z \in S$, denote by $\mathcal{O}^H_z \in \mathcal{O}^H_S$ the H-orbit of z.

Denote by $\mathbb{w}(y)$ the natural Weyl group action on Y and $Y \otimes \mathbf{Q}$ generated by the reflections $\mathbb{w}_\alpha$. The two lattices $Y_{Q,n}$ and $Y^{sc}_{Q,n}$ are both W-stable under this usual action. Let

$$ \rho := \frac{1}{2} \sum_{\alpha^\vee > 0} \alpha^\vee $$

be the half sum of all positive coroots of $\mathbf{G}$. We consider the twisted Weyl action

$$ \mathbb{w}[y] := \mathbb{w}(y - \rho) + \rho. $$

It induces a well-defined twisted action of W on

$$ \mathscr{X}_{Q,n} := Y / Y_{Q,n} $$

given by $\mathbb{w}[y + Y_{Q,n}] := \mathbb{w}[y] + Y_{Q,n}$, since $W\left(Y_{Q,n}\right) = Y_{Q,n}$ as already mentioned. Thus, we have a permutation representation

$$ \sigma^{\mathscr{X}} : W \longrightarrow \mathrm{Perm}\left(\mathscr{X}_{Q,n}\right), $$

which plays a pivotal role in the conjectural formulas on Whittaker space in both [18] and this paper.

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We note that the twisted Weyl action on $Y/Y_{Q,n}^{sc}$ is also well defined. For every $\alpha \in \Delta$, let $W_\alpha = \{1, \mathbb{w}_\alpha\} \subset W$. Arising from the surjection

$$Y/Y_{Q,n}^{sc} \twoheadrightarrow \mathscr{X}_{Q,n},$$

we have a map of sets

$$\phi_\alpha : \left(Y/Y_{Q,n}^{sc}\right)^{W_\alpha} \twoheadrightarrow \left(\mathscr{X}_{Q,n}\right)^{W_\alpha}.$$

Recall the following definition from [19, 18]:

Definition 2.1. A covering group $\overline{G}$ of a connected linear reductive group G is called saturated if $Y_{Q,n}^{sc} = Y_{Q,n} \cap Y^{sc}$. It is called of metaplectic type if there exists $\alpha \in \Delta$ such that ϕ_α is not surjective.

If G is semisimple simply connected, then $\overline{G}$ is saturated if and only if $\overline{G}^\vee$ is of adjoint type. On the other hand, covering groups of metaplectic type are rare. Indeed, it follows from [19, §4.5] that if G is almost simple, then $\overline{G}$ is of metaplectic type if and only if $\mathbf{G} = \mathrm{Sp}_{2r}$ and $n_\alpha \equiv 2 \pmod 4$ for the unique short simple coroot $\alpha^\vee$ of Sp_{2r}. In particular, the classical double cover of Sp_{2r} is such an example.

Throughout the paper, we denote

$$y_\rho := y - \rho \in Y \otimes \mathbf{Q}$$

for $y \in Y$. Clearly,

$$\mathbb{w}[y] - y = \mathbb{w}(y_\rho) - y_\rho.$$

By Weyl action or Weyl orbits in Y or $Y \otimes \mathbf{Q}$, we always refer to the ones with respect to the action $\mathbb{w}[y]$, unless specified otherwise. For simplicity, we will abuse notation and denote by y an element in $\mathscr{X}_{Q,n}$. We will also write $\mathcal{O}_{\mathscr{X}}$ for the set of W-orbits in $\mathscr{X}_{Q,n}$, and use the notation $\mathcal{O}_z := \mathcal{O}_z^W$, whenever we consider W-orbits with respect to the twisted action.

3. Unitary unramified principal series

Henceforth, we assume that $|n|_F = 1$. Let $K \subset G$ be the hyperspecial maximal compact subgroup generated by $\mathbf{T}(O)$ and $e_\alpha(O)$ for all roots α. With our assumption that η_n is trivial, the group $\overline{G}$ splits over K (see [14, Theorem 4.2]); we fix such a splitting s_K. If no confusion arises, we will omit s_K and write $K \subset \overline{G}$ instead. Call $\overline{G}$ an unramified covering group in this setting.

A genuine representation (π, V_π) is called unramified if $\dim V_\pi^K \neq 0$. Since $\overline{G}$ also splits uniquely over the unipotent subgroup $e_\alpha(O)$, we see that $\overline{h}_\alpha(u) \in s_K(K) \subset \overline{G}$ for every $u \in O^\times$.

3.1. Principal series representation

As $\overline{G}$ splits canonically over the unipotent radical U of the Borel subgroup B, we have $\overline{B} = \overline{T}U$. The covering torus $\overline{T}$ is a Heisenberg group. The centre $Z\left(\overline{T}\right)$ of the covering torus $\overline{T}$ is equal to $\phi^{-1}(\mathrm{Im}(i_{Q,n}))$ (see [51]), where

$$i_{Q,n}: Y_{Q,n} \otimes F^{\times} \to T$$

is the isogeny induced from the embedding $Y_{Q,n} \subset Y$.

Let $\chi \in \mathrm{Hom}_{\epsilon}\left(Z\left(\overline{T}\right), \mathbf{C}^{\times}\right)$ be a genuine character of $Z\left(\overline{T}\right)$ and write

$$i(\chi) := \mathrm{Ind}_{\overline{A}}^{\overline{T}} \chi'$$

for the induced representation of $\overline{T}$, where $\overline{A}$ is a maximal abelian subgroup of $\overline{T}$ and χ' is an extension of χ to $\overline{A}$. By the Stone–von Neumann theorem (see [51, Theorem 3.1]), the construction

$$\chi \mapsto i(\chi)$$

gives a bijection between the isomorphism classes of genuine representations of $Z\left(\overline{T}\right)$ and $\overline{T}$. Since we consider unramified covering group $\overline{G}$ in this paper, we take

$$\overline{A} := Z\left(\overline{T}\right) \cdot (K \cap T).$$

The left action of w on $i(\chi)$ is given by ${}^{w}i(\chi)\left(\bar{t}\right) = i(\chi)\left(w^{-1}\bar{t}w\right)$. The group W does not act on $i(\chi)$, only on its isomorphism classes. On the other hand, we have a well-defined action of W on χ:

$$\left({}^{w}\chi\right)\left(\bar{t}\right) := \chi\left(w^{-1}\bar{t}w\right).$$

Viewing $i(\chi)$ as a genuine representation of $\overline{B}$ by inflation from the quotient map $\overline{B} \to \overline{T}$, we denote by

$$I(i(\chi)) := \mathrm{Ind}_{\overline{B}}^{\overline{G}}\, i(\chi)$$

the normalised induced principal series representation of $\overline{G}$. For simplicity, we may also write $I(\chi)$ for $I(i(\chi))$. We know that $I(\chi)$ is unramified (i.e., $I(\chi)^{K} \neq 0$) if and only if χ is unramified – that is, χ is trivial on $Z\left(\overline{T}\right) \cap K$; here the 'if' part follows from [36, Lemma 2] and the 'only if' part from the Satake isomorphism for covers (see [53, Corollary 7.4]). In fact, the Satake isomorphism for $\overline{G}$ implies that a genuine representation is unramified if and only if it is a subquotient of an unramified principal series.

Setting $\overline{Y}_{Q,n} := Z\left(\overline{T}\right) / \left(Z\left(\overline{T}\right) \cap K\right)$, we have a natural abelian extension

$$\mu_n \hookrightarrow \overline{Y}_{Q,n} \stackrel{\varphi}{\twoheadrightarrow} Y_{Q,n} \tag{3.1}$$

such that unramified genuine characters of $Z\left(\overline{T}\right)$ correspond to genuine characters of $\overline{Y}_{Q,n}$. Since $\overline{A}/(T \cap K) \simeq \overline{Y}_{Q,n}$ as well, there is a canonical extension (also denoted by χ) of a χ to $\overline{A}$, by composing it with $\overline{A} \twoheadrightarrow \overline{Y}_{Q,n}$. Therefore, we will identify $i(\chi)$ with $\mathrm{Ind}_{\overline{A}}^{\overline{T}} \chi$ for this canonical extension χ.

3.2. γ-function

Let $\underline{\chi}: F^{\times} \to \mathbf{C}^{\times}$ be a linear character. Tate [50] defined a γ-factor $\gamma\left(s, \underline{\chi}, \psi\right), s \in \mathbf{C}$, which is essentially the ratio of two integrals of a test function and its Fourier transform

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(depending on a nontrivial character ψ). We have

$$\gamma\left(s,\underline{\chi},\psi\right) = \varepsilon\left(s,\underline{\chi},\psi\right) \cdot \frac{L\left(1-s,\underline{\chi}^{-1}\right)}{L\left(s,\underline{\chi}\right)},$$

where $L\left(s,\underline{\chi}\right)$ is the L-function of $\underline{\chi}$. If $\underline{\chi}$ is unramified and the conductor of ψ is O, then

$$\varepsilon\left(s,\underline{\chi},\psi\right) = 1 \text{ and } L\left(s,\underline{\chi}\right) = \left(1-q^{-s}\underline{\chi}(\varpi)\right)^{-1}.$$

In this case, we write

$$\gamma\left(s,\underline{\chi}\right) := \gamma\left(s,\underline{\chi},\psi\right) = \frac{1-q^{-s}\underline{\chi}(\varpi)}{1-q^{-1+s}\underline{\chi}(\varpi)^{-1}},$$

with the ψ omitted.

Let χ be a genuine unramified character of $Z\left(\overline{T}\right)$. For every $\alpha \in \Phi$,

$$\underline{\chi}_{\alpha} : F^{\times} \to \mathbf{C}^{\times}, \text{ given by } \underline{\chi}_{\alpha}(a) := \chi\left(\overline{h}_{\alpha}(a^{n_{\alpha}})\right), \tag{3.2}$$

is an unramified character of $F^{\times}$. We also use in this paper the shorthand notation

$$\chi_{\alpha} := \underline{\chi}_{\alpha}(\varpi) \in \mathbf{C}^{\times}$$

for every root α. For $\mathbb{w} = \mathbb{w}_{\alpha}$, we define the γ-factor $\gamma(w_{\alpha},\chi)$ to be such that

$$\gamma(w_{\alpha},\chi)^{-1} = \gamma\left(0,\underline{\chi}_{\alpha}\right)^{-1} = \frac{1-q^{-1}\chi\left(\overline{h}_{\alpha}(\varpi^{n_{\alpha}})\right)^{-1}}{1-\chi\left(\overline{h}_{\alpha}(\varpi^{n_{\alpha}})\right)}.$$

For general $\mathbb{w} \in \mathrm{W}$, define

$$\gamma(w,\chi) = \prod_{\alpha\in\Phi_{\mathbb{w}}} \gamma(w_{\alpha},\gamma),$$

where $\Phi_{\mathbb{w}} = \{\alpha > 0 : \mathbb{w}(\alpha) < 0\}$.

3.3. Intertwining operator

For $\mathbb{w} \in W$, let $A(w,\chi) : I(\chi) \to I({}^{w}\chi)$ be the intertwining operator defined by

$$A(w,\chi)(f)\left(\overline{g}\right) = \int_{U_w} f\left(\overline{w}^{-1}u\overline{g}\right) du$$

whenever it is absolutely convergent. It can be meromorphically continued for all χ (see [36, §7]). The operator $A(w,\chi)$ satisfies the cocycle relation as in the linear case. Let $f_0 \in I(\chi)$ and $f_0' \in I({}^{w}\chi)$ be the normalised unramified vectors. We have

$$A(w,\chi)(f_0) = c_{\mathsf{gk}}(w,\chi) f_0',$$

where

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$$c_{\mathsf{gk}}(w,\chi) = \prod_{\alpha \in \Phi_w} \frac{1-q^{-1}\chi\left(\overline{h}_\alpha(\varpi^{n_\alpha})\right)}{1-\chi\left(\overline{h}_\alpha(\varpi^{n_\alpha})\right)}.$$

The Plancherel measure $\mu(w,\chi)$ associated with $A(w,\chi)$ is such that

$$A\left(w^{-1},{}^w\chi\right) \circ A(w,\chi) = \mu(w,\chi)^{-1} \cdot \mathrm{id}; \tag{3.3}$$

more explicitly,

$$\mu(w,\chi)^{-1} = c_{\mathsf{gk}}\left(w^{-1},{}^w\chi\right) \cdot c_{\mathsf{gk}}(w,\chi).$$

To recall the interpretation of $c_{\mathsf{gk}}(w,\chi)$ in terms of L-functions, we first recall the setup on the dual side. Consider the adjoint representation

$$Ad_{\overline{\mathfrak{u}}^\vee} : {}^L\overline{T} \to \mathrm{GL}\left(\overline{\mathfrak{u}}^\vee\right),$$

where $\overline{\mathfrak{u}}^\vee$ is the Lie algebra of unipotent radical $\overline{U}^\vee$ of the Borel subgroup $\overline{T}^\vee\overline{U}^\vee \subset \overline{G}^\vee$. It factors through $Ad^{\mathbf{C}}_{\overline{\mathfrak{u}}^\vee}$:

$$\begin{array}{ccc} {}^L\overline{T} & \xrightarrow{Ad_{\overline{\mathfrak{u}}^\vee}} & \mathrm{GL}(\overline{\mathfrak{u}}^\vee), \\ \downarrow & \nearrow_{Ad^{\mathbf{C}}_{\overline{\mathfrak{u}}^\vee}} & \\ \overline{T}^\vee/Z\left(\overline{G}^\vee\right) & & \end{array}$$

as $Z\left(\overline{G}^\vee\right)$ acts trivially on $\overline{\mathfrak{u}}^\vee$.

Therefore, irreducible subspaces of $\overline{\mathfrak{u}}^\vee$ for $Ad_{\overline{\mathfrak{u}}^\vee}$ are in one-to-one correspondence with irreducible subspaces with respect to $Ad^{\mathbf{C}}_{\overline{\mathfrak{u}}^\vee}$, which are just the one-dimensional spaces associated to the positive roots of $\overline{G}^\vee$. More precisely, we have the decomposition of $Ad_{\overline{\mathfrak{u}}^\vee}$ into irreducible ${}^L\overline{T}$-modules:

$$Ad_{\overline{\mathfrak{u}}^\vee} = \bigoplus_{\alpha>0} Ad_\alpha,$$

where $(Ad_\alpha, V_\alpha) \subseteq \overline{\mathfrak{u}}^\vee$ is spanned by a basis vector $E_{\alpha^\vee_{Q,n}}$ in the Lie algebra $\overline{\mathfrak{u}}^\vee_\alpha$ associated to the positive root $\alpha^\vee_{Q,n}$ of $\overline{G}^\vee$.

By the local Langlands correspondence for covering tori (see [53, §10] or [14, §8]), associated to $i(\chi)$ we have a splitting

$$\phi_\chi : W_F \longrightarrow {}^L\overline{T}$$

of the L-group extension

$$\overline{T}^\vee \hookrightarrow {}^L\overline{T} \twoheadrightarrow W_F, \tag{3.4}$$

where $\overline{T}^\vee = \mathrm{Hom}\left(Y_{Q,n}, \mathbf{C}^\times\right)$ is the dual group of $\overline{T}$. Then $L(s,i(\chi),Ad_\alpha)$ is by definition equal to the local Artin L-function $L(s,Ad_\alpha \circ \phi_\chi)$ associated with $Ad_\alpha \circ \phi_\chi$ – that is,

$$L(s,i(\chi),Ad_\alpha) := L(s,Ad_\alpha \circ \phi_\chi) = \det\left(1-q^{-s}Ad_\alpha \circ \phi_\chi(\mathrm{Frob})|_{V_\alpha^I}\right)^{-1}.$$

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For unramified $i(\chi)$ (equivalently, unramified χ), the inertia group I acts trivially on V_α. It follows that

$$L(s,i(\chi),Ad_\alpha) = \det\left(1-q^{-s}Ad_\alpha \circ \phi_\chi(\varpi)|_{V_\alpha}\right)^{-1}.$$

Moreover, by [16, Theorem 7.8], we have

$$\phi_\chi \circ Ad(\varpi)\left(E_{\alpha^\vee_{Q,n}}\right) = \chi\left(\overline{h}_\alpha(\varpi^{n_\alpha})\right)\cdot E_{\alpha^\vee_{Q,n}}$$

and therefore

$$L(s,i(\chi),Ad_\alpha) = \left(1-q^{-s}\cdot\chi\left(\overline{h}_\alpha(\varpi^{n_\alpha})\right)\right)^{-1} = L\left(s,\underline{\chi}_\alpha\right).$$

Moreover, if we denote

$$E_{\mathbb{w}} := \bigoplus_{\mathbb{w}\in\Phi_{\mathbb{w}}} \mathbf{C}\cdot E_{\alpha^\vee_{Q,n}}$$

and let $Ad_{\mathbb{w}}$ be the restriction of the adjoint representation Ad to $E_{\mathbb{w}}$, then the Artin L-function associated to $Ad_{\mathbb{w}}$ is

$$L(s,Ad_{\mathbb{w}}\circ\phi_\chi) = \prod_{\alpha\in\Phi_{\mathbb{w}}} L(s,i(\chi),Ad_\alpha).$$

We also denote $L(s,i(\chi),Ad_{\mathbb{w}}) := L(s,Ad_{\mathbb{w}}\circ\phi_\chi)$. Thus,

$$c_{\mathsf{gk}}(w,\chi) = \frac{L(0,i(\chi),Ad_{\mathbb{w}})}{L(1,i(\chi),Ad_{\mathbb{w}})}. \tag{3.5}$$

3.4. Normalisation

For $\mathbb{w}\in W$, we normalise the intertwining operator by

$$\mathscr{A}(w,\chi) := c_{\mathsf{gk}}(w,\chi)^{-1}\cdot A(w,\chi) = \frac{L(1,i(\chi),Ad_{\mathbb{w}})}{L(0,i(\chi),Ad_{\mathbb{w}})}\cdot A(w,\chi).$$

Note that in our setting, $\varepsilon\left(s,\underline{\chi}_\alpha,\psi\right)=1$ for every α, and thus this normalisation is the same as the one for linear algebraic groups, first proposed by Langlands [30] and investigated, for example, in [28, 40].

Proposition 3.1. *Let χ be an unramified genuine character of $Z\left(\overline{T}\right)$. For every $\mathbb{w}_1,\mathbb{w}_2\in W$ (with no requirement on the length),*

$$\mathscr{A}(w_2w_1,\chi) = \mathscr{A}(w_2,{}^{w_1}\chi)\circ\mathscr{A}(w_1,\chi). \tag{3.6}$$

In particular, $\mathscr{A}\left(w^{-1},{}^{w}\chi\right)\circ\mathscr{A}(w,\chi) = id$ for every $\mathbb{w}\in W$. Moreover, if χ is also unitary, then $\mathscr{A}(w,\chi)$ is holomorphic.

Proof. The proof is essentially that of Winarsky [54, Page 951-952], which relies on an inductive argument on the length of $\mathbb{w}_2$ and also the basic step

$$\mathscr{A}(w_\alpha,{}^{w_\alpha}\chi)\circ\mathscr{A}(w_\alpha,\chi) = \mathrm{id}.$$

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However, this last equality follows from equation (3.3) and the normalisation of $\mathscr{A}(w_\alpha, \chi)$ in equation (3.6). See [36, Page 313-314] for some details of the argument in the context of covering groups. $\square$

4. R-groups

From now on, we assume that χ is a unitary unramified genuine character of $Z(\overline{T})$. Set

$$W_\chi := \{\mathbb{w} \in W : {}^w\chi = \chi\} \subset W.$$

Proposition 3.1 shows that

$$\mathbb{w} \mapsto \mathscr{A}(w, \chi)$$

gives rise to a group homomorphism

$$\tau_\chi : W_\chi \longrightarrow \mathrm{Isom}_{\overline{G}}(I(\chi)),$$

where $\mathrm{Isom}_{\overline{G}}(I(\chi))$ denotes the group of $\overline{G}$-isomorphisms of $I(\chi)$. Let $\mathrm{End}_{\overline{G}}(I(\chi))$ be the commuting algebra of $I(\chi)$. The group homomorphism τ_χ gives an algebra homomorphism which, by abuse of notation, is also denoted by

$$\tau_\chi : \mathbf{C}[W_\chi] \longrightarrow \mathrm{End}_{\overline{G}}(I(\chi)).$$

However, τ_χ is not an isomorphism in general.

We would like to define a subgroup $R_\chi \subset W_\chi$ such that τ_χ induces an algebra isomorphism

$$\mathbf{C}[R_\chi] \simeq \mathrm{End}_{\overline{G}}(I(\chi)).$$

For this purpose, consider the set

$$\Phi_\chi = \left\{\alpha > 0 : \underline{\chi}_\alpha = \mathbb{1}\right\} \subset \Phi,$$

where $\underline{\chi}_\alpha$ is as in formula (3.2). Let $W_\chi^0 \subset W$ be the subgroup generated by $\{\mathbb{w}_\alpha : \alpha \in \Phi_\chi\}$. It follows from [18, Lemma 3.1] that $\mathbb{w}_\alpha \in W_\chi$ if $\alpha \in \Phi_\chi$. Therefore,

$$W_\chi^0 \subseteq W_\chi$$

and we have a short exact sequence

$$W_\chi^0 \hookrightarrow W_\chi \twoheadrightarrow R_\chi,$$

where $R_\chi := W_\chi / W_\chi^0$. The sequence splits with a natural splitting $s : R_\chi \hookrightarrow W_\chi$, given as follows. Consider the group

$$W(\Phi_\chi) = \{\mathbb{w} \in W : \mathbb{w}(\Phi_\chi) = \Phi_\chi\}.$$

Then we have

$$R_\chi \simeq W_\chi \cap W(\Phi_\chi),$$

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or more explicitly,

$$\begin{aligned} R_\chi &\simeq \{\mathrm{w} \in W_\chi : \mathrm{w}(\Phi_\chi) = \Phi_\chi\} \\ &= \{\mathrm{w} \in W_\chi : \alpha > 0 \text{ and } \alpha \in \Phi_\chi \text{ imply that } \mathrm{w}(\alpha) > 0\} \\ &= \{\mathrm{w} \in W_\chi : \Phi_{\mathrm{w}} \cap \Phi_\chi = \emptyset\}. \end{aligned}$$

This gives $W_\chi \simeq W_\chi^0 \rtimes R_\chi$. We always identity R_χ with $W_\chi \cap W(\Phi_\chi)$.

Before we proceed, we first show that for $\mathrm{w} \in W_\chi$, the two factors $c_{\mathsf{gk}}(w,\chi)$ and $\gamma(w,\chi)^{-1}$ are actually equal.

Lemma 4.1. *Let $\overline{G}$ be an n-fold cover of a connected reductive group G, and χ a unitary unramified character of $Z(\overline{T})$. For every $\mathrm{w} \in W_\chi$, we have*

$$L(s, Ad_{\mathrm{w}} \circ \phi_\chi) = L(s, Ad_{\mathrm{w}}^\vee \circ \phi_\chi),$$

where $Ad_{\mathrm{w}}^\vee$ is the contragredient representation of Ad_{w}. Therefore, for every $\mathrm{w} \in W_\chi$,

$$c_{\mathsf{gk}}(w,\chi)^{-1} = \gamma(w,\chi),$$

which is nonzero if $\mathrm{w} \in R_\chi$.

Proof. The argument is already in [28, Lemma 4.2]. First, since $\mathrm{w} \in W_\chi$, the two representations $Ad_{\mathrm{w}} \circ \phi_\chi$ and $Ad_{\mathrm{w}}^\vee \circ \phi_\chi$ are equivalent. Therefore, $L(s, Ad_{\mathrm{w}} \circ \phi_\chi) = L(s, Ad_{\mathrm{w}}^\vee \circ \phi_\chi)$. Now for $\mathrm{w} \in W_\chi$, we have

$$c_{\mathsf{gk}}(w,\chi)^{-1} = \frac{L(1, Ad_{\mathrm{w}} \circ \phi_\chi)}{L(0, Ad_{\mathrm{w}} \circ \phi_\chi)} = \frac{L(1, Ad_{\mathrm{w}}^\vee \circ \phi_\chi)}{L(0, Ad_{\mathrm{w}} \circ \phi_\chi)} = \gamma(w,\chi).$$

The proof is completed in view of the fact that $1 - \underline{\chi}_\alpha(\varpi) \neq 0$ for every $\alpha \in \Phi_{\mathrm{w}}$, if $\mathrm{w} \in R_\chi$. □

The main theorem on R-groups is as follows:

Theorem 4.2 ([29, 44, 34, 33, 35])**.** *For a unitary unramified genuine character χ of $Z(\overline{T})$, we have*

$$W_\chi^0 = \{\mathrm{w} : \mathscr{A}(w,\chi) \text{ is a scalar}\}.$$

Moreover, the algebra $End_{\overline{G}}(I(\chi))$ has a basis given by $\{\mathscr{A}(w,\chi) : \mathrm{w} \in R_\chi\}$, and by restriction τ_χ gives a natural algebra isomorphism

$$\mathbf{C}[R_\chi] \simeq End_{\overline{G}}(I(\chi)).$$

For a general parabolic induction of linear algebraic groups, the analogous result was first shown by Knapp and Stein [29] for real groups, and by Silberger [44] for p-adic groups. The generalisation to covering groups includes the work of W.-W. Li [34, 33], which shows that Harish-Chandra's harmonic analysis extends to the covering setting; in particular, the Harish-Chandra c-function and Plancherel measure are discussed in detail. Finally, in the recent work of C. Luo [35], the Harish-Chandra commuting algebra theorem is proved for general parabolic induction on covering groups, and in particular, in the minimal parabolic case the isomorphism $\mathbf{C}[R_\chi] \simeq \mathrm{End}_{\overline{G}}(I(\chi))$ is established.

Let

$$I(\chi) = \bigoplus_{i=1}^{k} m_i \pi_i$$

be the decomposition of $I(\chi)$ into irreducible subrepresentations with multiplicities m_i. Denote

$$\Pi(\chi) := \{\pi_i : 1 \le i \le k\},$$

which constitutes the L-packet associated to the parameter ϕ_χ corresponding to $I(\chi)$. Some immediate consequences of Theorem 4.2 are:

(i) $\dim \mathrm{End}_{\overline{G}}(I(\chi)) = |R_\chi|$ and $\mathbf{C}[R_\chi] \simeq \bigoplus_{i=1}^{k} M_{m_i}(\mathbf{C})$.

(ii) $m_i = 1$ for all i if and only if R_χ is abelian.

(iii) in general, k is equal to the dimension of the centre of $\mathbf{C}[R_\chi]$, which is equal to the the number of conjugacy classes of R_χ. Thus, there is a bijective correspondence

$$\mathrm{Irr}\,(R_\chi) \longleftrightarrow \Pi(\chi).$$

Since we are dealing with unramified χ, we will show later (see Theorem 4.6) that R_χ is actually abelian, and thus $I(\chi)$ is multiplicity-free.

4.1. Parametrisation of $\Pi(\chi)$

Denote the correspondence

$$\mathrm{Irr}\,(R_\chi) \longleftrightarrow \Pi(\chi)$$

from (iii) in the foregoing by

$$\sigma \leftrightarrow \pi_\sigma \text{ and } \sigma_\pi \leftrightarrow \pi.$$

The correspondence is not canonical, and can be twisted by characters of R_χ. However, in the unramified setting, we have a natural parametrisation given as follows (see [27, Page 39]).

Denote by θ_σ the character of $\sigma \in \mathrm{Irr}\,(R_\chi)$. Consider the operator

$$P_\sigma := \frac{\dim \sigma}{|R_\chi|} \sum_{\mathbbm{w} \in R_\chi} \overline{\theta_\sigma(\mathbbm{w})} \cdot \mathscr{A}(w, \chi). \tag{4.1}$$

Since the characters $\{\theta_\sigma : \sigma \in \mathrm{Irr}\,(R_\chi)\}$ are orthogonal, the operators P_σ are orthogonal projections of $I(\chi)$ onto nonzero invariant subspaces. Denote

$$V(\sigma) := P_\sigma(I(\chi)).$$

Then $V(\sigma)$ are pairwise disjoint and we have

$$I(\chi) = \bigoplus_{\sigma \in \mathrm{Irr}(R_\chi)} V(\sigma).$$

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Since the number of inequivalent constituents of $I(\chi)$ is equal to $|\mathrm{Irr}(R_\chi)|$, it follows that $V(\sigma)$ is an isotypic sum of an irreducible representation. Thus, we write

$$V(\sigma) = m_{\pi_\sigma} \cdot \pi_\sigma,$$

and this gives a correspondence $\sigma \mapsto \pi_\sigma$. We also have

$$m_{\pi_\sigma} = \dim \sigma.$$

The inverse $\pi \mapsto \sigma_\pi$ can be described explicitly as well (see [27, Page 39]). The group $\mathrm{Hom}(R_\chi, \mathbf{C}^\times)$ acts on $\Pi(\chi)$ by transporting the obvious action on $\mathrm{Irr}(R_\chi)$ given by

$$\xi \otimes \sigma, \text{ where } \xi \in \mathrm{Hom}\left(R_\chi, \mathbf{C}^\times\right) \text{ and } \sigma \in \mathrm{Irr}(R_\chi).$$

This action is free and transitive on the elements $\pi \in \Pi(\chi)$ which occur with multiplicity 1 in $I(\chi)$. Thus the correspondence $\sigma \leftrightarrow \pi_\sigma$ is not canonical. However, the parametrisation given by definition (4.1) (without any further twisting) is already a natural one in view of the following:

Lemma 4.3. *With notations as before, $P_{\mathbb{1}}(I(\chi)) = \pi_\chi^{un}$ – that is, the unramified constituent π_χ^{un} of $I(\chi)$ is parametrised by the trivial representation $\mathbb{1} \in Irr(R_\chi)$.*

Proof. It suffices to show that $P_{\mathbb{1}}(f_0) = f_0$, where $f_0 \in \pi_\chi^{un} \subset I(\chi)$ is the normalised unramified vector. However, by definition (4.1) we have

$$P_{\mathbb{1}}(f_0) = \frac{1}{|R_\chi|} \sum_{\mathbb{w} \in R_\chi} \mathscr{A}(w,\chi)(f_0).$$

By the normalisation in $\mathscr{A}(w,\chi)$, we have $\mathscr{A}(w,\chi)(f_0) = f_0$ for every $\mathbb{w} \in R_\chi$ (in fact for every $\mathbb{w} \in W_\chi$). Thus, $P_{\mathbb{1}}(f_0) = f_0$, and this gives the desired conclusion. □

4.2. Some analysis on R_χ

From this subsection to Section 4.4, we will analyse the group R_χ by relating it to the work of Keys on simply connected Chevalley groups [26]. More precisely, we will show that there are groups W_χ^{sc} and R_χ^{sc} containing W_χ and R_χ, respectively, such that:

(i) We have $R_\chi^{sc}/R_\chi \simeq W_\chi^{sc}/W_\chi$, which can be understood from the dual side in terms of the L-parameter ϕ_χ. In particular, the size of R_χ^{sc}/R_χ is related to the centre $Z\left(\overline{G}^\vee\right)$ of the dual group $\overline{G}^\vee$.

(ii) The group R_χ^{sc} is equal to the R-group of a unramified principal series on a simply connected Chevalley group, which is determined in [26]. In particular, R_χ^{sc} is abelian, and this forces R_χ to be abelian. However, since R_χ is not equal to R_χ^{sc} in general, this shows that unitary genuine principal series of a covering group tend to be less reducible than the linear algebraic group.

For this purpose, we first define a group W_χ^{sc} such that

$$W_\chi \subseteq W_\chi^{sc} \subseteq W.$$

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Let $\mathbf{T}_{Q,n}$ and $\mathbf{T}_{Q,n}^{sc}$ be the split tori defined over F associated to the two lattices $Y_{Q,n}$ and $Y_{Q,n}^{sc}$, respectively. Denote by $T_{Q,n}$ and $T_{Q,n}^{sc}$ their F-rational points. Let $T_{Q,n}^{\dagger}$ and $T_{Q,n}^{sc,\dagger}$ be the images of $T_{Q,n}$ and $T_{Q,n}^{sc}$ in T with respect to the two maps

$$i_{Q,n} : T_{Q,n} \to T \text{ and } i^{sc} : T_{Q,n}^{sc} \to T,$$

which are induced from $Y_{Q,n} \hookrightarrow Y$ and $Y_{Q,n}^{sc} \hookrightarrow Y$, respectively. Recalling the projection

$$\phi : \overline{G} \twoheadrightarrow G,$$

we have from Section 3.1 that

$$Z\left(\overline{T}\right) = \phi^{-1}\left(T_{Q,n}^{\dagger}\right).$$

We denote

$$Z\left(\overline{T}\right)^{sc} := \phi^{-1}\left(T_{Q,n}^{sc,\dagger}\right) \subset Z\left(\overline{T}\right).$$

Let

$$\chi^{sc} := \chi|_{Z(\overline{T})^{sc}}$$

be the genuine character of $Z\left(\overline{T}\right)^{sc}$ obtained from the restriction of χ. Consider

$$W_{\chi}^{sc} := \{\mathbb{w} \in W : {}^{w}(\chi^{sc}) = \chi^{sc}\} \supset W_{\chi}$$

and analogously

$$R_{\chi}^{sc} := W_{\chi}^{sc}/W_{\chi}^{0},$$

which then contains R_{χ}. A splitting s^{sc} of R_{χ}^{sc} into W_{χ}^{sc} is given by

$$R_{\chi}^{sc} \simeq W_{\chi}^{sc} \cap W\left(\Phi_{\chi}\right).$$

In summary, we have a commutative diagram with exact rows and compatible splittings:

$$\begin{array}{ccccc} W_{\chi}^{0} & \hookrightarrow & W_{\chi} & \overset{\curvearrowleft s}{\twoheadrightarrow} & R_{\chi} \\ \| & & \downarrow & & \downarrow \\ W_{\chi}^{0} & \hookrightarrow & W_{\chi}^{sc} & \underset{\curvearrowleft s^{sc}}{\twoheadrightarrow} & R_{\chi}^{sc}, \end{array}$$

where s and s^{sc} are the aforementioned natural splittings of R_{χ} and R_{χ}^{sc}, respectively. It follows immediately that we have an isomorphism of finite groups:

$$R_{\chi}^{sc}/R_{\chi} \simeq W_{\chi}^{sc}/W_{\chi}.$$

4.3. The quotient R_{χ}^{sc}/R_{χ}

Recall the L-parameter

$$\phi_{\chi} : W_F \longrightarrow {}^{L}\overline{T}$$

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associated to $i(\chi)$, which is a splitting of the extension

$$\overline{T}^{\vee} \hookrightarrow {}^{L}\overline{T} \twoheadrightarrow W_F. \tag{4.2}$$

In fact, ϕ_χ depends only on χ, and this justifies our notation. Recall also that $\overline{T}^{\vee} = \mathrm{Hom}\left(Y_{Q,n}, \mathbf{C}^{\times}\right)$ is the dual group of $\overline{T}$ and

$$Z\left(\overline{G}^{\vee}\right) \simeq \mathrm{Hom}\left(Y_{Q,n}/Y_{Q,n}^{sc}, \mathbf{C}^{\times}\right).$$

Denote

$$\overline{T}_{ad}^{\vee} := \overline{T}^{\vee}/Z\left(\overline{G}^{\vee}\right) \simeq \mathrm{Hom}\left(Y_{Q,n}^{sc}, \mathbf{C}^{\times}\right).$$

Pushing out the exact sequence in formula (4.2) via the map $f : \overline{T}^{\vee} \to \overline{T}^{\vee}/Z\left(\overline{G}^{\vee}\right)$, we obtain a short exact sequence

$$\overline{T}_{ad}^{\vee} \hookrightarrow {}^{L}\overline{T}^{sc} := f_*\left({}^{L}\overline{T}\right) \twoheadrightarrow W_F. \tag{4.3}$$

Denote by $\mathrm{Spl}\left({}^{L}\overline{T}, W_F\right)$ the set of splittings of formula (4.2), and similarly for $\mathrm{Spl}\left({}^{L}\overline{T}^{sc}, W_F\right)$. Then f induces a map

$$f_* : \mathrm{Spl}\left({}^{L}\overline{T}, W_F\right) \longrightarrow \mathrm{Spl}\left({}^{L}\overline{T}^{sc}, W_F\right), \tag{4.4}$$

which arises from the obvious composite. The Weyl group W acts naturally on the two groups ${}^{L}\overline{T}$ and ${}^{L}\overline{T}^{sc}$, and the map f_* in formula (4.4) is W-equivariant. Here $f_*(\phi_\chi)$ is naturally associated with χ^{sc}. Since the local Langlands correspondence for a covering torus is W-equivariant (see [14, §9.3]), we have

$$W_\chi = \mathrm{Stab}_W(\phi_\chi)$$

and

$$W_\chi^{sc} = \mathrm{Stab}_W(f_*(\phi_\chi)).$$

Here $\mathrm{Stab}_W(x)$ denotes the subgroup of stabilisers of x in W.

As it might be more convenient to work with parameters valued in the dual group (instead of the L-group), we can have the following reduction. From Section 2.2, we have a distinguished genuine character χ_ψ of $Z\left(\overline{T}\right)$ depending on a nontrivial additive character ψ of F. This gives a splitting $\phi_{\chi_\psi} \in \mathrm{Spl}\left({}^{L}\overline{T}, W_F\right)$, which is fixed by W since χ_ψ is W-invariant (see [14, §6.5]). From this, we obtain a 'relative' version of map (4.4):

$$f_*^{\psi} : \mathrm{Hom}\left(W_F, \overline{T}^{\vee}\right) \longrightarrow \mathrm{Hom}\left(W_F, \overline{T}_{ad}^{\vee}\right). \tag{4.5}$$

If we denote

$$\phi_\chi^{\psi} := \phi_\chi/\phi_{\chi_\psi} \in \mathrm{Hom}\left(W_F, \overline{T}^{\vee}\right),$$

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then f_*^ψ sends ϕ_χ^ψ to

$$\frac{f_*(\phi_\chi)}{f_*(\phi_{\chi_\psi})} \in \mathrm{Hom}\left(W_F, \overline{T}_{ad}^\vee\right).$$

The kernel of f_*^ψ is given by

$$\mathrm{Ker}\left(f_*^\psi\right) = \mathrm{Hom}\left(W_F, Z\left(\overline{G}^\vee\right)\right).$$

Since f_*^ψ is also W-equivariant, we have

$$W_\chi = \mathrm{Stab}_W\left(\phi_\chi^\psi\right), \qquad W_\chi^{sc} = \mathrm{Stab}_W\left(f_*^\psi\left(\phi_\chi^\psi\right)\right).$$

Assume henceforth that the conductor of ψ is O. In this case, χ_ψ is unramified; thus ϕ_χ^ψ is unramified and $\phi_\chi^\psi(\varpi) \in \overline{T}^\vee$ is the relative Satake parameter for χ. Also, f_*^ψ is determined by the map

$$f^\vee : \overline{T}^\vee \to \overline{T}_{ad}^\vee$$

such that

$$f^\vee\left(\phi_\chi^\psi(\varpi)\right) = \left(f_*^\psi\left(\phi_\chi^\psi\right)\right)(\varpi).$$

For every $t \in \overline{T}^\vee$ such that $f^\vee(t)$ is fixed by W_χ^{sc}, the action of W_χ^{sc} on the set $t \cdot Z\left(\overline{G}^\vee\right)$ is well defined. For such t, let

$$\mathcal{O}^{W_\chi^{sc}}\left(t \cdot Z\left(\overline{G}^\vee\right)\right)$$

be the set of W_χ^{sc}-orbits in $t \cdot Z\left(\overline{G}^\vee\right) \subset \overline{T}^\vee$.

Proposition 4.4. *Assume G is a semisimple group. Let χ be a unitary unramified genuine character of $Z\left(\overline{T}\right)$. Then*

$$\left[R_\chi^{sc} : R_\chi\right] = \frac{\left|Z\left(\overline{G}^\vee\right)\right|}{\left|\mathcal{O}^{W_\chi^{sc}}\left(\phi_\chi^\psi(\varpi) \cdot Z\left(\overline{G}^\vee\right)\right)\right|}.$$

Proof. For a finite group H acting on a finite set X, the orbit counting formula reads

$$|X/H| = \frac{1}{|H|}\sum_{h \in H} \left|X^h\right|,$$

where $X^h \subset X$ is the set of h-fixed points. To apply this to the case $X = \phi_\chi^\psi(\varpi) \cdot Z\left(\overline{G}^\vee\right)$ and $H = W_\chi^{sc}$, we first note that the action of W_χ^{sc} on the set $\phi_\chi^\psi(\varpi) \cdot Z\left(\overline{G}^\vee\right)$ is well defined, as already mentioned. Since W (and thus in particular W_χ^{sc}) acts trivially on $Z\left(\overline{G}^\vee\right)$, we see

$$X^{\mathsf{w}} = \begin{cases} X & \text{if } \mathsf{w} \in W_\chi, \\ \emptyset & \text{if } \mathsf{w} \notin W_\chi. \end{cases}$$

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Thus,

$$\left|\mathcal{O}^{W_\chi^{sc}}\left(\phi_\chi^\psi(\varpi)\cdot Z\left(\overline{G}^\vee\right)\right)\right| = \frac{1}{\left|W_\chi^{sc}\right|}\cdot\left|Z\left(\overline{G}^\vee\right)\right|\cdot|W_\chi|.$$

The result follows from the equality $\left[R_\chi^{sc}:R_\chi\right] = \left[W_\chi^{sc}:W_\chi\right]$. □

It is clear that the index $\left[R_\chi^{sc}:R_\chi\right] = \left[W_\chi^{sc}:W_\chi\right]$ is bounded above by $\left|Z\left(\overline{G}^\vee\right)\right|$ for covers of semisimple groups. In particular, if the dual group of $\overline{G}$ is of adjoint type, then $R_\chi^{sc} = R_\chi$. For example, if G is a simply connected Chevalley group and $n=1$, then there is no difference between R_χ^{sc} and R_χ. For another nontrivial example, consider the n-fold cover $\overline{\mathrm{SL}}_{n+1}^{(n)}$, which has dual group PGL_{n+1}, or the odd-degree cover of Sp_{2r} whose dual group is SO_{2r+1}.

4.4. The group R_χ^{sc}

Let $\overline{G}$ be an n-fold cover of a general linear algebraic group. Let $\mathbf{H}$ be the connected linear reductive group over F such that its root datum is obtained from inverting that of $\overline{\mathbf{G}}_{Q,n}^\vee$ – that is, the Langlands dual group of $\mathbf{H}$ is isomorphic to $\overline{\mathbf{G}}_{Q,n}^\vee$. If $n=1$, then $\mathbf{H} = \mathbf{G}$. Let

$$\mathbf{H}^{sc} \twoheadrightarrow \mathbf{H}_{der} \hookrightarrow \mathbf{H}$$

be the simply connected cover of the derived subgroup $\mathbf{H}_{der} \subset \mathbf{H}$. Thus, $Y_{Q,n}^{sc}$ is the cocharacter (and also the coroot) lattice of $\mathbf{H}^{sc}$. Denote by H, H^{sc} the F-rational points of $\mathbf{H}$ and $\mathbf{H}^{sc}$, respectively. Here H is the principal endoscopic group for $\overline{G}$.

We see that $T_{Q,n}^{sc}$ is just the torus of H^{sc}. The genuine character χ^{sc} of $Z\left(\overline{T}\right)^{sc}$ gives rise to a linear unramified character

$$\underline{\chi}^{sc}: T_{Q,n}^{sc} \to \mathbf{C}^\times \text{ given by } \underline{\chi}^{sc}\left(\alpha_{Q,n}^\vee(a)\right) := \chi^{sc}\left(\overline{h}_\alpha(a^{n_\alpha})\right)$$

for all $\alpha \in \Delta$. In fact, the covering

$$\mu_n \hookrightarrow Z\left(\overline{T}\right)^{sc} \twoheadrightarrow T_{Q,n}^{sc,\dagger}$$

has a splitting ρ^{sc} given by $\alpha_{Q,n}^\vee(a) \mapsto \overline{h}_\alpha(a^{n_\alpha})$ for all $\alpha \in \Delta$, and we have

$$\underline{\chi}^{sc} = \chi^{sc} \circ \rho^{sc} \circ i^{sc}.$$

One could form the unramified principal series $I\left(\underline{\chi}^{sc}\right)$ of H^{sc} and thus have the R-group $R_{\underline{\chi}^{sc}}$ for $I\left(\underline{\chi}^{sc}\right)$.

Proposition 4.5. *With notations as before, we have*

$$R_\chi^{sc} \simeq R_{\underline{\chi}^{sc}}.$$

Proof. It suffices to show that $W_{\chi^{sc}} = W_{\underline{\chi}^{sc}}$ and $\Phi_{\chi^{sc}} = \Phi_{\underline{\chi}^{sc}}$. By the definition of $\underline{\chi}^{sc}$, we have

$$\underline{\chi}^{sc}\left(\alpha_{Q,n}^\vee(a)\right) = \chi^{sc}\left(\overline{h}_\alpha(a^{n_\alpha})\right) \tag{4.6}$$

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for all $\alpha \in \Delta$. We claim that the equality holds for all $\alpha \in \Phi$. By induction on the length of $\mathbb{w}$ such that $\mathbb{w}(\alpha) \in \Delta$, it suffices to prove that if

$$\underline{\chi}^{sc}\left(\beta_{Q,n}^{\vee}(a)\right) = \chi^{sc}\left(\overline{h}_{\beta}(a^{n_{\beta}})\right), \quad \beta \in \Phi,$$

and $\gamma^{\vee} = \mathbb{w}_{\alpha}(\beta^{\vee}), \alpha \in \Delta$, then

$$\underline{\chi}^{sc}\left(\gamma_{Q,n}^{\vee}(a)\right) = \chi^{sc}\left(\overline{h}_{\gamma}(a^{n_{\gamma}})\right), \quad \beta \in \Phi. \tag{4.7}$$

As shown in [16, Page 112], we have

$$\overline{h}_{\gamma}(a^{n_{\gamma}}) = w_{\alpha}^{-1} \cdot \overline{h}_{\beta}(a^{n_{\beta}}) \cdot w_{\alpha} = w_{\alpha} \cdot \overline{h}_{\beta}(a^{n_{\beta}}) \cdot w_{\alpha}^{-1},$$

which is also equal to $\overline{h}_{\beta}(a^{n_{\beta}}) \cdot \overline{h}_{\alpha}(a^{n_{\alpha}})^{-\langle \alpha_{Q,n}, \beta_{Q,n}^{\vee} \rangle}$ by equation (2.2). Thus by the induction hypothesis, we have

$$\begin{aligned}
\underline{\chi}^{sc}\left(\gamma_{Q,n}^{\vee}(a)\right) =& \underline{\chi}^{sc}\left(\beta_{Q,n}^{\vee}(a)\right) \cdot \underline{\chi}^{sc}\left(\alpha_{Q,n}^{\vee}(a)\right)^{-\langle \alpha_{Q,n}, \beta_{Q,n}^{\vee} \rangle} \\
=& \chi^{sc}\left(\overline{h}_{\beta}(a^{n_{\beta}})\right) \cdot \chi^{sc}\left(\overline{h}_{\alpha}(a^{n_{\alpha}})\right)^{-\langle \alpha_{Q,n}, \beta_{Q,n}^{\vee} \rangle} \\
=& \chi^{sc}\left(\overline{h}_{\gamma}(a^{n_{\gamma}})\right).
\end{aligned}$$

This proves that equation (4.6) holds for all $\alpha \in \Phi$, and thus $\Phi_{\underline{\chi}^{sc}} = \Phi_{\chi^{sc}}$.

Let $\mathbb{w} = \mathbb{w}_l \mathbb{w}_{l-1} \cdots \mathbb{w}_i$ be a minimal decomposition of $\mathbb{w}$, with $\mathbb{w}_i = \mathbb{w}_{\alpha_i}$ for some $\alpha_i \in \Delta$. Let $w = w_1 \cdots w_2 w_1$, where $w_i := w_{\alpha_i}$, be the representative of $\mathbb{w}_{\alpha_i}$. The foregoing argument also shows inductively that

$$w \cdot \overline{h}_{\alpha}(\varpi^{n_{\alpha}}) \cdot w^{-1} = \overline{h}_{\mathbb{w}(\alpha)}(\varpi^{n_{\mathbb{w}(\alpha)}})$$

for all $\alpha \in \Phi$. It follows that

$$w^{-1} \cdot \overline{h}_{\alpha}(\varpi^{n_{\alpha}}) \cdot w = \overline{h}_{\mathbb{w}^{-1}(\alpha)}(\varpi^{n_{\mathbb{w}^{-1}(\alpha)}})$$

for all $\alpha \in \Phi$, and thus

$${}^{w}\left(\underline{\chi}^{sc}\right)\left(\alpha_{Q,n}^{\vee}(a)\right) = {}^{w}\left(\chi^{sc}\right)\left(\overline{h}_{\alpha}(a^{n_{\alpha}})\right)$$

for all $\mathbb{w} \in W$ and $\alpha \in \Delta$. Therefore, $W_{\underline{\chi}^{sc}} = W_{\chi^{sc}}$. Thus by the definition of R-groups, we have $R_{\underline{\chi}^{sc}} \simeq R_{\chi}^{sc}$. □

In [26, §3], all possibilities for the group $R_{\chi}^{sc} \simeq R_{\underline{\chi}^{sc}}$ are tabulated as in Tables 1 and 2.

Immediately, we have the following theorem:

TABLE 1. The group R_{χ}^{sc} for classical group $\mathbf{H}^{sc}$.

$\mathbf{H}^{sc}$	A_r	B_r	C_r	D_r, r even	D_r, r odd
R_{χ}^{sc}	$\mathbf{Z}/d\mathbf{Z}, d \mid (r+1)$	$\mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$ or $(\mathbf{Z}/2\mathbf{Z})^2$	$\mathbf{Z}/2\mathbf{Z}$ or $\mathbf{Z}/4\mathbf{Z}$

TABLE 2. The group R_χ^{sc} for exceptional group $\mathbf{H}^{sc}$.

$\mathbf{H}^{sc}$	E_6	E_7	E_8	F_4	G_2
R_χ^{sc}	$\mathbf{Z}/3\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$	$\{1\}$	$\{1\}$	$\{1\}$

Theorem 4.6. *Let $\overline{G}$ be an n-fold cover of a connected reductive group G and χ be a unitary unramified genuine character of $Z\left(\overline{T}\right)$. Then $R_\chi \subset R_\chi^{sc}$ is abelian, and therefore*

$$I(\chi) = \bigoplus_{\sigma \in \mathrm{Irr}(R_\chi)} \pi_\sigma.$$

That is, the decomposition of $I(\chi)$ is multiplicity-free.

Corollary 4.7. *For every $\mathsf{w} \in R_\chi$, we have*

$$\mathscr{A}(w,\chi) = \bigoplus_{\sigma \in \mathrm{Irr}(R_\chi)} \sigma(\mathsf{w}) \cdot \mathrm{id}_{\pi_\sigma}.$$

Therefore, for every $f \in C_c^\infty\left(\overline{G}\right)$,

$$\mathrm{Trace}\ \mathscr{A}(w,\chi) I(\chi)(f) = \sum_{\sigma \in \mathrm{Irr}(R_\chi)} \sigma(\mathsf{w}) \cdot \mathrm{Trace}\ \pi_\sigma(f).$$

Proof. It suffices to verify the first equality, as the second will follow from it. For $\sigma \in \mathrm{Irr}(R_\chi)$, recall the projection P_σ of $I(\chi)$ on π_σ. By Schur's lemma, $\mathscr{A}(w,\chi)$ acts on π_σ as a homothety given by $c_{\pi_\sigma}(\mathsf{w}) \in \mathbf{C}$. However,

$$\begin{aligned} P_{\sigma'} &= \frac{1}{|R_\chi|} \sum_{\mathsf{w} \in R_\chi} \overline{\sigma'(\mathsf{w})} \left(\bigoplus_{\sigma \in \mathrm{Irr}(R_\chi)} c_{\pi_\sigma}(\mathsf{w}) \cdot \mathrm{id}_{\pi_\sigma} \right) \\ &= \bigoplus_{\sigma \in \mathrm{Irr}(R_\chi)} \left(\frac{1}{|R_\chi|} \sum_{\mathsf{w} \in R_\chi} \overline{\sigma'(\mathsf{w})} \cdot c_{\pi_\sigma}(\mathsf{w}) \right) \cdot \mathrm{id}_{\pi_\sigma}. \end{aligned}$$

That is, the pairing of the class function $c_{\pi_\sigma}(-)$ on R_χ with $\sigma'(-)$ is equal to the Kronecker delta function $\delta_{\sigma,\sigma'}$. Since the characters $\mathrm{Irr}(R_\chi)$ form an orthonormal basis for class functions on R_χ, this shows that $c_{\pi_\sigma} = \sigma$ for every $\sigma \in \mathrm{Irr}(R_\chi)$. □

4.5. Covers of SL_2

We illustrate the previous discussion on R-groups by considering the n-fold cover $\overline{G} = \overline{\mathrm{SL}}_2^{(n)}$ arising from $Q(\alpha^\vee) = 1$. We show that our analysis agrees with certain results in [49].

If n is even, then the dual group $\overline{G}^\vee$ is SL_2. Write

$$s_\zeta := \phi_\chi^\psi(\varpi) = \begin{pmatrix} \zeta & \\ & \zeta^{-1} \end{pmatrix} \in \overline{G}^\vee$$

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for the relative Satake parameter of χ discussed in Section 4.3, where $\zeta \in \mathbf{C}^\times$ depends on χ. There are three cases:

- $\zeta^4 \neq 1$. In this case, $\Phi_\chi = \emptyset$ and thus $W(\Phi_\chi) = W$. Also, $W_\chi^{sc} = W_\chi = \{1\}$. Therefore, $R_\chi^{sc} = R_\chi = \{1\}$, and in particular $\left|\mathcal{O}^{W_\chi^{sc}}\left(s_\zeta \cdot Z\left(\overline{G}^\vee\right)\right)\right| = 2$.
- $\zeta^4 = 1$ but $\zeta^2 \neq 1$. In this case, $\Phi_\chi = \emptyset$ and thus $W(\Phi_\chi) = W$. However, we have $W_\chi^{sc} = W$, while $W_\chi = \{1\}$. Therefore $R_\chi^{sc} = W$ and $R_\chi = \{1\}$. Indeed, we have $\left|\mathcal{O}^{W_\chi^{sc}}\left(s_\zeta \cdot Z\left(\overline{G}^\vee\right)\right)\right| = 1$ in this case.
- $\zeta^2 = 1$. In this case, $\Phi_\chi = \{\alpha\}$ and thus $W(\Phi_\chi) = \{1\}$. Also, $W_\chi^{sc} = W_\chi = W$. Therefore $R_\chi^{sc} = R_\chi = \{1\}$, and also $\left|\mathcal{O}^{W_\chi^{sc}}\left(s_\zeta \cdot Z\left(\overline{G}^\vee\right)\right)\right| = 2$.

Hence for n even, $I(\chi)$ is always irreducible for a unitary unramified genuine character χ.

Now we consider n odd, in which case the dual group of $\overline{\mathrm{SL}}_2^{(n)}$ is $\overline{G}^\vee = \mathrm{PGL}_2$ and thus we always have $R_\chi^{sc} = R_\chi$, by Proposition 4.4. Write

$$s_\zeta := \rho_\chi^\psi(\varpi) = \begin{pmatrix} \zeta & \\ & 1 \end{pmatrix} \in \overline{G}^\vee$$

for the relative Satake parameter for $I(\chi)$. There are three cases:

- $\zeta^2 \neq 1$. In this case, $\Phi_\chi = \emptyset$ and thus $W(\Phi_\chi) = W$. Also, $W_\chi = \{1\}$. Therefore $R_\chi = \{1\}$.
- $\zeta = -1$. Then $\Phi_\chi = \emptyset$ and thus $W(\Phi_\chi) = W$. But $W_\chi = W$. Therefore, $R_\chi = W$.
- $\zeta = 1$. Then $\Phi_\chi = \{\alpha\}$ and thus $W(\Phi_\chi) = \{1\}$. But $W_\chi = W$. In this case, $R_\chi = \{1\}$.

Combining the even and odd cases, we see that the only reducibility point for $I(\chi)$ is when n is odd and χ is such that $\underline{\chi}_\alpha$ is a nontrivial quadratic character; in this case,

$$I(\chi) = \pi_\chi^{un} \oplus \pi.$$

The result agrees with [49, Proposition 5.1].

4.6. Comparison of R_χ and $\mathcal{S}_{\phi_\chi}$

For linear algebraic groups, it was shown by Keys [27] that the R-group R_χ is naturally identified with the component group of the centraliser of the parameter ϕ_χ. In this subsection, we establish the same identification for covering groups.

Let

$$\phi : W_F \to {}^L\overline{G}$$

be an L-parameter. Let $S_\phi \subset \overline{G}^\vee$ be the centraliser of $\phi(W_F)$ in $\overline{G}^\vee$ – that is,

$$S_\phi := \left\{ g \in \overline{G}^\vee : g \cdot \phi(a) \cdot g^{-1} = \phi(a) \text{ for every } a \in W_F \right\}.$$

It is a reductive subgroup of $\overline{G}^\vee$ but not necessarily connected. Let S_ϕ^0 be the connected component of S_ϕ. Define

$$\mathcal{S}_\phi := \frac{S_\phi}{Z\left(\overline{G}^\vee\right) \cdot S_\phi^0}. \tag{4.8}$$

There is a dual-group (instead of L-group) relative version which is more closely related to linear algebraic groups. Recall that depending on the choice of a distinguished genuine character χ_ψ, we have an isomorphism

$${}^L\overline{G} \simeq_{\chi_\psi} \overline{G}^\vee \times W_F$$

(see formula (2.5)). We obtain an unramified parameter

$$\phi[\chi_\psi] : W_F \xrightarrow{\phi} {}^L\overline{G} \longrightarrow\!\!\!\!\!\rightarrow \overline{G}^\vee,$$

where the second map is the projection depending on χ_ψ. In fact, by the local Langlands correspondence for covering tori, the distinguished genuine character χ_ψ gives rise to a splitting of a certain fundamental central extension

$$Z\left(\overline{G}^\vee\right) \hookrightarrow E \longrightarrow\!\!\!\!\!\rightarrow W_F$$

(see [14, Proposition 6.5], where E is represented by $E_1 + E_2$ using notations there). Since by definition ${}^L\overline{G}$ equals the pushout of E via the inclusion $Z\left(\overline{G}^\vee\right) \hookrightarrow \overline{G}^\vee$ (see [14, §5.2]), we have that χ_ψ yields an L-parameter ϕ_{χ_ψ}, which takes values in $Z\left({}^L\overline{G}\right)$. We have

$$\phi[\chi_\psi] = \phi \cdot \phi_{\chi_\psi}^{-1}.$$

Analogously, let $S_{\phi[\chi_\psi]} \subset \overline{G}^\vee$ be the centraliser of $\phi[\chi_\psi](W_F)$ in $\overline{G}^\vee$. Define

$$\mathcal{S}_{\phi[\chi_\psi]} := \frac{S_{\phi[\chi_\psi]}}{Z\left(\overline{G}^\vee\right) \cdot S_{\phi[\chi_\psi]}^0}.$$

Lemma 4.8. *With notations as before, $S_\phi = S_{\phi[\chi_\psi]}$ for every distinguished genuine character χ_ψ. Hence $\mathcal{S}_\phi = \mathcal{S}_{\phi[\chi_\psi]}$ as well.*

Proof. It follows from the isomorphism ${}^L\overline{G} \simeq_{\chi_\psi} \overline{G}^\vee \times W_F$ that

$$\phi(a) = (\phi[\chi_\psi](a), a) \in \overline{G}^\vee \times W_F.$$

Thus, $g \in \overline{G}^\vee$ centralises $\phi(a)$ if and only if it centralises $\phi[\chi_\psi](a)$; this gives the equality $S_\phi = S_{\phi[\chi_\psi]}$ and completes the proof. □

Theorem 4.9. *Let $\overline{G}$ be an n-fold cover of a connected reductive group G. Let χ be a unitary unramified genuine character of $Z\left(\overline{T}\right)$, and let ϕ_χ be the L-parameter associated to χ. We have an isomorphism*

$$R_\chi \simeq \mathcal{S}_{\phi_\chi}.$$

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Proof. The idea is to reduce to the linear algebraic case, where the isomorphism is proved in [27, Page 42].

As in Section 4.4, let $\mathbf{H}$ be the connected split linear algebraic group whose dual group is $\overline{\mathbf{G}}_{Q,n}^{\vee}$. Let $\mathbf{T}_{Q,n}$ be its split torus whose cocharacter lattice is $Y_{Q,n}$. We have $H, T_{Q,n}$ denoting the F-rational points of $\mathbf{H}, \mathbf{T}_{Q,n}$, respectively.

Let χ be a unitary unramified genuine character of $Z\left(\overline{T}\right)$. For simplicity, denote by ϕ (instead of ϕ_χ) the associated L-parameter valued in ${}^L\overline{T}$. By choosing a distinguished genuine character χ_ψ, we have the unramified parameter

$$\phi[\chi_\psi]: W_F \to \overline{T}^{\vee} \hookrightarrow \overline{G}^{\vee},$$

which is just ϕ_χ^ψ in the notation of Section 4.3. Note that $\overline{T}^{\vee} = T_{Q,n}^{\vee}$ and $\overline{G}^{\vee} = H^{\vee}$. The parameter $\phi[\chi_\psi]$ is thus associated to an unramified character

$$\underline{\chi}: T_{Q,n} \to \mathbf{C}^{\times}$$

such that

$$\phi_{\underline{\chi}} = \phi[\chi_\psi],$$

where $\phi_{\underline{\chi}}$ is the L-parameter associated to $\underline{\chi}$ by the local Langlands correspondence for linear tori. We can explicate $\underline{\chi}$ as follows. First, $\chi \cdot \chi_\psi^{-1}: T_{Q,n}^{\dagger} \to \mathbf{C}^{\times}$ is a linear character, where $T_{Q,n}^{\dagger}$ is the image of the isogeny

$$i_{Q,n}: T_{Q,n} \to T$$

(see Section 4.2). Then $\underline{\chi}$ is just the pullback of $\chi \cdot \chi_\psi^{-1}$ via $i_{Q,n}$; that is, $\underline{\chi} = \left(\chi \cdot \chi_\psi^{-1}\right) \circ i_{Q,n}$. In any case, we have

$$\mathcal{S}_{\phi_{\underline{\chi}}} = \mathcal{S}_{\phi[\chi_\psi]}.$$

Now we consider $R_{\underline{\chi}}$ and R_χ. Since χ_ψ is Weyl-invariant, we have

$$W_\chi = W_{\chi \cdot \chi_\psi^{-1}} = W_{\underline{\chi}}. \tag{4.9}$$

By the construction of a distinguished genuine character, we have

$$\chi_\psi\left(\overline{h}_\alpha(a^{n_\alpha})\right) = 1$$

for all $\alpha \in \Delta$ (see [14, §6.1]). However, from the proof of Proposition 4.5 we have $w \cdot \overline{h}_\alpha(\varpi^{n_\alpha}) \cdot w^{-1} = \overline{h}_{\mathsf{w}(\alpha)}(\varpi^{n_{\mathsf{w}(\alpha)}})$ for all $\mathsf{w} \in W$ and $\alpha \in \Delta$. It follows that $\chi_\psi\left(\overline{h}_\alpha(a^{n_\alpha})\right) = 1$ for every $\alpha \in \Phi$. Thus,

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$$\begin{aligned}\Phi_{\underline{\chi}} &= \left\{\alpha_{Q,n} > 0 : \underline{\chi}\left(a^{\alpha^\vee_{Q,n}}\right) = 1\right\} \\ &= \left\{\alpha > 0 : \left(\chi \cdot \chi_\psi^{-1}\right)\left(\overline{h}_\alpha(a^{n_\alpha})\right) = 1\right\} \\ &= \left\{\alpha > 0 : \chi\left(\overline{h}_\alpha(a^{n_\alpha})\right) = 1\right\} \\ &= \left\{\alpha > 0 : \underline{\chi}_\alpha = \mathbb{1}\right\} \\ &= \Phi_\chi.\end{aligned} \tag{4.10}$$

We deduce from equations (4.9) and (4.10) that

$$R_{\underline{\chi}} \simeq R_\chi.$$

It is proved in [27, Proposition 2.6] that we have the isomorphism $R_{\underline{\chi}} \simeq \mathcal{S}_{\phi_{\underline{\chi}}}$. From this we see that $R_\chi \simeq \mathcal{S}_{\phi[\chi_\psi]}$, which is also isomorphic to $\mathcal{S}_\phi$ by Lemma 4.8. This completes the proof. □

In view of the isomorphism $R_\chi \simeq \mathcal{S}_{\phi_\chi}$, the character relation in Corollary 4.7 can be interpreted in terms of $\mathrm{Irr}\left(\mathcal{S}_{\phi_\chi}\right)$ as well.

Corollary 4.10. *If the dual group $\overline{G}^\vee$ is semisimple simply connected, then $R_\chi = \{1\}$ for every unitary unramified genuine character χ of $Z\left(\overline{T}\right)$.*

Proof. This follows from the isomorphism $R_\chi \simeq \mathcal{S}_{\phi_\chi}$ and the well-known result of Steinberg that the centraliser of a semisimple element inside a simply connected group (such as $\overline{G}^\vee$ by our assumption) is connected.

Alternatively (and equivalently), from the proof of Theorem 4.9 we have $R_\chi = R_{\underline{\chi}}$, where $I\left(\underline{\chi}\right)$ is an unramified principal series of H. If $\overline{G}^\vee$ is simply connected, then H is of adjoint type, and it can be argued directly (see [32, Corollary 1.6] or [8]) that $R_{\underline{\chi}} = \{1\}$ in this case. □

Example 4.11. Let $\mathbf{G} = \mathrm{Sp}_{2r}$ and let Q be a Weyl-invariant quadratic form on its cocharacter lattice. If $n_\alpha = 0 \pmod 2$ for the unique short simple coroot $\alpha^\vee$ of Sp_{2r}, then

$$\overline{G}^\vee = \mathrm{Sp}_{2r};$$

in this case $R_\chi = \{1\}$ and $I(\chi)$ is always irreducible. This applies in particular to the case of the metaplectic-type group $\overline{\mathrm{Sp}}_{2r}^{(n)}$ – that is, when $n_\alpha = 2 \mod 4$ (see Definition 2.1). For $\overline{\mathrm{SL}}_2^{(n)}$, this is compatible with the discussion in Section 4.5.

5. Whittaker space and the main conjecture

The main goal of this section is to investigate $\dim \mathrm{Wh}_\psi(\pi)$, where $\pi \in \Pi(\chi)$ is any irreducible constituent of $I(\chi)$.

5.1. The Whittaker space

Let $\psi : F \to \mathbf{C}^\times$ be an additive character of conductor O. Let

$$\psi_U : U \to \mathbf{C}^\times$$

be the character of U such that its restriction to every $U_\alpha, \alpha \in \Delta$ is given by $\psi \circ e_\alpha^{-1}$. We may write ψ instead of ψ_U for simplicity.

Definition 5.1. For a genuine representation (π, V_π) of $\overline{G}$, a linear functional $\ell : V_\pi \to \mathbf{C}$ is called a ψ-Whittaker functional if $\ell(\pi(u)v) = \psi(u) \cdot v$ for all $u \in U$ and $v \in V_\pi$. Write $\mathrm{Wh}_\psi(\pi)$ for the space of ψ-Whittaker functionals for π.

The space $\mathrm{Wh}_\psi(I(\chi))$ for an unramified principal series $I(\chi)$ could be described as follows:

- First, let $\mathrm{Ftn}(i(\chi))$ be the vector space of functions $\mathbf{c}$ on $\overline{T}$ satisfying

$$\mathbf{c}\left(\bar{t} \cdot \bar{z}\right) = \mathbf{c}\left(\bar{t}\right) \cdot \chi\left(\bar{z}\right) \text{ for } \bar{t} \in \overline{T} \text{ and } \bar{z} \in \overline{A}.$$

  The support of any $\mathbf{c} \in \mathrm{Ftn}(i(\chi))$ is a disjoint union of cosets in $\overline{T}/\overline{A}$. We have

$$\dim \mathrm{Ftn}(i(\chi)) = |\mathscr{X}_{Q,n}|,$$

  since $\overline{T}/\overline{A} \simeq Y/Y_{Q,n} = \mathscr{X}_{Q,n}$.
- Second, for every $\gamma \in \overline{T}$, let $\mathbf{c}_\gamma \in \mathrm{Ftn}(i(\chi))$ be the unique element satisfying

$$\mathrm{supp}(\mathbf{c}_\gamma) = \gamma \cdot \overline{A} \text{ and } \mathbf{c}_\gamma(\gamma) = 1.$$

  Clearly, $\mathbf{c}_{\gamma \cdot a} = \chi(a)^{-1} \cdot \mathbf{c}_\gamma$ for every $a \in \overline{A}$. If $\{\gamma_i\} \subset \overline{T}$ is a chosen set of representatives of $\overline{T}/\overline{A}$, then $\{\mathbf{c}_{\gamma_i}\}$ forms a basis for $\mathrm{Ftn}(i(\chi))$. Let $i(\chi)^\vee$ be the vector space of functionals of $i(\chi)$. The set $\{\gamma_i\}$ gives rise to linear functionals $l_{\gamma_i} \in i(\chi)^\vee$ such that $l_{\gamma_i}\left(\phi_{\gamma_j}\right) = \delta_{ij}$, where $\phi_{\gamma_j} \in i(\chi)$ is the unique element such that

$$\mathrm{supp}\left(\phi_{\gamma_j}\right) = \overline{A} \cdot \gamma_j^{-1} \text{ and } \phi_{\gamma_j}\left(\gamma_j^{-1}\right) = 1.$$

  It is easy to see that for any $\gamma \in \overline{T}$ and $a \in \overline{A}$, we have

$$\phi_{\gamma a} = \chi(a) \cdot \phi_\gamma, \qquad l_{\gamma a} = \chi(a)^{-1} \cdot l_\gamma.$$

  Moreover, there is a natural isomorphism of vector spaces

$$\mathrm{Ftn}(i(\chi)) \simeq i(\chi)^\vee$$

  given by

$$\mathbf{c} \mapsto l_{\mathbf{c}} := \sum_{\gamma_i \in \overline{T}/\overline{A}} \mathbf{c}(\gamma_i) \cdot l_{\gamma_i}.$$

  It can be checked easily that this isomorphism does not depend on the choice of representatives for $\overline{T}/\overline{A}$.

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- Third, there is an isomorphism between $i(\chi)^\vee$ and the space $\mathrm{Wh}_\psi(I(\chi))$ of ψ-Whittaker functionals of $I(\chi)$ (see [37, §6]), given by $\lambda \mapsto W_\lambda$ with
$$W_\lambda : I(\chi) \to \mathbf{C}, \qquad f \mapsto \lambda\left(\int_U f\left(\overline{w}_G^{-1}u\right)\psi(u)^{-1}\mu(u)\right),$$
where $f \in I(\chi)$ is an $i(\chi)$-valued function on $\overline{G}$. Here $\overline{w}_G = \overline{w}_{\alpha_1}\overline{w}_{\alpha_2}\cdots\overline{w}_{\alpha_k} \in K$ is the representative of $\mathbb{w}_G$, where $\mathbb{w}_G = \mathbb{w}_{\alpha_1}\mathbb{w}_{\alpha_2}\cdots\mathbb{w}_{\alpha_k}$ is a minimum decomposition of $\mathbb{w}_G$.

Thus, we have a chain of natural isomorphisms of vector spaces all of dimension $|\mathscr{X}_{Q,n}|$:
$$\mathrm{Ftn}(i(\chi)) \simeq i(\chi)^\vee \simeq \mathrm{Wh}_\psi(I(\chi)).$$
For every $\mathbf{c} \in \mathrm{Ftn}(i(\chi))$, by abuse of notation we will write $\lambda_{\mathbf{c}}^\chi \in \mathrm{Wh}_\psi(I(\chi))$ for the resulting ψ-Whittaker functional of $I(\chi)$ obtained from this isomorphism.

The operator $A(w,\chi) : I(\chi) \to I({}^w\chi)$ induces a homomorphism of vector spaces
$$A(w,\chi)^* : \mathrm{Wh}_\psi(I({}^w\chi)) \to \mathrm{Wh}_\psi(I(\chi))$$
given by
$$\left\langle \lambda_{\mathbf{c}}^{{}^w\chi}, - \right\rangle \mapsto \left\langle \lambda_{\mathbf{c}}^{{}^w\chi}, A(w,\chi)(-) \right\rangle,$$
where $\mathbf{c} \in \mathrm{Ftn}(i({}^w\chi))$.

5.2. The scattering matrix $\mathcal{S}_{\mathfrak{R}}(w,i(\chi))$

Let $\mathfrak{R},\mathfrak{R}' \subset \overline{T}$ be two ordered sets of representatives of $\overline{T}/\overline{A} = \mathscr{X}_{Q,n}$. Let
$$\left\{\lambda_\gamma^{{}^w\chi} : \gamma \in \mathfrak{R}\right\}$$
be the ordered basis for $\mathrm{Wh}_\psi(I({}^w\chi))$ and
$$\left\{\lambda_{\gamma'}^{\chi} : \gamma' \in \mathfrak{R}'\right\}$$
be the ordered basis for $\mathrm{Wh}_\psi(I(\chi))$. The map $A(w,\chi)^*$ is then determined by the so-called scattering matrix
$$\mathcal{S}_{\mathfrak{R},\mathfrak{R}'}(w,i(\chi)) = \left[\tau\left(w,\chi,\gamma,\gamma'\right)\right]_{\gamma\in\mathfrak{R},\gamma'\in\mathfrak{R}'}$$
such that
$$A(w,\chi)^*\left(\lambda_\gamma^{{}^w\chi}\right) = \sum_{\gamma'\in\mathfrak{R}'} \tau(w,\chi,\gamma,\gamma')\cdot\lambda_{\gamma'}^{\chi}. \tag{5.1}$$

We briefly describe the matrix $\mathcal{S}_{\mathfrak{R},\mathfrak{R}'}(w,i(\chi))$. First we have the following:

- For $\mathbb{w} \in W$ and $\overline{z},\overline{z}' \in \overline{A}$, the identity
$$\tau\left(w,\chi,\gamma\cdot\overline{z},\gamma'\cdot\overline{z}'\right) = \left({}^w\chi\right)^{-1}(\overline{z})\cdot\tau(w,\chi,\gamma,\gamma')\cdot\chi\left(\overline{z}'\right) \tag{5.2}$$
holds.

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- For $w_1, w_2 \in W$ such that $l(w_2 w_1) = l(w_2) + l(w_1)$, we have
$$\tau(w_2 w_1, \chi, \gamma, \gamma') = \sum_{\gamma'' \in \overline{T}/\overline{A}} \tau(w_2, {}^{w_1}\chi, \gamma, \gamma'') \cdot \tau(w_1, \chi, \gamma'', \gamma'), \tag{5.3}$$
which is referred to as the cocycle relation. By equation (5.2), this sum is independent of the choice of representatives γ''.

This cocycle relation implies that in principle it suffices to understand $\tau(w_\alpha, \chi, \gamma, \gamma')$ for $\gamma, \gamma' \in \overline{T}$ and $\alpha \in \Delta$. For this purpose, let du be the self-dual Haar measure of F with respect to ψ such that $du(O) = 1$; thus, $du(O^\times) = 1 - q^{-1}$. Consider the Gauss sum
$$G_\psi(a,b) := \int_{O^\times} (u, \varpi)_n^a \cdot \psi\left(\varpi^b u\right) du \text{ for } a, b \in \mathbf{Z}.$$
We also write
$$\mathbf{g}_\psi(k) := G_\psi(k, -1),$$
where $k \in \mathbf{Z}$ is any integer. Denote henceforth
$$\varepsilon := (-1, \varpi)_n \in \mathbf{C}^\times.$$
It is known that
$$\mathbf{g}_\psi(k) = \begin{cases} \varepsilon^k \cdot \overline{\mathbf{g}_\psi(-k)} & \text{for every } k \in \mathbf{Z}, \\ -q^{-1} & \text{if } n | k, \\ \mathbf{g}_\psi(k) & \text{with } |\mathbf{g}_\psi(k)| = q^{-1/2} \text{ if } n \nmid k. \end{cases} \tag{5.4}$$
Here $\overline{z}$ denotes the complex conjugation of a complex number z.

It is shown in [24, 37] (with some refinement from [15]) that $\tau(w_\alpha, \chi, \gamma, \gamma')$ is determined as follows:

Theorem 5.2. *Suppose that $\gamma = \mathbf{s}_{y_1}$ and $\gamma' = \mathbf{s}_y$, with $y_1, y \in Y$. First, we can write $\tau(w_\alpha, \chi, \gamma, \gamma') = \tau^1(w_\alpha, \chi, \gamma, \gamma') + \tau^2(w_\alpha, \chi, \gamma, \gamma')$ satisfying the following properties:*

- $\tau^i(w_\alpha, \chi, \gamma \cdot \overline{z}, \gamma' \cdot \overline{z}') = ({}^{w_\alpha}\chi)^{-1}(\overline{z}) \cdot \tau^i(w_\alpha, \chi, \gamma, \gamma') \cdot \chi(\overline{z}')$ *for all* $\overline{z}, \overline{z}' \in \overline{A}$;
- $\tau^1(w_\alpha, \chi, \gamma, \gamma') = 0$ *unless* $y_1 \equiv y \mod Y_{Q,n}$;
- $\tau^2(w_\alpha, \chi, \gamma, \gamma') = 0$ *unless* $y_1 \equiv w_\alpha[y] \mod Y_{Q,n}$.

Second:

- *if* $y_1 = y$, *then*
$$\tau^1(w_\alpha, \chi, \gamma, \gamma') = \left(1 - q^{-1}\right) \frac{\chi\left(\overline{h}_\alpha(\varpi^{n_\alpha})\right)^{k_{y,\alpha}}}{1 - \chi(\overline{h}_\alpha(\varpi^{n_\alpha}))}, \text{ where } k_{y,\alpha} = \left\lceil \frac{\langle y, \alpha \rangle}{n_\alpha} \right\rceil;$$
- *if* $y_1 = w_\alpha[y]$, *then*
$$\tau^2(w_\alpha, \chi, \gamma, \gamma') = \varepsilon^{\langle y_\rho, \alpha \rangle \cdot D(y, \alpha^\vee)} \cdot \mathbf{g}_{\psi^{-1}}(\langle y_\rho, \alpha \rangle Q(\alpha^\vee)).$$

5.3. The main conjecture

If $\mathfrak{R} = \mathfrak{R}'$, then

$$\mathcal{S}_{\mathfrak{R}}(w, i(\chi)) := \mathcal{S}_{\mathfrak{R},\mathfrak{R}}(w, i(\chi))$$

is a square matrix with entries indexed by a single ordered set $\mathfrak{R}$. Since W acts on $\mathscr{X}_{Q,n} = Y/Y_{Q,n}$ with respect to $\mathbb{w}[-]$, there is a decomposition

$$\mathscr{X}_{Q,n} = \bigsqcup_{\mathcal{O}_y \in \mathcal{O}_{\mathscr{X}}} \mathcal{O}_y$$

into W-orbits. This gives a natural partition

$$\mathfrak{R} = \bigsqcup_{\mathcal{O}_y \in \mathcal{O}_{\mathscr{X}}} \mathfrak{R}_y,$$

where $\mathfrak{R}_y \subset \mathfrak{R}$ is the subset of representatives of $\mathcal{O}_y$. For each W-orbit $\mathcal{O}_y$, denote

$$\mathcal{S}_{\mathfrak{R}}(w, i(\chi))_{\mathcal{O}_y} := [\tau(w, \chi, \mathbf{s}_z, \mathbf{s}_{z'})]_{z,z' \in \mathfrak{R}_y}.$$

It follows from the cocycle relation (5.3) and Theorem 5.2 that $\mathcal{S}_{\mathfrak{R}}(w, i(\chi))$ is a block-diagonal matrix with blocks $\mathcal{S}_{\mathfrak{R}}(w, i(\chi))_{\mathcal{O}_y}$ for $\mathcal{O}_y \in \mathcal{O}_{\mathscr{X}}$, which we write as

$$\mathcal{S}_{\mathfrak{R}}(w, i(\chi)) = \bigoplus_{\mathcal{O}_y \in \mathcal{O}_{\mathscr{X}}} \mathcal{S}_{\mathfrak{R}}(w, i(\chi))_{\mathcal{O}_y}. \tag{5.5}$$

In fact, for $y \in \mathscr{X}_{Q,n}$, let

$$\mathrm{Wh}_\psi(I(\chi))_{\mathcal{O}_y} = \mathrm{Span}\left\{\lambda^{\chi}_{\mathbf{s}_z} : z \in \mathfrak{R}_y\right\} \subset \mathrm{Wh}_\psi(I(\chi)) \tag{5.6}$$

be the '$\mathcal{O}_y$-subspace' of the Whittaker space of $I(\chi)$. It is well defined and independent of the representatives for $y \in \mathscr{X}_{Q,n}$, and moreover,

$$\dim \mathrm{Wh}_\psi(I(\chi))_{\mathcal{O}_y} = |\mathcal{O}_y|.$$

We have a decomposition

$$\mathrm{Wh}_\psi(I(\chi)) = \bigoplus_{\mathcal{O}_y \in \mathcal{O}_{\mathscr{X}}} \mathrm{Wh}_\psi(I(\chi))_{\mathcal{O}_y}.$$

For every $\sigma \in \mathrm{Irr}(R_\chi)$, the inclusion $h_\sigma : \pi_\sigma \hookrightarrow I(\chi)$ induces a surjection of vector spaces

$$h_\sigma^* : \mathrm{Wh}_\psi(I(\chi)) \twoheadrightarrow \mathrm{Wh}_\psi(\pi_\sigma).$$

Denote

$$\mathrm{Wh}_\psi(\pi_\sigma)_{\mathcal{O}_y} := h_\sigma^*\left(\mathrm{Wh}_\psi(I(\chi))_{\mathcal{O}_y}\right). \tag{5.7}$$

Consider the natural permutation representation

$$\sigma^{\mathscr{X}} : W \longrightarrow \mathrm{Perm}(\mathscr{X}_{Q,n})$$

of W given by $\sigma_{\mathscr{X}}(\mathbb{w})(y) = \mathbb{w}[y]$. Clearly, there is a decomposition

$$\sigma^{\mathscr{X}} = \bigoplus_{\mathcal{O}_y \in \mathcal{O}_{\mathscr{X}}} \sigma^{\mathscr{X}}_{\mathcal{O}_y},$$

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where

$$\sigma_{\mathcal{O}_y}^{\mathscr{X}} : W \longrightarrow \mathrm{Perm}(\mathcal{O}_y)$$

is the permutation representation on the W-orbit $\mathcal{O}_y$. As we always identify R_χ as a subgroup of $W_\chi \subset W$, we can thus view $\sigma_{\mathcal{O}_y}^{\mathscr{X}}$ as a representation of R_χ by restriction.

Conjecture 5.3. *Let $\overline{G}$ be a saturated n-fold cover of a semisimple simply connected group G with $\overline{G}^\vee \simeq G^\vee$. Let χ be a unitary unramified genuine character of $Z(\overline{T})$. Then for the natural correspondence (as discussed in Section 4.1)*

$$\mathrm{Irr}(R_\chi) \longrightarrow \Pi(\chi), \qquad \sigma \mapsto \pi_\sigma,$$

we have

$$\dim \mathrm{Wh}_\psi(\pi_\sigma)_{\mathcal{O}_y} = \left\langle \sigma, \sigma_{\mathcal{O}_y}^{\mathscr{X}} \right\rangle_{R_\chi}$$

for every W-orbit $\mathcal{O}_y$, where $\langle \sigma_1, \sigma_2 \rangle_{R_\chi}$ is the pairing of the two representations σ_1, σ_2 of R_χ. In particular, $\dim \mathrm{Wh}_\psi(\pi_\sigma) = \langle \sigma, \sigma^{\mathscr{X}} \rangle_{R_\chi}$ *for every $\sigma \in \mathrm{Irr}(R_\chi)$.*

Note that the conjecture is trivially true for arbitrary $\overline{G}$ if $R_\chi = \{1\}$. This should apply in particular to all Brylinski–Deligne covers of GL_r, for which R_χ is expected to be trivial. We also recall that if G is simply connected, then by Definition 2.1 $\overline{G}$ is saturated if and only if the dual group $\overline{G}^\vee$ is of adjoint type – that is, $Y_{Q,n} = Y_{Q,n}^{sc}$ in this case. In fact, it follows from $\overline{G}^\vee \simeq G^\vee$ that $\overline{G}$ is necessarily saturated.

Remark 5.4. Conjecture 5.3 is compatible with the decomposition

$$I(\chi) = \bigoplus_{\sigma \in \mathrm{Irr}(R_\chi)} \pi_\sigma.$$

Indeed, we have

$$\sum_{\sigma \in \mathrm{Irr}(R_\chi)} \left\langle \sigma, \sigma_{\mathcal{O}_y}^{\mathscr{X}} \right\rangle_{R_\chi} = \left\langle \mathbf{C}[R_\chi], \sigma_{\mathcal{O}_y}^{\mathscr{X}} \right\rangle_{R_\chi} = |\mathcal{O}_y|,$$

which is equal to $\dim \mathrm{Wh}_\psi(I(\chi))_{\mathcal{O}_y}$.

Remark 5.5. Ginzburg proposed in [21, Conjecture 1] that if π is an irreducible unramified representation of $\overline{G}$ which is nongeneric, then there exists a nongeneric theta representation Θ of a Levi subgroup $\overline{M} \subset \overline{P} \subset \overline{G}$ such that $\pi \hookrightarrow \mathrm{Ind}_{\overline{P}}^{\overline{G}}(\Theta)$. On the other hand, Conjecture 5.3 implies that for simply connected G, we have $\dim \mathrm{Wh}_\psi\left(\pi_\chi^{un}\right) = \left|\mathcal{O}_{\mathscr{X}}^{R_\chi}\right|$ (the number of R_χ-orbits in $\mathscr{X}_{Q,n}$), since $\pi_\chi^{un} = \pi_{\mathbb{1}}$. That is, Ginzburg's conjecture is vacuously true for such an unramified representation π_χ^{un}. For a comparison with the case of regular unramified χ, see [18, Remark 7.4].

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5.4. A formula for $\dim \mathrm{Wh}_\psi(\pi_\sigma)$

In this subsection, let G be a general connected reductive group and let $\overline{G}$ be an n-fold cover, unless specified otherwise. Consider the decomposition

$$I(\chi) = \bigoplus_{\sigma \in \mathrm{Irr}(R_\chi)} \pi_\sigma.$$

Set $\mathbb{w} \in W_\chi$. We have ${}^w\chi \simeq \chi$, and thus an endomorphism

$$\mathscr{A}(w,\chi)^* : \mathrm{Wh}_\psi(I(\chi)) \to \mathrm{Wh}_\psi(I(\chi))$$

induced from $\mathscr{A}(w,\chi) = \gamma(w,\chi)\cdot A(w,\chi)$ (see Lemma 4.1). In fact, if $\mathbb{w} \in R_\chi$, then the normalising factor is nonzero, and thus $A(w,\chi)$ is already holomorphic for every $\mathbb{w} \in R_\chi$. In any case, it follows from Corollary 4.7 that for $\mathbb{w} \in R_\chi$, we have

$$\mathscr{A}(w,\chi)^* = \bigoplus_{\sigma \in \mathrm{Irr}(R_\chi)} \sigma(\mathbb{w}) \cdot \mathrm{id}_{\mathrm{Wh}_\psi(\pi_\sigma)},$$

and therefore the characteristic polynomial of $\mathscr{A}(w,\chi)^*$ is

$$\begin{aligned} \det(X\cdot \mathrm{id} - \mathscr{A}(w,\chi)^*) &= \det\left(X \cdot I_{|\mathscr{X}_{Q,n}|} - \mathscr{A}(w,\chi)^*\right) \\ &= \prod_{\sigma \in \mathrm{Irr}(R_\chi)} (X - \sigma(\mathbb{w}))^{\dim \mathrm{Wh}_\psi(\pi_\sigma)}. \end{aligned}$$

For every $\mathbb{w} \in W$ and $\mathcal{O}_y \in \mathcal{O}_{\mathscr{X}}$, the restriction of $\mathscr{A}(w,\chi)^* : \mathrm{Wh}_\psi(I({}^w\chi)) \to \mathrm{Wh}_\psi(I(\chi))$ to $\mathrm{Wh}_\psi(I({}^w\chi))_{\mathcal{O}_y}$ gives a well-defined homomorphism

$$\mathscr{A}(w,\chi)^*_{\mathcal{O}_y} : \mathrm{Wh}_\psi(I({}^w\chi))_{\mathcal{O}_y} \to \mathrm{Wh}_\psi(I(\chi))_{\mathcal{O}_y}$$

represented by $\mathcal{S}_{\mathfrak{R}}(w,i(\chi))_{\mathcal{O}_y}$ (see [18, Proposition 4.4]). Moreover,

$$\mathscr{A}(w,\chi)^* = \bigoplus_{\mathcal{O}_y \in \mathcal{O}_{\mathscr{X}}} \mathscr{A}(w,\chi)^*_{\mathcal{O}_y}, \tag{5.8}$$

where the sum is taken over all W-orbits in $\mathscr{X}_{Q,n}$. For $\mathbb{w} \in R_\chi$, this gives

$$\mathscr{A}(w,\chi)^*_{\mathcal{O}_y} = \bigoplus_{\sigma \in \mathrm{Irr}(R_\chi)} \sigma(\mathbb{w}) \cdot \mathrm{id}_{\mathrm{Wh}_\psi(\pi_\sigma)_{\mathcal{O}_y}}. \tag{5.9}$$

For every W-orbit $\mathcal{O}_y \subset \mathscr{X}_{Q,n}$, consider the map

$$\sigma^{\mathrm{Wh}}_{\mathcal{O}_y} : R_\chi \longrightarrow \mathrm{GL}\left(\mathrm{Wh}_\psi(I(\chi))_{\mathcal{O}_y}\right)$$

given by

$$\sigma^{\mathrm{Wh}}_{\mathcal{O}_y}(\mathbb{w}) := \mathscr{A}(w,\chi)^*_{\mathcal{O}_y}.$$

Also let

$$\sigma^{\mathrm{Wh}} : R_\chi \longrightarrow \mathrm{GL}(\mathrm{Wh}_\psi(I(\chi))) \tag{5.10}$$

be the map given by

$$\sigma^{\mathrm{Wh}}(\mathbb{w}) := \mathscr{A}(w,\chi)^*.$$

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It is then clear from equation (5.8) that

$$\sigma^{\mathrm{Wh}} = \bigoplus_{\mathcal{O}_y \in \mathcal{O}_{\mathscr{X}}} \sigma^{\mathrm{Wh}}_{\mathcal{O}_y}.$$

Theorem 5.6. *Let $\overline{G}$ be an n-fold cover of a connected reductive group G. Then for every $\mathcal{O}_y \in \mathcal{O}_{\mathscr{X}}$, the map $\sigma^{\mathrm{Wh}}_{\mathcal{O}_y}$ is a well-defined group homomorphism and*

$$\dim \mathrm{Wh}_\psi(\pi_\sigma)_{\mathcal{O}_y} = \left\langle \sigma, \sigma^{\mathrm{Wh}}_{\mathcal{O}_y} \right\rangle_{R_\chi}$$

for every $\sigma \in \mathrm{Irr}(R_\chi)$. Hence, $\dim \mathrm{Wh}_\psi(\pi_\sigma) = \langle \sigma, \sigma^{\mathrm{Wh}} \rangle_{R_\chi}$.

Proof. For $\mathbb{w}, \mathbb{w}' \in R_\chi$, we have

$$\begin{aligned}
\sigma^{\mathrm{Wh}}_{\mathcal{O}_y}(\mathbb{w}) \circ \sigma^{\mathrm{Wh}}_{\mathcal{O}_y}(\mathbb{w}') &= \mathscr{A}(w,\chi)^*_{\mathcal{O}_y} \circ \mathscr{A}(w',\chi)^*_{\mathcal{O}_y} \\
&= \mathscr{A}(w,\chi)^*_{\mathcal{O}_y} \circ \mathscr{A}(w', {}^w\chi)^*_{\mathcal{O}_y} \\
&= (\mathscr{A}(w', {}^w\chi) \circ \mathscr{A}(w,\chi))^*_{\mathcal{O}_y} \\
&= \mathscr{A}(w'w,\chi)^*_{\mathcal{O}_y} \\
&= \sigma^{\mathrm{Wh}}_{\mathcal{O}_y}(\mathbb{w}'\mathbb{w}) = \sigma^{\mathrm{Wh}}_{\mathcal{O}_y}(\mathbb{w}\mathbb{w}'),
\end{aligned}$$

where the last equality follows from the fact that R_χ is abelian (see Theorem 4.6). This shows that $\sigma^{\mathrm{Wh}}_{\mathcal{O}_y}$ is a representation of R_χ on $\mathrm{Wh}_\psi(I(\chi))_{\mathcal{O}_y}$. Clearly, equation (5.9) gives

$$\sigma^{\mathrm{Wh}}_{\mathcal{O}_y} = \bigoplus_{\sigma \in \mathrm{Irr}(R_\chi)} \dim \mathrm{Wh}_\psi(\pi_\sigma)_{\mathcal{O}_y} \cdot \sigma,$$

and thus it follows that

$$\dim \mathrm{Wh}_\psi(\pi_\sigma)_{\mathcal{O}_y} = \left\langle \sigma, \sigma^{\mathrm{Wh}}_{\mathcal{O}_y} \right\rangle_{R_\chi}.$$

This completes the proof. □

If $\mathbb{w} \in R_\chi$, then $c_{\mathsf{gk}}(w,\chi)^{-1} = \gamma(w,\chi)$ – see Lemma 4.1 – is actually holomorphic and nonzero at χ. Denote by $\theta_{\sigma^{\mathrm{Wh}}_{\mathcal{O}_y}}$ the character of $\sigma^{\mathrm{Wh}}_{\mathcal{O}_y}$. For $\mathbb{w} \in R_\chi$, we have

$$\theta_{\sigma^{\mathrm{Wh}}_{\mathcal{O}_y}}(\mathbb{w}) = \gamma(w,\chi) \cdot \mathrm{Tr}\left(A(w,\chi)^*_{\mathcal{O}_y}\right)$$

and thus

$$\dim \mathrm{Wh}_\psi(\pi_\sigma)_{\mathcal{O}_y} = \frac{\gamma(w,\chi)}{|R_\chi|} \cdot \sum_{\mathbb{w} \in R_\chi} \overline{\sigma(\mathbb{w})} \cdot \mathrm{Tr}\left(A(w,\chi)^*_{\mathcal{O}_y}\right).$$

Recall the permutation representation

$$\sigma^{\mathscr{X}}_{\mathcal{O}_y} : W \longrightarrow \mathrm{Perm}(\mathcal{O}_y)$$

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associated to the W-orbit $\mathcal{O}_y$. Let $\theta_{\sigma^{\mathscr{X}}_{\mathcal{O}_y}}$ be the character of $\sigma^{\mathscr{X}}_{\mathcal{O}_y}$. Then for every $\mathbbm{w} \in W$, we have

$$\theta_{\sigma^{\mathscr{X}}_{\mathcal{O}_y}}(\mathbbm{w}) = |(\mathcal{O}_y)^{\mathbbm{w}}|,$$

where

$$(\mathcal{O}_y)^{\mathbbm{w}} = \{y \in \mathcal{O}_y : \mathbbm{w}[y] = y\}.$$

By restriction, we view both $\sigma^{\mathscr{X}}$ and $\sigma^{\mathscr{X}}_{\mathcal{O}_y}$ as representations of R_χ. Conjecture 5.3 can be reformulated as follows:

Conjecture 5.7. *Let $\overline{G}$ be a saturated n-fold cover of a semisimple simply connected group G with $\overline{G}^\vee \simeq G^\vee$. Let χ be a unitary unramified genuine character of $Z(\overline{T})$. Then for every W-orbit $\mathcal{O}_y$, we have*

$$\sigma^{\mathrm{Wh}}_{\mathcal{O}_y} \simeq \sigma^{\mathscr{X}}_{\mathcal{O}_y};$$

equivalently but more explicitly, for every $\mathbbm{w} \in R_\chi$,

$$\mathrm{Tr}\left(A(w,\chi)^*_{\mathcal{O}_y}\right) = |(\mathcal{O}_y)^{\mathbbm{w}}| \cdot \gamma(w,\chi)^{-1}.$$

The equivalence between Conjectures 5.3 and 5.7 follows from Theorem 5.6. The computation of $\mathrm{Tr}\left(A(w,\chi)^*_{\mathcal{O}_y}\right)$ is equivalent to that of $\mathrm{Tr}\left(\mathcal{S}_{\mathfrak{R}}(w,i(\chi))_{\mathcal{O}_y}\right)$ for any representative set $\mathfrak{R} \subset Y$ of $\mathscr{X}_{Q,n}$.

Remark 5.8. Conjecture 5.7 answers in a special case for the scattering matrix the analogous question raised in [19, §3.2] regarding the trace of a local coefficient matrix. We note that if ${}^w\chi = \chi$, then both the local coefficient matrix and the scattering matrix associated to the operator $A(w,\chi)^*$ give invariants of the operator, albeit different. In this paper, it is the latter that is used and plays a crucial role in determining $\dim \mathrm{Wh}_\psi(\pi_\sigma)$. We hope that this phenomenon also helps justify our viewpoint in [19] that both the local coefficient matrix and the scattering matrix are important objects and should be studied together.

5.5. Double cover of $\mathbf{GSp}_{2r}$

In this subsection, we apply Theorem 5.6 to the double cover of GSp_{2r} and show that it recovers [48, Corollary 6.6]. Meanwhile, we also show that the analogue of Conjecture 5.7 fails for such covers. This example shows that the conjecture cannot be extended in a naive way to covers of a reductive group whose derived subgroup is simply connected.

Let GSp_{2r} be the group of similitudes of symplectic type, and let $(X,\Delta,Y,\Delta^\vee)$ be its root data, given as follows. The character group $X \simeq \mathbf{Z}^{r+1}$ has a standard basis

$$\{e_i^* : 1 \le i \le r\} \cup \{e_0^*\},$$

where the simple roots are

$$\Delta = \left\{e_i^* - e_{i+1}^* : 1 \le i \le r-1\right\} \cup \{2e_r^* - e_0^*\}.$$

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The cocharacter group $Y \simeq \mathbf{Z}^{r+1}$ is given with a basis

$$\{e_i : 1 \le i \le r\} \cup \{e_0\}.$$

The simple coroots are

$$\Delta^\vee = \{e_i - e_{i+1} : 1 \le i \le r-1\} \cup \{e_r\}.$$

Write $\alpha_i = e_i^* - e_{i+1}^*$, $\alpha_i^\vee = e_i - e_{i+1}$ for $1 \le i \le r-1$, and also $\alpha_r = 2e_r^* - e_0^*$, $\alpha_r^\vee = e_r$. Consider the covering $\overline{\mathrm{GSp}}_{2r}$ incarnated by $(D, \mathbb{1})$. We are interested in those $\overline{\mathrm{GSp}}_{2r}$ whose restriction to Sp_{2r} is the one with $Q(\alpha_r^\vee) = 1$. That is, we assume

$$Q(\alpha_i^\vee) = 2 \text{ for } 1 \le i \le r-1, \text{ and } Q(\alpha_r^\vee) = 1.$$

Since $\Delta^\vee \cup \{e_0\}$ gives a basis for Y, to determine Q it suffices to specify $Q(e_0)$. For $n = 2$, we will obtain a double cover $\overline{\mathrm{GSp}}_{2r}$ which restricts to the classical metaplectic double cover $\overline{\mathrm{Sp}}_{2r}$. The number $Q(e_0) \in \mathbf{Z}/2\mathbf{Z}$ determines whether the similitude factor $F^\times$ corresponding to the cocharacter e_0 splits into $\overline{\mathrm{GSp}}_{2r}$ or not. To recover the classical double cover of GSp_{2r} (see [48]), we take $Q(e_0)$ to be an even number in this subsection.

In this case, we have

$$Y_{Q,2} = \left\{ \sum_{i=1}^{r} k_i \alpha_i^\vee + k e_0 \in Y : k_i \in \mathbf{Z} \text{ for } 1 \le i \le r-1, k_r, k \in 2\mathbf{Z} \right\}.$$

The sublattice $Y_{Q,2}^{sc}$ is spanned by $\{\alpha_{i,Q,2}^\vee\}_{1 \le i \le r}$ – that is,

$$\{\alpha_1^\vee, \alpha_2^\vee, \ldots, \alpha_{r-1}^\vee, 2\alpha_r^\vee\}.$$

Regarding the dual group of the double cover $\overline{\mathrm{GSp}}_{2r}$, we have

$$\overline{\mathrm{GSp}}_{2r}^\vee = \begin{cases} \mathrm{GSp}_{2r}(\mathbf{C}) & \text{if } r \text{ is odd,} \\ \mathrm{PGSp}_{2r}(\mathbf{C}) \times \mathrm{GL}_1(\mathbf{C}) & \text{if } r \text{ is even.} \end{cases}$$

Thus, $\mathbf{H}$ is GSp_{2r} (resp., $\mathrm{Spin}_{2r+1} \times \mathrm{GL}_1$) if r is odd (resp., even). Note that

$$\mathscr{X}_{Q,2} = Y/Y_{Q,2} = \{0, e_r, e_0, e_r + e_0\},$$

which is isomorphic to $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$.

If r is odd, then the torus $T_{Q,2}$ of $\mathbf{H}$ acts transitively on all nondegenerate characters of the unipotent subgroup of the Borel subgroup of $\mathbf{H}$. Thus, $R_{\underline{\chi}} = \{1\}$ for every unramified unitary character $\underline{\chi}$ (see [32, Lemma 2.5]). Therefore, the R-group R_χ for $\overline{\mathrm{GSp}}_{2r}$ with r odd is trivial for every unitary unramified genuine character χ.

We assume that r is even. For every $\alpha \in \Phi$, recall that we have the notation (see Section 3.2)

$$\chi_\alpha := \underline{\chi}_\alpha(\varpi) = \chi\left(\overline{h}_\alpha(\varpi^{n_\alpha})\right).$$

It follows from [26, Page 399] that the only nontrivial R_χ is $\{1, \mathbb{w}\} \simeq \mathbf{Z}/2\mathbf{Z}$, which is generated by

$$\mathbb{w} := \mathbb{w}_{\alpha_1} \mathbb{w}_{\alpha_3} \cdots \mathbb{w}_{\alpha_{r-1}},$$

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with the character χ satisfying

$$\chi_{\alpha_i} = -1 \text{ for all } i = 2k-1, 1 \le k \le r/2.$$

Proposition 5.9. *Assume r is even and χ is an unramified character satisfying the foregoing condition. Then as a representation of R_χ,*

$$\sigma^{\mathrm{Wh}} \simeq 2 \cdot \mathbb{1} \oplus 2 \cdot \varepsilon,$$

where ε denotes the nontrivial character of R_χ.

Proof. It suffices to compute the trace of σ^{Wh}. It is easy to see that for every $y \in \mathscr{X}_{Q,2}$ and $\mathsf{w}_{\alpha_i}, i = 1,3,\dots,r-1$, we have

$$\mathsf{w}_{\alpha_i}[y] = y \in \mathscr{X}_{Q,2}.$$

Thus, by [19, Proposition 4.12], the (y,y)-entry of $A(w,y)^*$ is given by

$$\tau(w,\chi,\mathbf{s}_y,\mathbf{s}_y) = \gamma(w,\chi)^{-1} \cdot \prod_{i=1,3,\dots,r-1} \chi_{\alpha_i}^{\langle y,\alpha_i\rangle} = \gamma(w,\chi)^{-1} \cdot (-1)^{\langle y,\alpha_i\rangle}.$$

We see

$$\gamma(w,\chi) \cdot \tau(w,\chi,\mathbf{s}_y,\mathbf{s}_y) = \begin{cases} 1 & \text{if } y = 0, \\ -1 & \text{if } y = e_r, \\ 1 & \text{if } y = e_0, \\ -1 & \text{if } y = e_r + e_0. \end{cases}$$

This shows that $\mathrm{Tr}\left(\sigma^{\mathrm{Wh}}\right)(w) = 0$. Thus, $\sigma^{\mathrm{Wh}} = 2 \cdot \mathbb{1} \oplus 2 \cdot \varepsilon$, as claimed. □

Theorem 5.10 ([48, Corollary 6.6])**.** *If r is odd, then every unitary unramified genuine principal series $I(\chi)$ for the double cover $\overline{GSp}_{2r}$ is irreducible. If r is even, then the only reducibility of $I(\chi)$ occurs when $R_\chi \simeq \mathbf{Z}/2\mathbf{Z}$; in this case, $I(\chi) = \pi_\chi^{un} \oplus \pi_\varepsilon$, and*

$$\dim \mathrm{Wh}_\psi\left(\pi_\chi^{un}\right) = \dim \mathrm{Wh}_\psi(\pi_\varepsilon) = 2.$$

Proof. We only need to show the last two equalities, which follow from combining Theorem 5.6 and Proposition 5.9. □

Remark 5.11. It follows from the proof of Proposition 5.9 that $\mathsf{w}[y] = y$ for every $y \in \mathscr{X}_{Q,2}$. Thus,

$$\sigma^{\mathscr{X}} = 4 \cdot \mathbb{1};$$

in particular, it is not isomorphic to σ^{Wh}. We see that the (naive) analogue of Conjecture 5.7 fails for such $\overline{\mathrm{GSp}}_{2r}$.

6. On the dimension of $\mathbf{Wh}_\psi\left(\pi_\chi^{un}\right)$

6.1. Lower bound for $\dim \mathbf{Wh}_\psi\left(\pi_\chi^{un}\right)$

In this subsection, we will prove Conjecture 5.7 for π_χ^{un} in a special case, which gives a lower bound of $\dim \mathrm{Wh}_\psi\left(\pi_\chi^{un}\right)$.

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Let $\overline{G}$ be an n-fold covering group of a connected reductive group G. Assume that $\overline{G}$ is not of metaplectic type (see Definition 2.1). We call $z \in Y$ an exceptional point (see [19, Definition 5.1]) if

$$\langle z_\rho, \alpha \rangle = -n_\alpha$$

for every $\alpha \in \Delta$ – that is,

$$\mathbb{w}_\alpha[z] = z + \alpha_{Q,n}^\vee \text{ for every } \alpha \in \Delta.$$

Note that the definition here is the same as [19, Definition 5.1], since we have assumed that G is not of metaplectic type.

For $\overline{G}$ not of metaplectic type, denote by $Y_n^{\text{exc}} \subset Y$ the set of exceptional points. Let

$$f : Y \twoheadrightarrow \mathscr{X}_{Q,n}$$

be the quotient map, and denote

$$\mathscr{X}_{Q,n}^{\text{exc}} := f\left(Y_n^{\text{exc}}\right).$$

If $y \in Y$ is exceptional, then $y \in f^{-1}\left(\mathscr{X}_{Q,n}^W\right)$; that is,

$$\mathscr{X}_{Q,n}^{\text{exc}} \subset \left(\mathscr{X}_{Q,n}\right)^W.$$

Denoting

$$\rho_{Q,n} := \frac{1}{2} \sum_{\alpha > 0} \alpha_{Q,n}^\vee \in Y \otimes \mathbf{Q},$$

we always have

$$(\{\rho - \rho_{Q,n}\} \cap Y) \subseteq Y_n^{\text{exc}}.$$

If G is a semisimple group and $\overline{G}$ is not of metaplectic type, then (see [19, Lemma 5.2])

$$Y_n^{\text{exc}} = \{\rho - \rho_{Q,n}\} \cap Y;$$

that is, Y_n^{exc} contains the unique element $\rho - \rho_{Q,n}$ if it lies in Y. The dependence of Y_n^{exc} on $\overline{G}$ for covers of simply connected groups is also determined explicitly in [19, §6–§7].

Theorem 6.1. *Let $\overline{G}$ be an n-fold cover of a connected reductive group G. Assume that $\overline{G}$ is not of metaplectic type. Then for every $z \in \mathscr{X}_{Q,n}^{\text{exc}}$ and $\mathbb{w} \in R_\chi$, we have $\mathcal{S}_{\mathfrak{R}}(w, i(\chi))_{\mathcal{O}_z} = \gamma(w,\chi)^{-1}$. Therefore,*

$$\sigma_{\mathcal{O}_z}^{\text{Wh}} = \sigma_{\mathcal{O}_z}^{\mathscr{X}} = \mathbb{1}_{R_\chi},$$

and thus

$$\dim \text{Wh}_\psi\left(\pi_\chi^{un}\right) \geq \left|\mathscr{X}_{Q,n}^{\text{exc}}\right|.$$

In particular, if $\rho - \rho_{Q,n} \in Y$, then π_χ^{un} is generic. Moreover, Conjecture 5.7 holds for such $\mathcal{O}_z$.

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Proof. Since $\mathcal{O}_z = \{z\}$, we have

$$\mathcal{S}_{\mathfrak{R}}(w,i(\chi))_{\mathcal{O}_z} = \tau(w,\chi,\mathbf{s}_z,\mathbf{s}_z).$$

First, we note that by equation (5.2) and the fact that χ is fixed by $\mathbb{w} \in R_\chi$, the entry $\tau(w,\chi,\mathbf{s}_z,\mathbf{s}_z)$ is independent of the representative for $z \in \mathscr{X}_{Q,n}^{\text{exc}}$. It follows from [19, Proposition 4.12] (as $\overline{G}$ is not of metaplectic type) that

$$\tau(w,\chi,\mathbf{s}_z,\mathbf{s}_z) = \gamma(w,\chi)^{-1}$$

for every $z \in \mathscr{X}_{Q,n}^{\text{exc}}$, and Conjecture 5.7 holds for such $\mathcal{O}_z$. In fact, $\sigma_{\mathcal{O}_z}^{\text{Wh}} = \sigma_{\mathcal{O}_z}^{\mathscr{X}} = \mathbb{1}_{R_\chi}$, and thus

$$\dim \text{Wh}_\psi \left(\pi_\chi^{un}\right)_{\mathcal{O}_z} = \left\langle \mathbb{1}, \sigma_{\mathcal{O}_z}^{\text{Wh}} \right\rangle_{R_\chi} = 1.$$

Therefore, $\dim \text{Wh}_\psi \left(\pi_\chi^{un}\right) \geq \left|\mathscr{X}_{Q,n}^{\text{exc}}\right|$. This completes the proof. □

For covers of a semisimple and simply connected group G, it follows from [19, Theorem 6.3] that

$$0 \leq \left|\mathscr{X}_{Q,n}^{\text{exc}}\right| \leq \left|(\mathscr{X}_{Q,n})^W\right| \leq 1.$$

In fact, in [19, §7] we determined explicitly the size of the two sets $\mathscr{X}_{Q,n}^{\text{exc}}$ and $(\mathscr{X}_{Q,n})^W$. On the other hand, if G is semisimple but not simply connected, then it is possible to have

$$\left|\mathscr{X}_{Q,n}^{\text{exc}}\right| \leq 1 < \left|(\mathscr{X}_{Q,n})^W\right|.$$

See [19] for details.

We note that the equality $\sigma_{\mathcal{O}_z}^{\text{Wh}} = \sigma_{\mathcal{O}_z}^{\mathscr{X}}$ in Theorem 6.1 might fail for $z \in (\mathscr{X}_{Q,n})^W - \mathscr{X}_{Q,n}^{\text{exc}}$ for covers of a semisimple group; we will consider such an example from n-fold covers of SO_3 in the next section. This example shows that the naive analogue of Conjecture 5.3 does not hold for general semisimple groups.

Remark 6.2. If G is almost simple and $\overline{G}$ is of metaplectic type, then it follows from the discussion after Definition 2.1 that $\overline{G} = \overline{\text{Sp}}_{2r}$ with $n_\alpha \equiv 2 \pmod 4$. Moreover, Example 4.11 shows that $R_\chi = \{1\}$ in this case, and thus the (in-)equalities $\dim \text{Wh}_\psi \left(\pi_\chi^{un}\right) = |\mathscr{X}_{Q,n}| \geq \left|\mathscr{X}_{Q,n}^{\text{exc}}\right|$ hold trivially.

6.2. Unramified Whittaker function

We note that Theorem 6.1 applied to the case $n = 1$ shows that π_χ^{un} is the only generic constituent of $I(\chi)$ for the linear algebraic group G. This fact also follows from the Casselman–Shalika formula [12]. Motivated by this, for covering groups we consider in this subsection the relation between $\dim \text{Wh}_\psi \left(\pi_\chi^{un}\right)$ and the unramified Whittaker function, which is also the approach taken in [21], but for a general unramified genuine character.

First, by restriction, we obtain a surjection of vector spaces

$$h^{un} : \text{Wh}_\psi(I(\chi)) \twoheadrightarrow \text{Wh}_\psi\left(\pi_\chi^{un}\right).$$

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Let $f^0 \in \pi_\chi^{un} \subset I(\chi)$ be the normalised unramified function. For $\mathbf{c} \in \mathrm{Ftn}(i(\chi))$, let $\lambda_{\mathbf{c}} \in \mathrm{Wh}_\psi(I(\chi))$ be the associated Whittaker functional, also viewed as an element in $\mathrm{Wh}_\psi\left(\pi_\chi^{un}\right)$; that is, we may still use $\lambda_{\mathbf{c}}$ for $h^{un}(\lambda_{\mathbf{c}})$ if no confusion arises. The unramified Whittaker function on $\overline{G}$ associated with $\mathbf{c}$ is given by

$$\mathcal{W}_{\mathbf{c}}(\overline{g}) = \lambda_{\mathbf{c}}\left(\pi_\chi^{un}(\overline{g}) f^0\right) = \lambda_{\mathbf{c}}\left(I(\chi)(\overline{g}) f^0\right).$$

We have a decomposition $\overline{G} = \overline{B}K = U\overline{T}K$ and

$$\mathcal{W}_{\mathbf{c}}\left(u\overline{t}k\right) = \psi(u)\cdot \mathcal{W}_{\mathbf{c}}\left(\overline{t}\right) \text{ for } u \in U, \overline{t} \in \overline{T}, k \in K.$$

Thus, the value of $\mathcal{W}_{\mathbf{c}}$ is determined by its restriction to $\overline{T}$. If $\mathbf{c} = \mathbf{c}_\gamma$ for $\gamma \in \overline{T}$, then we denote

$$\mathcal{W}_\gamma := \mathcal{W}_{\mathbf{c}_\gamma}.$$

Recall that $\overline{t} \in \overline{T}$ is called dominant if

$$\overline{t}\cdot (U \cap K)\cdot \overline{t}^{-1} \subset K.$$

Let

$$Y^+ = \{y \in Y : \langle y, \alpha\rangle \geq 0 \text{ for all } \alpha \in \Delta\}.$$

Then an element $\mathbf{s}_y \in \overline{T}$ is dominant if and only if $y \in Y^+$. The following result regarding $\mathcal{W}_\gamma\left(\overline{t}\right)$ is shown in [24, 38, 13] for coverings of GL_r. For a general covering group, the idea is the same; it is implicit in [37] and explicated in [17].

Proposition 6.3. *We have* $\mathcal{W}_\gamma\left(\overline{t}\right) = 0$ *unless* $\overline{t} \in \overline{T}$ *is dominant. Moreover, for dominant* $\overline{t}$*,*

$$\mathcal{W}_\gamma\left(\overline{t}\right) = \delta_B^{1/2}\left(\overline{t}\right)\cdot \sum_{\mathsf{w}\in W} c_{\mathsf{gk}}\left(w_G w^{-1}, \chi\right)\cdot \tau\left(w, {}^{w^{-1}}\chi, \gamma, w_G\cdot \overline{t}\cdot w_G^{-1}\right),$$

where δ_B *is the modular character of* B.

It follows that for $z \in Y$ and $y \in Y$,

$$\mathcal{W}_{\mathbf{s}_z}(\mathbf{s}_y) = \begin{cases} \delta_B^{1/2}(\mathbf{s}_y)\cdot \sum_{\mathsf{w}\in W} c_{\mathsf{gk}}\left(w_G w^{-1}, \chi\right)\cdot \tau\left(w, {}^{w^{-1}}\chi, \mathbf{s}_z, w_G\cdot \mathbf{s}_y\cdot w_G^{-1}\right) & \text{if } y \in Y^+, \\ 0 & \text{otherwise.} \end{cases}$$

For every $\gamma, \overline{t} \in \overline{T}$, we define

$$\mathcal{W}_\gamma^*\left(\overline{t}\right) := \delta_B^{1/2}\left(\overline{t}\right)^{-1}\cdot \sum_{\mathsf{w}\in W} c_{\mathsf{gk}}\left(w_G w^{-1}, \chi\right)\cdot \tau\left(w, {}^{w^{-1}}\chi, \gamma, \overline{t}\right).$$

We emphasise that here $\overline{t}$ is not required to be dominant. In particular,

$$\mathcal{W}_{\mathbf{s}_z}^*(\mathbf{s}_y) = \delta_B^{1/2}(\mathbf{s}_y)^{-1}\cdot \sum_{\mathsf{w}\in W} c_{\mathsf{gk}}\left(w_G w^{-1}, \chi\right)\cdot \tau\left(w, {}^{w^{-1}}\chi, \mathbf{s}_z, \mathbf{s}_y\right) \tag{6.1}$$

for every $z, y \in Y$. We can extend by linearity and define $\mathcal{W}_{\mathbf{c}}^*\left(\overline{t}\right)$ for every $\mathbf{c} \in \mathrm{Ftn}(i(\chi))$. If $w_G^{-1}\overline{t}w_G$ is dominant, then

$$\mathcal{W}_{\mathbf{c}}^*\left(\overline{t}\right) = \mathcal{W}_{\mathbf{c}}\left(w_G^{-1}\overline{t}w_G\right).$$

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We denote

$$\mathcal{W}^* := \{\mathcal{W}^*_{\mathbf{c}} : \mathbf{c} \in \mathrm{Ftn}(i(\chi))\}.$$

If $\mathcal{W}^*_{\mathbf{c}} = 0$ as a function of $\overline{T}$, then $\mathcal{W}_{\mathbf{c}} = 0$, and it follows that $\lambda_{\mathbf{c}}(v) = 0$ for every $v \in \pi^{un}_{\chi}$ as the unramified vector f^0 generates π^{un}_{χ}. Thus, we have an injection of vector spaces

$$\mathrm{Wh}_{\psi}\left(\pi^{un}_{\chi}\right) \hookrightarrow \mathcal{W}^* \tag{6.2}$$

given by

$$\lambda_{\mathbf{c}} \mapsto \mathcal{W}^*_{\mathbf{c}}.$$

For $z \in \mathscr{X}_{Q,n}$, let $\mathcal{W}^*_{\mathcal{O}_z} \subset \mathcal{W}^*$ be the subspace spanned by $\left\{\mathcal{W}^*_{\mathbf{s}_{z'}} : z' \in \mathfrak{R}_z\right\}$, where $\mathfrak{R}_z \subset Y$ is a set of representatives of $\mathcal{O}_z$. Note that $\mathcal{W}^*_{\mathbf{s}_{z'}}$ depends on the representative z'; however, the space $\mathcal{W}^*_{\mathcal{O}_z}$ depends only on $\mathcal{O}_z$. From the restriction of formula (6.2), we obtain

$$\mathrm{Wh}_{\psi}\left(\pi^{un}_{\chi}\right)_{\mathcal{O}_z} \hookrightarrow \mathcal{W}^*_{\mathcal{O}_z}. \tag{6.3}$$

Let $\mathbf{C}^{|\mathcal{O}_z|}$ be the $|\mathcal{O}_z|$-dimensional complex vector space. Endow it with the coordinates indexed by $\mathfrak{R}_z$. Thus we write $(c_y)_{y \in \mathfrak{R}_z}$ for a general vector in $\mathbf{C}^{|\mathcal{O}_z|}$. Depending on $\mathfrak{R}_z$, there is an evaluation map

$$\nu_{\mathfrak{R}_z} : \mathcal{W}^*_{\mathcal{O}_z} \longrightarrow \mathbf{C}^{|\mathcal{O}_z|},$$

with the yth coordinate of $\nu_{\mathfrak{R}_z}\left(\mathcal{W}_{\mathbf{s}_{z'}}\right), y, z' \in \mathfrak{R}_z$ given by

$$\nu_{\mathfrak{R}_z}\left(\mathcal{W}^*_{\mathbf{s}_{z'}}\right)_y = \mathcal{W}^*_{\mathbf{s}_{z'}}(\mathbf{s}_y);$$

that is,

$$\nu_{\mathfrak{R}_z}\left(\mathcal{W}^*_{\mathbf{s}_{z'}}\right) = \left(\mathcal{W}^*_{\mathbf{s}_{z'}}(\mathbf{s}_y)\right)_{y \in \mathfrak{R}_z} \in \mathbf{C}^{|\mathcal{O}_z|}.$$

If $\mathfrak{R} \subset Y$ is a set of representatives for $\mathscr{X}_{Q,n}$, then we have a unique subset $\mathfrak{R}_z \subset \mathfrak{R}$ representing $\mathcal{O}_z$. By combining all the $\nu_{\mathfrak{R}_z}$, we obtain an evaluation map

$$\nu : \mathcal{W}^* \longrightarrow \mathbf{C}^{|\mathscr{X}_{Q,n}|},$$

which depends on the chosen $\mathfrak{R}$. In particular, for a general $\mathbf{c} \in \mathrm{Ftn}(i(\chi))$, we have

$$\nu\left(\mathcal{W}^*_{\mathbf{c}}\right)_y = \mathcal{W}^*_{\mathbf{c}}(\mathbf{s}_y).$$

For every $\mathcal{O}_z \in \mathcal{O}_{\mathscr{X}}$, composing formula (6.3) with $\nu_{\mathfrak{R}_z}$ gives a vector space homomorphism

$$\nu^{\chi}_{\mathfrak{R}_z} : \mathrm{Wh}_{\psi}\left(\pi^{un}_{\chi}\right)_{\mathcal{O}_z} \longrightarrow \mathbf{C}^{|\mathcal{O}_z|}.$$

Similarly, we have

$$\nu^{\chi} : \mathrm{Wh}_{\psi}\left(\pi^{un}_{\chi}\right) \longrightarrow \mathbf{C}^{|\mathscr{X}_{Q,n}|}$$

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with

$$\nu^{\chi} = \bigoplus_{\mathcal{O}_z \in \mathcal{O}_{\mathscr{X}}} \nu^{\chi}_{\mathfrak{R}_z}.$$

Conjecture 6.4. *For every W-orbit $\mathcal{O}_z$ and every choice of $\mathfrak{R}_z$, the homomorphism $\nu^{\chi}_{\mathfrak{R}_z} : \mathrm{Wh}_{\psi}\left(\pi^{un}_{\chi}\right)_{\mathcal{O}_z} \to \mathbf{C}^{|\mathcal{O}_z|}$ is injective. Thus,* $\dim \mathrm{Wh}_{\psi}\left(\pi^{un}_{\chi}\right)_{\mathcal{O}_z} = \mathrm{rank}\left(\nu^{\chi}_{\mathfrak{R}_z}\right)$ *and*

$$\dim \mathrm{Wh}_{\psi}\left(\pi^{un}_{\chi}\right) = \sum_{\mathcal{O}_z \in \mathcal{O}_{\mathscr{X}}} \mathrm{rank}\left(\nu^{\chi}_{\mathfrak{R}_z}\right).$$

For application purposes – that is, to determine $\dim \mathrm{Wh}_{\psi}\left(\pi^{un}_{\chi}\right)_{\mathcal{O}_z}$ – it is sufficient to consider one $\mathfrak{R}_z$. More precisely, for low-rank groups, or low-degree covering groups, we can verify the injectivity in Conjecture 6.4 for a particular $\mathfrak{R}_z$ and compute $\mathrm{rank}\left(\nu^{\chi}_{\mathfrak{R}_z}\right)$ explicitly.

Theorem 6.5. *Let $\overline{G}$ be an n-fold cover of a linear algebraic group G, which is not of metaplectic type (see Definition 2.1). For every $z \in Y_n^{\mathrm{exc}}$, taking $\mathfrak{R}_z = \{z\}$, we have*

$$\mathrm{rank}\left(\nu^{\chi}_{\mathfrak{R}_z}\right) = 1.$$

Hence, $\dim \mathrm{Wh}_{\psi}\left(\pi^{un}_{\chi}\right) \geq \left|\mathscr{X}^{\mathrm{exc}}_{Q,n}\right|$.

Proof. If $z \in Y_n^{\mathrm{exc}}$, then $-z$ is dominant and it follows from [19, Theorem 5.6] that we have a Casselman–Shalika formula for $\overline{G}$ which reads

$$\mathcal{W}^{*}_{\mathbf{s}_z}(\mathbf{s}_z) = \delta_B^{1/2}(\mathbf{s}_z)^{-1} \cdot \prod_{\alpha > 0}\left(1 - q^{-1}\chi_{\alpha}\right).$$

Since χ is unitary, we see that

$$\nu^{\chi}_{\mathfrak{R}_z}(\lambda_{\mathbf{s}_z}) = \mathcal{W}^{*}_{\mathbf{s}_z}(\mathbf{s}_z) \neq 0.$$

This shows that $\nu^{\chi}_{\mathfrak{R}_z}$ is an isomorphism between the 1-dimensional vector spaces. The result follows. □

Theorem 6.5 is compatible with Theorem 6.1, though the approach in the former highlights the role of unramified Whittaker functions.

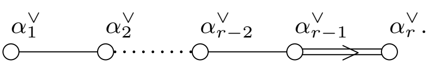
7. Covers of symplectic groups

The goal of this section is to show that Conjecture 5.7 (equivalently Conjecture 5.3) holds for $\overline{\mathrm{Sp}}^{(n)}_{2r}, r \geq 1$, and $\overline{\mathrm{SL}}^{(2)}_3$. Recall that for every $\alpha \in \Phi$, we denote

$$\chi_{\alpha} := \underline{\chi}_{\alpha}(\varpi) = \chi\left(\overline{h}_{\alpha}(\varpi^{n_{\alpha}})\right).$$

7.1. Covers of $\mathrm{Sp}_{2r}, r \geq 1$

Consider the Dynkin diagram for the simple coroots of Sp_{2r}:



Let $Y = Y^{\mathrm{sc}} = \langle \alpha_1^\vee, \alpha_2^\vee, \dots, \alpha_{r-1}^\vee, \alpha_r^\vee \rangle$ be the cocharacter lattice of Sp_{2r}, where $\alpha_r^\vee$ is the short coroot as shown in the diagram. For simplicity, let Q be the Weyl-invariant quadratic form on Y such that $Q(\alpha_r^\vee) = 1$. The bilinear form B_Q is given by

$$B_Q(\alpha_i^\vee, \alpha_j^\vee) = \begin{cases} 2 & \text{if } i = j = r, \\ 4 & \text{if } 1 \le i = j \le r-1, \\ -2 & \text{if } j = i+1, \\ 0 & \text{if } \alpha_i^\vee, \alpha_j^\vee \text{ are not adjacent.} \end{cases}$$

Let $\overline{G} := \overline{\mathrm{Sp}}_{2r}^{(n)}$ be the n-fold cover of Sp_{2r}. We have

$$\overline{G}^\vee = \begin{cases} \mathrm{Sp}_{2r} & \text{if } n \text{ is even,} \\ \mathrm{SO}_{2r+1} & \text{if } n \text{ is odd.} \end{cases}$$

By Corollary 4.10, we have $R_\chi = \{1\}$ if n is even. For odd n, it is clear that $n_{\alpha_i} = n$ for all $\alpha_i \in \Delta$ and

$$Y_{Q,n} = Y_{Q,n}^{sc} = nY.$$

Following notations in [10, Page 267], we consider the map

$$\bigoplus_{i=1}^{r} \mathbf{Z}\alpha_i^\vee \to \bigoplus_{i=1}^{r} \mathbf{Z}e_i$$

given by

$$(x_1, x_2, x_3, \dots, x_r) \mapsto (x_1, x_2 - x_1, x_3 - x_2, \dots, x_{r-1} - x_{r-2}, x_r - x_{r-1}),$$

which is an isomorphism. The Weyl group is $W = S_r \rtimes (\mathbf{Z}/2\mathbf{Z})^r$, where S_r is the permutation group on $\bigoplus_i \mathbf{Z}e_i$ and each $(\mathbf{Z}/2\mathbf{Z})_i$ acts by $e_i \mapsto \pm e_i$. In particular, $\mathbb{w}_{\alpha_i}, 1 \le i \le r-1$, acts on $(y_1, y_2, \dots, y_r) \in \bigoplus_i \mathbf{Z}e_i$ by exchanging y_i and y_{i+1}, while $\mathbb{w}_{\alpha_r}$ acts by (-1) on $\mathbf{Z}e_r$.

For odd n, it follows from Propositions 4.4 and 4.5 and [26, §3] that the only possible nontrivial R-group (up to isomorphism) for $\overline{G}$ is

$$R_\chi = \{1, \mathbb{w}_{\alpha_r}\},$$

where χ is the unramified genuine character of $Z(\overline{T}) \subset \overline{G}$ such that

- $\underline{\chi}_{\alpha_i}$ is any unitary unramified linear character for all $1 \le i \le r-1$, and
- furthermore,

$$\underline{\chi}_{\alpha_r}^2 = \mathbb{1} \text{ and } \underline{\chi}_{\alpha_r} \neq \mathbb{1}.$$

In particular, $\chi_{\alpha_r} = -1$. Denote by ε the nontrivial character of R_χ. We have a decomposition

$$I(\chi) = \pi_\chi^{un} \oplus \pi_\varepsilon,$$

where π_ε is nonisomorphic to $\pi_\chi^{un} = \pi_{\mathbb{1}}$.

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Theorem 7.1. *For odd n and χ as before, Conjecture 5.7 holds for $\overline{Sp}_{2r}^{(n)}$; in this case,*

$$\dim \mathrm{Wh}_\psi\left(\pi_\chi^{un}\right) = \frac{n^r + n^{r-1}}{2}, \qquad \dim \mathrm{Wh}_\psi(\pi_\varepsilon) = \frac{n^r - n^{r-1}}{2}.$$

Proof. For n odd,

$$\mathscr{X}_{Q,n} = Y/Y_{Q,n} \simeq (\mathbf{Z}/n\mathbf{Z})^r.$$

For every W-orbit $\mathcal{O}_y \subset \mathscr{X}_{Q,n}$, we will compute and check explicitly that

$$\mathrm{Tr}\left(A(w_{\alpha_r},\chi)^*_{\mathcal{O}_y}\right) = |(\mathcal{O}_y)^{\mathrm{w}_{\alpha_r}}| \cdot \gamma(w_{\alpha_r},\chi)^{-1}. \tag{7.1}$$

There is a decomposition

$$\mathcal{O}_y = \bigsqcup_{i\in I} \mathcal{O}_{z_i}^{R_\chi}$$

of $\mathcal{O}_y$ into R_χ-orbits, where

$$\mathcal{O}_z^{R_\chi} = \{z\} \text{ or } \mathcal{O}_z^{R_\chi} = \{z, \mathrm{w}_{\alpha_r}[z]\}.$$

To show equation (7.1), it suffices to prove that for every R_χ-orbit $\mathcal{O}_z^{R_\chi} \subset \mathscr{X}_{Q,n}$, we have

$$\sum_{z'\in\mathcal{O}_z^{R_\chi}} \tau(w_{\alpha_r},\chi,\mathbf{s}_{z'},\mathbf{s}_{z'}) = \left|\left(\mathcal{O}_z^{R_\chi}\right)^{\mathrm{w}_{\alpha_r}}\right| \cdot \gamma(w_{\alpha_r},\chi)^{-1}. \tag{7.2}$$

First, if $\mathcal{O}_z^{R_\chi} = \{z\}$, then $\mathrm{w}_{\alpha_r}[z] = z \in \mathscr{X}_{Q,n}$. Write $z = \sum_{i=1}^r z_i e_i \in \mathscr{X}_{Q,n}$, with $0 \le z_i \le n-1$. The equality $\mathrm{w}_{\alpha_r}[z] = z$ is equivalent to $z_r = (n+1)/2$. It follows from [19, Proposition 4.12] that

$$\tau(w_{\alpha_r},\chi,\mathbf{s}_z,\mathbf{s}_z) = \chi_{\alpha_r}^2 \cdot \gamma(w_{\alpha_r},\chi)^{-1} = 1 \cdot \gamma(w_{\alpha_r},\chi)^{-1}.$$

That is, equation (7.2) holds for such $\mathcal{O}_z^{R_\chi}$. In fact, we also see that the character $\theta_{\sigma^{\mathscr{X}}}$ of the representation $\sigma^{\mathscr{X}} : R_\chi \to \mathrm{Perm}(\mathscr{X}_{Q,n})$ is given by

$$\theta_{\sigma^{\mathscr{X}}}(1) = n^r, \qquad \theta_{\sigma^{\mathscr{X}}}(\mathrm{w}_{\alpha_r}) = n^{r-1}.$$

Second, assume $\mathcal{O}_z^{R_\chi} = \{z, \mathrm{w}_{\alpha_r}[z]\}$; then $n \nmid \langle z_\rho, \alpha_r\rangle$. It follows from the proof of [15, Lemma 3.9] that

$$k_{z,\alpha_r} + k_{\mathrm{w}_{\alpha_r}[z],\alpha_r} = 1.$$

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We obtain

$$\begin{aligned}
&\tau\left(w_{\alpha_r},\chi,\mathbf{s}_z,\mathbf{s}_z\right)+\tau\left(w_{\alpha_r},\chi,\mathbf{s}_{\mathbb{w}_{\alpha_r}[z]},\mathbf{s}_{\mathbb{w}_{\alpha_r}[z]}\right)\\
&=\frac{1-q^{-1}}{1-\chi_{\alpha_r}}\left(\chi_{\alpha_r}\right)^{k_{z,\alpha_r}}+\frac{1-q^{-1}}{1-\chi_{\alpha_r}}\left(\chi_{\alpha_r}\right)^{k_{\mathbb{w}_{\alpha_r}[z],\alpha_r}}\\
&=\frac{1-q^{-1}}{2}(-1)^{k_{z,\alpha_r}}+\frac{1-q^{-1}}{2}(-1)^{1-k_{z,\alpha_r}}\\
&=0.
\end{aligned}$$

On the other hand, $\left|\left(\mathcal{O}_z^{R_\chi}\right)^{\mathbb{w}_{\alpha_r}}\right|=0$; this shows that equation (7.2) holds in this case.

Therefore, Conjecture 5.7 holds for $\overline{\mathrm{Sp}}_{2r}^{(n)}$. The desired dimension formula for the Whittaker spaces of π_χ^{un} and π_ε follows from Theorem 5.6 and the character $\theta_{\sigma^{\mathscr{X}}}$ already computed. This completes the proof. □

Remark 7.2. Equation (7.2) might fail for covers of a general simply connected group. More precisely, for a general $\overline{G}$, we have the decomposition of a W-orbit $\mathcal{O}_y$ into R_χ-orbits

$$\mathcal{O}_y=\bigsqcup_{i\in I}\mathcal{O}_{z_i}^{R_\chi}.$$

Conjecture 5.7 predicts that for every $\mathbb{w}\in R_\chi$, we have

$$\sum_{i\in I}\sum_{z'\in\mathcal{O}_{z_i}^{R_\chi}}\tau(w,\chi,\mathbf{s}_{z'},\mathbf{s}_{z'})=\sum_{i\in I}\left|\left(\mathcal{O}_{z_i}^{R_\chi}\right)^{\mathbb{w}}\right|\cdot\gamma(w,\chi)^{-1}. \tag{7.3}$$

However, the inner summands indexed by I on the two sides may not be equal. A counterexample arises from considering $\overline{\mathrm{SL}}_4^{(3)}$ with $y=\sum_{\alpha\in\Delta}\alpha^\vee$, in which case $|\mathcal{O}_y|=6$.

This subtlety is the main difficulty with verifying Conjecture 5.7 by direct computation. Indeed, it follows from Tables 1 and 2 (or, more precisely, [26, §3]) that the nontrivial unramified group R_χ for covers of simply connected groups of type B_r,D_r,E_6 and E_7 is small; thus the orbits $\mathcal{O}_{z_i}^{R_\chi}$ are all small. However, as just noted, one needs to consider the whole W-orbit $\mathcal{O}_y$, whose size could be large, depending on W and n. This hinders direct computation in the general situation.

Theorem 7.1 could also be obtained from the consideration in Subject 6.2, especially Conjecture 6.4. We illustrate this by considering the case of $\overline{\mathrm{SL}}_2^{(n)}$. Write $n=2d+1$ and $\Delta=\{\alpha\}$. The twisted Weyl action on $\mathscr{X}_{Q,n}=\mathbf{Z}/n\mathbf{Z}$ is given by

$$\mathbb{w}_\alpha[k\alpha^\vee]=(1-k)\alpha^\vee\in\mathscr{X}_{Q,n}.$$

In total there are $(d+1)$-many W-orbits. Every orbit except that of $-d\alpha^\vee$ is free. We choose a set of representatives of $\mathscr{X}_{Q,n}$ as

$$\mathfrak{R}=\{i\cdot\alpha^\vee:-d\le i\le d\}.$$

The W-orbits in $\mathscr{X}_{Q,n}$ are

$$\mathcal{O}_{\mathscr{X}}=\{\mathcal{O}_{i\alpha^\vee}:1\le i\le d\}\cup\{\mathcal{O}_{-d\alpha^\vee}\},$$

with

$$\mathfrak{R}_{i\alpha^\vee} = \{i\alpha^\vee, (1-i)\alpha^\vee\} \text{ for } 1 \le i \le d \text{ and } \mathfrak{R}_{-d\alpha^\vee} = \{-d\alpha^\vee\}.$$

Proposition 7.3. *Conjecture 6.4 holds for $\mathfrak{R}_{i\alpha^\vee}$, and moreover,*

$$\operatorname{rank}\left(\nu^\chi_{\mathfrak{R}_{i\alpha^\vee}}\right) = 1$$

for every $1 \le i \le d$ and $i = -d$.

Proof. Since $-d\alpha^\vee \in Y$ is an exceptional point, in view of Theorem 6.5 it suffices to deal with the case $1 \le i \le d$. We denote $z := i\alpha^\vee$. Let $\lambda_{\mathbf{c}} \in \mathrm{Wh}_\psi(I(\chi))_{\mathcal{O}_z}$, viewed as an element in $\mathrm{Wh}_\psi\left(\pi^{un}_\chi\right)_{\mathcal{O}_z}$ by restriction. Then

$$\operatorname{supp}(\mathbf{c}) \subset \mathbf{s}_z \cdot \overline{A} \cup \mathbf{s}_{\mathsf{w}[z]} \cdot \overline{A}.$$

We have

$$\mathcal{W}^*_{\mathbf{c}} = \mathbf{c}(\mathbf{s}_z) \cdot \mathcal{W}^*_{\mathbf{s}_z} + \mathbf{c}\left(\mathbf{s}_{\mathsf{w}[z]}\right) \cdot \mathcal{W}^*_{\mathbf{s}_{\mathsf{w}[z]}}.$$

Recall the projection map $h^{un} : \mathrm{Wh}_\psi(I(\chi))_{\mathcal{O}_z} \twoheadrightarrow \mathrm{Wh}_\psi\left(\pi^{un}_\chi\right)$ from Section 6.2. Assume $\mathbf{c}$ is such that

$$\nu^\chi_{\mathfrak{R}_z}(\lambda_{\mathbf{c}}) = \left(\mathcal{W}^*_{\mathbf{c}}(\mathbf{s}_z), \mathcal{W}^*_{\mathbf{c}}\left(\mathbf{s}_{\mathsf{w}[z]}\right)\right) = (0,0). \tag{7.4}$$

We want to show that $h^{un}(\lambda_{\mathbf{c}}) = 0$. For each $z = i\alpha^\vee$, we denote

$$\mathcal{M}_{\mathfrak{R}_z} := \begin{pmatrix} \mathcal{W}^*_{\mathbf{s}_z}(\mathbf{s}_z) & \mathcal{W}^*_{\mathbf{s}_z}\left(\mathbf{s}_{\mathsf{w}[z]}\right) \\ \mathcal{W}^*_{\mathbf{s}_{\mathsf{w}[z]}}(\mathbf{s}_z) & \mathcal{W}^*_{\mathbf{s}_{\mathsf{w}[z]}}\left(\mathbf{s}_{\mathsf{w}[z]}\right) \end{pmatrix}.$$

It is easy to see that

$$\nu^\chi_{\mathfrak{R}_z}(\lambda_{\mathbf{c}}) = \left(\mathbf{c}(\mathbf{s}_z), \mathbf{c}\left(\mathbf{s}_{\mathsf{w}[z]}\right)\right) \mathcal{M}_{\mathfrak{R}_z}. \tag{7.5}$$

Now by equation (6.1) we have

$$\mathcal{W}^*_{\mathbf{s}_z}(\mathbf{s}_y) \cdot \delta_B^{1/2}(\mathbf{s}_y) = c_{\mathsf{gk}}(w_\alpha, \chi) \cdot \tau(\mathrm{id}, \chi, \mathbf{s}_z, \mathbf{s}_y) + \tau(w_\alpha, {}^{w_\alpha}\chi, \mathbf{s}_z, \mathbf{s}_y).$$

Since we are in the case where $\underline{\chi}_\alpha^2 = \mathbb{1}$ but $\underline{\chi}_\alpha \neq \mathbb{1}$, we have

$$\underline{\chi}_\alpha(\varpi) = \chi\left(\overline{h}_\alpha(\varpi^n)\right) = -1.$$

We also note that ${}^{w_\alpha}\chi = \chi$. Thus a straightforward computation gives

$$\mathcal{M}_{\mathfrak{R}_{i\alpha^\vee}} = \begin{pmatrix} \mathcal{W}^*_{\mathbf{s}_{i\alpha^\vee}}(\mathbf{s}_{i\alpha^\vee}) & \mathcal{W}^*_{\mathbf{s}_{i\alpha^\vee}}\left(\mathbf{s}_{(1-i)\alpha^\vee}\right) \\ \mathcal{W}^*_{\mathbf{s}_{(1-i)\alpha^\vee}}(\mathbf{s}_{i\alpha^\vee}) & \mathcal{W}^*_{\mathbf{s}_{(1-i)\alpha^\vee}}\left(\mathbf{s}_{(1-i)\alpha^\vee}\right) \end{pmatrix} = \begin{pmatrix} q^{i-1} & q^{-i-1}\mathbf{g}_{\psi^{-1}}(1-2i) \\ q^i \mathbf{g}_{\psi^{-1}}(2i-1) & q^{-i-1} \end{pmatrix}.$$

Combining equations (7.4) and (7.5), we get

$$q^{-1} \cdot \mathbf{c}(\mathbf{s}_{i\alpha^\vee}) + \mathbf{g}_{\psi^{-1}}(2i-1) \cdot \mathbf{c}\left(\mathbf{s}_{(1-i)\alpha^\vee}\right) = 0. \tag{7.6}$$

Consider the map

$$P^*_{\mathbb{1}} : \mathrm{Wh}_\psi(I(\chi)) \longrightarrow \mathrm{Wh}_\psi(I(\chi))$$

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induced from the projection $P_{\mathbb{1}}: I(\chi) \to I(\chi)$. Showing that $h^{un}(\lambda_{\mathbf{c}}) = 0$ is equivalent to proving $P_{\mathbb{1}}^*(\lambda_{\mathbf{c}}) = 0$. Now

$$\begin{aligned}
&P_{\mathbb{1}}^*(\lambda_{\mathbf{c}}) \\
&= \frac{1}{2} \cdot (\lambda_{\mathbf{c}} + \mathscr{A}(w,\chi)^*(\lambda_{\mathbf{c}})) \\
&= \frac{1}{2} \left(\mathbf{c}(\mathbf{s}_z) + \mathbf{c}(\mathbf{s}_z)\gamma(w,\chi)\tau(w,\chi,z,z) + \mathbf{c}\left(\mathbf{s}_{\mathbb{w}[z]}\right)\gamma(w,\chi)\tau(w,\chi,\mathbb{w}[z],z)\right) \cdot \lambda_{\mathbf{s}_z} \\
&+ \frac{1}{2} \left(\mathbf{c}\left(\mathbf{s}_{\mathbb{w}[z]}\right) + \mathbf{c}(\mathbf{s}_z)\gamma(w,\chi)\tau(w,\chi,z,\mathbb{w}[z]) + \mathbf{c}\left(\mathbf{s}_{\mathbb{w}[z]}\right)\gamma(w,\chi)\tau(w,\chi,\mathbb{w}[z],\mathbb{w}[z])\right) \cdot \lambda_{\mathbf{s}_{\mathbb{w}[z]}}.
\end{aligned}$$

A simplification gives that the coefficient in front of $\lambda_{\mathbf{s}_z}$ is

$$\frac{1}{1+q^{-1}} \cdot \left(\mathbf{c}(\mathbf{s}_{i\alpha^\vee})q^{-1} + \mathbf{c}\left(\mathbf{s}_{(1-i)\alpha^\vee}\right)\mathbf{g}_{\psi^{-1}}(2i-1)\right),$$

which is equal to 0 by equation (7.6). Similarly, it can be checked easily that the coefficient in front of $\lambda_{\mathbf{s}_{\mathbb{w}[z]}}$ is also 0. This shows that Conjecture 6.4 holds.

It is clear that for every $1 \le i \le d$, we have

$$\operatorname{rank}\left(\nu^{\chi}_{\mathfrak{R}_{i\alpha^\vee}}\right) = \operatorname{rank}\left(\mathcal{M}_{\mathfrak{R}_{i\alpha^\vee}}\right) = 1.$$

The proof is now completed. □

It follows from Proposition 7.3 that

$$\dim \mathrm{Wh}_\psi\left(\pi_\chi^{un}\right) = \left\langle \mathbb{1}, \sigma^{\mathscr{X}} \right\rangle_{R_\chi} = |\mathcal{O}_{\mathscr{X}}| = d+1 = \frac{n+1}{2}.$$

Consequently,

$$\dim \mathrm{Wh}_\psi(\pi_\varepsilon) = \left\langle \varepsilon, \sigma^{\mathscr{X}} \right\rangle_{R_\chi} = d = \frac{n-1}{2},$$

the number of free R_χ-orbits in $\mathscr{X}_{Q,n}$.

Remark 7.4. Let n be odd and χ be the nontrivial quadratic genuine character of $Z\left(\overline{T}\right) \subset \overline{\mathrm{Sp}}_{2r}$. The Whittaker dimension for the constituents in $\pi_\chi^{un} \oplus \pi'_\chi = \mathrm{Ind}_{\overline{B}}^{\overline{\mathrm{Sp}}_{2r}}(i(\chi))$ can be deduced from that of $\overline{\mathrm{SL}}_2^{(n)}$ as follows. Here we write π'_χ for π_ε. Each of the rank 1 lattice $(\mathbf{Z}e_j) \subset Y$ gives rise to an n-fold covering $\overline{\mathrm{GL}}_{1,e_j}$ of the torus $\mathrm{GL}_1 \simeq F^\times$, by restriction from $\overline{T}$. We have an isomorphism (see [20, §5.1.3])

$$\prod_{1 \le j \le r} \overline{\mathrm{GL}}_{1,e_j} / H \simeq \overline{T}, \tag{7.7}$$

where $H = \left\{(\zeta_j) \in (\mu_n)^r : \prod_j \zeta_j = 1\right\}$; that is, block-commutativity holds for coverings of Levi subgroups of $\overline{\mathrm{Sp}}_{2r}$. Thus, we can write

$$i(\chi) = \prod_{j=1}^{r} i(\chi_j),$$

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where $i(\chi_j) \in \operatorname{Irr}\left(\overline{\mathrm{GL}}_{1,e_j}\right)$ is of dimension n. Note also that $\overline{T}_0 := \overline{\mathrm{GL}}_{1,e_r}$ is just the covering torus of $\overline{\mathrm{SL}}_2$ associated to α_r. Let $\overline{M} = \prod_{1\le j\le r-1} \overline{\mathrm{GL}}_{1,e_j} \times \overline{\mathrm{SL}}_2$ be the Levi subgroup of the parabolic subgroup $\overline{P} \subset \overline{\mathrm{Sp}}_{2r}^{(n)}$ associated to α_r. The character χ_r is a nontrivial quadratic character of $Z\left(\overline{T}_0\right)$ and thus we have

$$\operatorname{Ind}_{\overline{T}_0}^{\overline{\mathrm{SL}}_2}(i(\chi_r)) = \pi^{un}_{\chi_r} \oplus \pi'_{\chi_r}.$$

By induction in stages, we have

$$\pi^{un}_{\chi} = \operatorname{Ind}_{\overline{P}}^{\overline{\mathrm{Sp}}_{2r}} \left(\boxtimes_{1\le j\le r-1} i(\chi_j)\right) \boxtimes \pi^{un}_{\chi_r}$$

and similarly,

$$\pi'_{\chi} = \operatorname{Ind}_{\overline{P}}^{\overline{\mathrm{Sp}}_{2r}} \left(\boxtimes_{1\le j\le r-1} i(\chi_j)\right) \boxtimes \pi'_{\chi_r}.$$

Now it follows from the equalities

$$\dim i(\chi_j) = n, \qquad \dim \mathrm{Wh}_\psi\left(\pi^{un}_{\chi_r}\right) = \frac{n+1}{2}, \qquad \dim \mathrm{Wh}_\psi\left(\pi'_{\chi_r}\right) = \frac{n-1}{2}$$

and Rodier's heredity that

$$\dim \mathrm{Wh}_\psi\left(\pi^{un}_{\chi}\right) = n^{r-1}\cdot \frac{n+1}{2}, \qquad \dim \mathrm{Wh}_\psi\left(\pi'_{\chi}\right) = n^{r-1}\cdot \frac{n-1}{2},$$

which agrees with Theorem 7.1.

7.2. Double cover of SL_3

Before we proceed, we recall some observations from Section 5.4. By Tables 1 and 2 and Theorem 4.6, the group R_χ is always cyclic for all semisimple type except the D_r case when r is even. Recall that

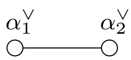
$$\mathscr{A}(w,\chi)^* = \gamma(w,\chi)\cdot A(w,\chi)^*.$$

Assume R_χ is cyclic and let w be a generator; then $\sigma(\mathsf{w}), \sigma \in \operatorname{Irr}(R_\chi)$ are distinct and $\dim \mathrm{Wh}_\psi(\pi_\sigma), \sigma \in \operatorname{Irr}(R_\chi)$ are just the multiplicities of the distinct eigenvalues $\sigma(\mathsf{w})\cdot \gamma(w,\chi)^{-1}$ of the polynomial

$$\det\left(X\cdot \mathrm{id} - A(w,\chi)^*\right) = \det\left(X \cdot I_{|\mathscr{X}_{Q,n}|} - \mathcal{S}_{\mathfrak{R}}(w, i(\chi))\right).$$

This will be the observation we apply to the double cover $\overline{\mathrm{SL}}_3^{(2)}$ in this subsection.

Let $\alpha_1^\vee, \alpha_2^\vee$ be the two simple coroots of SL_3:



For convenience, we write $\mathsf{w}_i = \mathsf{w}_{\alpha_i}$ for $i = 1,2$. Let $\alpha_3 = \alpha_1^\vee + \alpha_2^\vee \in \Phi^+$. Let $Q : Y \to \mathbf{Z}$ be the unique Weyl-invariant quadratic form such that $Q\left(\alpha_1^\vee\right) = 1$. Taking $n = 2$, we get

$$Y_{Q,2} = Y_{Q,2}^{sc} = 2Y$$

and thus $\overline{\mathrm{SL}}_3^\vee = \mathrm{PGL}_3$ and also

$$\mathscr{X}_{Q,2} \simeq (\mathbf{Z}/2\mathbf{Z}) \oplus (\mathbf{Z}/2\mathbf{Z}).$$

The ordered set

$$\mathfrak{R} = \{0, \alpha_1^\vee, \alpha_2^\vee, \alpha_3^\vee\} \subset Y$$

is a set of representatives of $\mathscr{X}_{Q,2}$. There are two W-orbits

$$\mathcal{O}_0 = \{0, \alpha_1^\vee, \alpha_2^\vee\}, \qquad \mathcal{O}_{\alpha_3^\vee} = \{\alpha_3^\vee\}.$$

Let χ be a unitary unramified genuine character of $Z(\overline{T})$. Since the dual group of $\overline{\mathrm{SL}}_3^{(2)}$ is PGL_3, we see that $R_\chi^{sc} = R_\chi$ by Proposition 4.4; moreover, R_χ is either trivial or $\mathbf{Z}/3\mathbf{Z}$.

Assuming $R_\chi = \mathbf{Z}/3\mathbf{Z}$, we see that $\Phi_\chi = \emptyset$ and $R_\chi = W_\chi = \langle \mathbb{w}_1\mathbb{w}_2 \rangle = \langle \mathbb{w}_2\mathbb{w}_1 \rangle \subset W$ – that is,

$$^{\mathbb{w}_2\mathbb{w}_1}\chi = \chi.$$

This implies that

$$\chi_{\alpha_1} = \zeta = \chi_{\alpha_2} \text{ and } \chi_{\alpha_3} = \zeta^2, \tag{7.8}$$

where $\zeta \in \mathbf{C}^\times$ is a primitive third root of unity. For such χ, we have the decomposition

$$I(\chi) = \pi_\chi^{un} \oplus \pi_1 \oplus \pi_2$$

of $I(\chi)$ into nonisomorphic irreducible components. We have $\dim \mathrm{Wh}_\psi(I(\chi)) = 4$, and the permutation representation

$$\sigma^{\mathscr{X}} : R_\chi \longrightarrow \mathrm{Perm}(\mathscr{X}_{Q,2})$$

is such that in $\mathscr{X}_{Q,2}$:

$$\mathbb{w}_2\mathbb{w}_1[\alpha_3^\vee] = \alpha_3^\vee, \qquad \mathbb{w}_2\mathbb{w}_1[0] = \alpha_1^\vee, \qquad \mathbb{w}_2\mathbb{w}_1[\alpha_1^\vee] = \alpha_2^\vee, \qquad \mathbb{w}_2\mathbb{w}_1[\alpha_2^\vee] = \alpha_0^\vee.$$

We have $\mathrm{Irr}(R_\chi) = \{\mathbb{1}, \sigma, \sigma^2\}$, where σ is the generator given by

$$\sigma(\mathbb{w}_2\mathbb{w}_1) = \zeta.$$

It then follows easily that

$$\sigma^{\mathscr{X}} = (2 \cdot \mathbb{1}) \oplus \sigma \oplus \sigma^2.$$

Thus, we could label constituents of $I(\chi)$ as

$$\pi_\chi^{un} = \pi_{\mathbb{1}} = \pi_{\sigma^0}, \qquad \pi_1 = \pi_\sigma, \qquad \pi_2 = \pi_{\sigma^2}.$$

Since R_χ is cyclic and $\mathbb{w}_2\mathbb{w}_1$ is a generator of $\mathrm{Irr}(R_\chi)$, to determine $\dim \mathrm{Wh}_\psi(\pi_{\sigma^i})$ it suffices to compute the characteristic polynomial of $A(w_2w_1,\chi)^*$ which takes the form

$$\det(X \cdot \mathrm{id} - A(w_2w_1,\chi)^*) = \prod_{0 \le i \le 2} \left(X - \zeta^i \cdot \gamma(w_2w_1,\chi)^{-1}\right)^{\dim \mathrm{Wh}_\psi(\pi_{\sigma^i})},$$

where

$$\gamma(w_2w_1,\chi)^{-1} = \frac{\left(1-q^{-1}\chi_{\alpha_1}^{-1}\right)\left(1-q^{-1}\chi_{\alpha_3}\right)^{-1}}{(1-\chi_{\alpha_1})(1-\chi_{\alpha_3})} = \frac{1+q^{-1}+q^{-2}}{3}.$$

Let $\mathcal{S}_{\mathfrak{R}}(w_2w_1,i(\chi))$ be the scattering matrix associated to the ordered set $\mathfrak{R}$. For simplicity of computation, we assume $\mu_4 \subset F^\times$, and hence $\varepsilon = 1$. Using equations (5.2) and (5.3) and Theorem 5.2, we obtain in this case an explicit form (again, using the shorthand notation $\chi_{\alpha_1},\chi_{\alpha_2},\chi_{\alpha_3}$):

$$\mathcal{S}_{\mathfrak{R}}(w_2w_1,i(\chi))$$
$$= \begin{pmatrix} \frac{(1-q^{-1})^2}{(1-\chi_{\alpha_1})(1-\chi_{\alpha_3})} & \mathbf{g}_{\psi^{-1}}(-1)\gamma\left(\underline{\chi}_{\alpha_3}\right)^{-1} & \mathbf{g}_{\psi^{-1}}(-1)\frac{1-q^{-1}}{1-\chi_{\alpha_1}} & 0 \\ \mathbf{g}_{\psi^{-1}}(1)\frac{1-q^{-1}}{1-\chi_{\alpha_3}} & \chi_{\alpha_1}\frac{1-q^{-1}}{1-\chi_{\alpha_1}}\gamma\left(\underline{\chi}_{\alpha_3}\right)^{-1} & q^{-1} & 0 \\ \mathbf{g}_{\psi^{-1}}(1)\gamma\left(\underline{\chi}_{\alpha_1}\right)^{-1} & 0 & \chi_{\alpha_3}\frac{1-q^{-1}}{1-\chi_{\alpha_3}}\gamma\left(\underline{\chi}_{\alpha_1}\right)^{-1} & 0 \\ 0 & 0 & 0 & \chi_{\alpha_1}\chi_{\alpha_3}\gamma\left(\underline{\chi}_{\alpha_1}\right)^{-1}\gamma\left(\underline{\chi}_{\alpha_3}\right)^{-1} \end{pmatrix}.$$

A straightforward computation gives

$$\det\left(X\cdot I_4 - \mathcal{S}_{\mathfrak{R}}(w_2w_1,i(\chi))\right)$$
$$= \left(X - \frac{1+q^{-1}+q^{-2}}{3}\right)\cdot\left(X^3 - \left(\frac{1+q^{-1}+q^{-2}}{3}\right)^3\right),$$

and thus

$$\det\left(X\cdot I_4 - \mathscr{A}(w_2w_1,\chi)^*\right) = (X-1)^2\cdot(X-\zeta)\cdot\left(X-\zeta^2\right).$$

Therefore,

$$\dim \mathrm{Wh}_\psi(\pi_{\mathbb{1}}) = 2 \text{ and } \dim \mathrm{Wh}_\psi(\pi_{\sigma^i}) = 1 \text{ for } i = 1,2.$$

Clearly,

$$\sigma^{\mathrm{Wh}} = (2\cdot\mathbb{1})\oplus\sigma\oplus\sigma^2.$$

Proposition 7.5. *For $\overline{SL}_3^{(2)}$, we have*

$$\sigma^{\mathrm{Wh}} = \sigma^{\mathscr{X}} = (2\cdot\mathbb{1})\oplus\sigma\oplus\sigma^2.$$

Moreover, Conjecture 5.7 holds.

Proof. The equalities are clear. It suffices to prove that Conjecture 5.7 holds for the two orbits $\mathcal{O}_0$ and $\mathcal{O}_{\alpha_3^\vee}$, which is a priori stronger than the equalities. This follows from a direct computation, or we can argue alternatively by using the fact that $\mathcal{O}_{\mathscr{X}} = \{\mathcal{O}_0,\mathcal{O}_{\alpha_3^\vee}\}$, with $\alpha_3^\vee \in \mathscr{X}_{Q,2}^{\mathrm{exc}}$. Indeed, by Theorem 6.1, $\sigma^{\mathrm{Wh}}_{\mathcal{O}_{\alpha_3^\vee}} = \sigma^{\mathrm{Wh}}_{\mathcal{O}_{\alpha_3^\vee}} = \mathbb{1}$. However, since

$$\sigma^{\mathrm{Wh}} = \sigma^{\mathrm{Wh}}_{\mathcal{O}_{\alpha_3^\vee}}\oplus\sigma^{\mathrm{Wh}}_{\mathcal{O}_0} = \sigma^{\mathscr{X}}_{\mathcal{O}_{\alpha_3^\vee}}\oplus\sigma^{\mathscr{X}}_{\mathcal{O}_0} = \sigma^{\mathscr{X}},$$

it enforces

$$\sigma^{\mathrm{Wh}}_{\mathcal{O}_0} = \sigma^{\mathrm{Wh}}_{\mathcal{O}_0} = \mathbb{1}\oplus\sigma\oplus\sigma^2.$$

The proof is completed. □

8. Two remarks

In this section, we consider two examples to justify the necessary constraints imposed on G and $\overline{G}$ in Conjecture 5.7. First, we consider $\overline{\mathrm{SO}}_3^{(n)}$ and show that a naive analogous formula does not hold for general covers of semisimple groups which are not simply connected. Second, we consider the double cover of the simply connected $\mathrm{Spin}_6 \simeq \mathrm{SL}_4$, whose dual group is SL_4/μ_2, and show that analogous Conjecture 5.7 does not hold. This shows that it is necessary to require the cover $\overline{G}$ to be saturated.

8.1. Covers of SO_3

Let $Y = \mathbf{Z} \cdot e$ be the cocharacter lattice of SO_3 with $\alpha^\vee = 2e$ generating the coroot lattice Y^{sc}. Let $Q : Y \to \mathbf{Z}$ be the Weyl-invariant quadratic form such that $Q(e) = 1$. Thus, $Q(\alpha^\vee) = 4$. We get

$$\alpha^\vee_{Q,n} = \frac{n}{\gcd(4,n)} \cdot \alpha^\vee.$$

On the other hand,

$$Y_{Q,n} = \mathbf{Z} \cdot \frac{n}{\gcd(2,n)} e.$$

The equality ${}^{w_\alpha}\chi = \chi$ is equivalent to

$$\chi\left(\overline{h}_\alpha\left(a^{n/\gcd(2,n)}\right)\right) = 1$$

for all $a \in F^\times$.

Lemma 8.1. *Let χ be a unitary unramified genuine character of $Z\left(\overline{T}\right)$. Then $R_\chi = W$ if and only if $4|n$ and $\underline{\chi}_\alpha$ is a nontrivial quadratic character.*

Proof. Clearly, $R_\chi = W$ if and only if $\Phi_\chi = \emptyset$ and $W_\chi = W$. We discuss case by case. First, if $4 \nmid n$, then $\gcd(4,n) = \gcd(2,n)$, and in this case $R_\chi = \{1\}$. Second, if $n = 4m$, then $\alpha^\vee_{Q,n} = m\alpha^\vee$ and $n/\gcd(2,n) = 2m$. In this case, if $\underline{\chi}_\alpha$ is a nontrivial quadratic character, we have $R_\chi = W$. □

Remark 8.2. For G of adjoint type, if χ is a unitary unramified character of T, then $I(\chi)$ is always irreducible (see Corollary 4.10). The result shows that this may fail for covers of groups of adjoint type.

Now we assume that $n = 4m$ and $\underline{\chi}_\alpha$ is a nontrivial quadratic character – that is,

$$\chi_\alpha = \chi\left(\overline{h}_\alpha(\varpi^{n_\alpha})\right) = -1.$$

In this case, $R_\chi = W$ and

$$I(\chi) = \pi^{un}_\chi \oplus \pi.$$

We have

$$Y_{Q,n} = Y^{sc}_{Q,n} = \mathbf{Z} \cdot \alpha^\vee_{Q,n} = \mathbf{Z} \cdot (m\alpha^\vee) = \mathbf{Z} \cdot (2me).$$

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Therefore, $\overline{G}^\vee \simeq \mathrm{SO}_3$ is of adjoint type. It is clear that

$$\mathscr{X}_{Q,n} \simeq \mathbf{Z}/(2m)\mathbf{Z},$$

with the twisted Weyl action given by

$$\mathbb{w}_\alpha[ie] = (-i)e + 2e = (2-i)e.$$

We have $|\mathcal{O}_{\mathscr{X}}| = m+1$ – that is, there are $m+1$ many W-orbits in $\mathscr{X}_{Q,n}$. Let $\mathfrak{R} \subset Y$ be the following set of representatives of $\mathscr{X}_{Q,n}$:

$$\mathfrak{R} = \{ie : -m+1 \le i \le m\}.$$

The two trivial W-orbits are

$$\mathcal{O}_e = \{e\}, \qquad \mathcal{O}_{(-m+1)e} = \{(-m+1)e\};$$

while for all other $ie \in \mathfrak{R}$ with $2 \le i \le m$, the orbit

$$\mathcal{O}_{ie} = \{ie, (2-i)e\} \subset \mathscr{X}_{Q,n}$$

is W-free. We thus have $\mathfrak{R}_e, \mathfrak{R}_{(-m+1)e}$ and $\mathfrak{R}_{ie} \subset \mathfrak{R}$ to represent the three families of orbits.

Proposition 8.3. *Assume $n = 4m$ and $\underline{\chi}_\alpha$ is a nontrivial quadratic character. Then*

$$\sigma^{\mathrm{Wh}} = m \cdot \mathbb{1}_W \oplus m \cdot \varepsilon_W, \text{ and } \sigma^{\mathscr{X}} = (m+1) \cdot \mathbb{1}_W \oplus (m-1) \cdot \varepsilon_W.$$

Hence,

$$\dim \mathrm{Wh}_\psi\left(\pi_\chi^{un}\right) = m = \dim \mathrm{Wh}_\psi(\pi).$$

Proof. Choosing $\mathfrak{R}$ as before, the scattering matrix $\mathcal{S}_{\mathfrak{R}}(w_\alpha, i(\chi))$ is the block-diagonal matrix with blocks $\mathcal{S}_{\mathfrak{R}}(w_\alpha, i(\chi))_{\mathcal{O}_{ie}}$ for $i = -m+1$ and $1 \le i \le m$. Here,

$$\mathcal{S}_{\mathfrak{R}}(w_\alpha, i(\chi))_{\mathcal{O}_{(-m+1)e}} = \gamma(w_\alpha, \chi)^{-1} = \frac{1 - q^{-1}\chi_\alpha}{1 - \chi_\alpha} = \frac{1 + q^{-1}}{2}$$

and

$$\mathcal{S}_{\mathfrak{R}}(w_\alpha, i(\chi))_{\mathcal{O}_e} = \chi_\alpha \cdot \gamma(w_\alpha, \chi)^{-1} = -\frac{1 + q^{-1}}{2};$$

also, for $2 \le i \le m$,

$$\mathcal{S}_{\mathfrak{R}}(w_\alpha, i(\chi))_{\mathcal{O}_{ie}} = \begin{pmatrix} \frac{1-q^{-1}}{1-\chi_\alpha}\chi_\alpha & \mathbf{g}_{\psi^{-1}}((i-1)4) \\ \mathbf{g}_{\psi^{-1}}((1-i)4) & \frac{1-q^{-1}}{1-\chi_\alpha} \end{pmatrix}.$$

It follows that for $2 \le i \le m$,

$$\sigma^{\mathrm{Wh}}_{\mathcal{O}_{ie}} = \sigma^{\mathscr{X}}_{\mathcal{O}_{ie}} = \mathbb{1} \oplus \varepsilon$$

and

$$\sigma^{\mathrm{Wh}}_{\mathcal{O}_{(-m+1)e}} = \sigma^{\mathscr{X}}_{\mathcal{O}_{(-m+1)e}} = \mathbb{1};$$

however,

$$\sigma_{\mathcal{O}_e}^{\mathrm{Wh}} = \varepsilon_W, \qquad \sigma_{\mathcal{O}_e}^{\mathscr{X}} = \mathbb{1}.$$

The last result on the Whittaker dimension follows from Theorem 5.6. □

Proposition 8.3 can also be proved by using the method described in Section 5.4. That is, we can compute $\dim \mathrm{Wh}_\psi\left(\pi_\chi^{un}\right)$ by showing that

(i) Conjecture 6.4 holds for $\overline{\mathrm{SO}}_3$ and

(ii) we have

$$\operatorname{rank}\left(\nu_{\mathfrak{R}_{ie}}^{\chi}\right) = \begin{cases} 1 & \text{if } 2 \le i \le m \text{ or } i = -m+1, \\ 0 & \text{if } i = 1. \end{cases}$$

Here (i) can be verified exactly in the same way as Proposition 7.3, and thus we omit the details. We discuss (ii) for the three cases $i = -m+1$, $i = 1$ and $2 \le i \le m$ separately.

- First, since $(-m+1)e = \rho - \rho_{Q,n}$, it is the unique element in Y_n^{exc}. In this case, the equality $\operatorname{rank}\left(\nu_{\mathfrak{R}_{(1-m)e}}^{\chi}\right) = 1$ follows from Theorem 6.5.
- Second, for $\mathfrak{R}_e = \{e\}$, we have $\operatorname{rank}\left(\nu_{\mathfrak{R}_e}^{\chi}\right) = 1$ if and only if $\mathcal{W}_{\mathbf{s}_e}^*(\mathbf{s}_e) \neq 0$. A straightforward computation from equation (6.1) gives

$$\mathcal{W}_{\mathbf{s}_e}^*(\mathbf{s}_e) = 0.$$

  Thus, $\operatorname{rank}\left(\nu_{\mathcal{O}_e}^{\chi}\right) = 0$.

- Third, we deal with free W-orbits $\mathcal{O}_{ie}, 2 \le i \le m$. Similar to the case of $\overline{\mathrm{SL}}_2^{(n)}$, we have $\operatorname{rank}\left(\nu_{\mathfrak{R}_{ie}}^{\chi}\right) = \operatorname{rank}(\mathcal{M}_{\mathfrak{R}_{ie}})$, where

$$\mathcal{M}_{\mathfrak{R}_{ie}} := \begin{pmatrix} \mathcal{W}_{\mathbf{s}_{ie}}^*(\mathbf{s}_{ie}) & \mathcal{W}_{\mathbf{s}_{ie}}^*\left(\mathbf{s}_{(2-i)e}\right) \\ \mathcal{W}_{\mathbf{s}_{(2-i)e}}^*(\mathbf{s}_{ie}) & \mathcal{W}_{\mathbf{s}_{(2-i)e}}^*\left(\mathbf{s}_{(2-i)e}\right) \end{pmatrix}.$$

  Again, since ${}^{w_\alpha}\chi = \chi$ and $\chi_\alpha = -1$, it follows from equation (6.1) that

$$\mathcal{M}_{\mathfrak{R}_{ie}} = \begin{pmatrix} q^{(i-2)/2} & q^{-(2+i)/2}\mathbf{g}_{\psi^{-1}}((1-i)4) \\ q^{i/2}\mathbf{g}_{\psi^{-1}}((i-1)4) & q^{-(2+i)/2} \end{pmatrix}.$$

  Note that we have $1 \le i-1 \le m-1$ and thus $\det(\mathcal{M}_{\mathfrak{R}_{ie}}) = 0$. Clearly, this implies that $\operatorname{rank}(\mathcal{M}_{\mathfrak{R}_{ie}}) = 1$.

Combining the foregoing gives $\dim \mathrm{Wh}_\psi\left(\pi_\chi^{un}\right) = m$. It follows from this example of $\overline{\mathrm{SO}}_3^{(n)}$ that a naive analogue of Conjecture 5.7 does not hold for coverings of a general semisimple group. Here the difference between $\mathscr{X}_{Q,n}^{\mathrm{exc}}$ and $\left(\mathscr{X}_{Q,n}\right)^W$ plays a sensitive role and accounts for the failure. Indeed, in the case of $\overline{\mathrm{SO}}_3^{(n)}$, we have

$$\left|\mathscr{X}_{Q,n}^{\mathrm{exc}}\right| = 1 \text{ and } \left|\left(\mathscr{X}_{Q,n}\right)^W\right| = 2.$$

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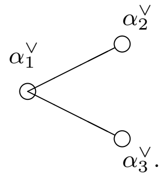
8.2. Double cover of Spin_6

We consider in this subsection only the double cover of $\mathrm{Spin}_6 \simeq \mathrm{SL}_4$, though the phenomenon appears for general $2m$-fold covers of Spin_{2k} with m and k being both odd. For this reason, we would like to consider the situation from the perspective of spin groups.

Consider the Dynkin diagram of simple coroots for the simply connected $G = \mathrm{Spin}_6$:



Let Q be the Weyl-invariant quadratic form Q of Y such that $Q(\alpha_i^\vee) = 1$ for all $1 \leq i \leq 3$. Let $\overline{G}$ be the double cover of G arising from Q. We have $Y_{Q,2}^{sc} = 2 \cdot Y$ and

$$Y_{Q,2} = \left\{ \sum_{i=1}^{3} y_i \alpha_i^\vee : 2|y_i \text{ for all } i \text{ and } 2(y_1 + y_2 + y_3) \right\}.$$

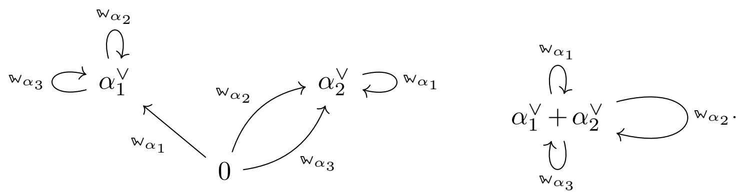
Thus we have

$$\overline{G}^\vee = \mathrm{SO}_6,$$

and the principal endoscopic group H for $\overline{G}$ is SO_6. We have

$$\mathscr{X}_{Q,2} = \{0, \alpha_1^\vee, \alpha_2^\vee, \alpha_1^\vee + \alpha_2^\vee\}.$$

Note that $\alpha_2^\vee = \alpha_3^\vee \in \mathscr{X}_{Q,n}$. There are two W-orbits of $\mathscr{X}_{Q,2}$ represented by the following graph:



We have $\mathscr{X}_{Q,n} = \mathcal{O}_0 \cup \mathcal{O}_{\alpha_1^\vee + \alpha_2^\vee}$.

It then follows from [22, Theorme 6.8] that the only nontrivial unramified R_χ is $\{1, \mathbb{w} = \mathbb{w}_{\alpha_{r-1}} \mathbb{w}_{\alpha_r}\}$, with

$$\chi_{\alpha_2} = \chi_{\alpha_3} = -1.$$

A direct computation using equations (5.2) and (5.3) and Theorem 5.2 gives

$$\begin{aligned} \tau(w, \chi, \mathbf{s}_0, \mathbf{s}_0) + \tau\left(w, \chi, \mathbf{s}_{\alpha_2^\vee}, \mathbf{s}_{\alpha_2^\vee}\right) &= \gamma(w, \chi) - \gamma(w, \chi) = 0 \\ \tau\left(w, \chi, \mathbf{s}_{\alpha_1^\vee}, \mathbf{s}_{\alpha_1^\vee}\right) &= \gamma(w, \chi) \\ \tau\left(w, \chi, \mathbf{s}_{\alpha_1 + \alpha_2^\vee}, \mathbf{s}_{\alpha_1 + \alpha_2^\vee}\right) &= -\gamma(w, \chi). \end{aligned}$$

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Denoting $\mathrm{Irr}(R_\chi) = \{\mathbb{1}, \varepsilon\}$, it then follows that

$$\sigma^{\mathrm{Wh}}_{\mathcal{O}_0} = (2 \cdot \mathbb{1}) \oplus \varepsilon, \qquad \sigma^{\mathrm{Wh}}_{\mathcal{O}_{\alpha_1^\vee + \alpha_2^\vee}} = \varepsilon.$$

In particular, writing $I(\chi) = \pi_\chi^{un} \oplus \pi$, we have

$$\dim \mathrm{Wh}_\psi\left(\pi_\chi^{un}\right) = 2 \text{ and } \dim \mathrm{Wh}_\psi(\pi) = 2.$$

On the other hand, it is clear from the diagram that

$$\sigma^{\mathscr{X}}_{\mathcal{O}_0} = 3 \cdot \mathbb{1}, \qquad \sigma^{\mathscr{X}}_{\mathcal{O}_{\alpha_1^\vee + \alpha_2^\vee}} = \mathbb{1}.$$

We see that the analogous Conjecture 5.7 does not hold in this case. The constraint that $\overline{G}^\vee$ be of adjoint type seems to be necessary.

For a low-rank group and 'small' R_χ, Conjecture 5.7 should be computable and explicitly verifiable. However, for general n-fold covers of a simply connected group G, in view of the difficulty highlighted in Remark 7.2, it is desirable to approach the problem from a more uniform and conceptual perspective. In any case, we will leave the investigation of this to a future work, as a continuation of the present paper.

Acknowledgments The author is very grateful to the referee for a careful reading and many helpful comments on an earlier version of the paper.

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