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APPROXIMATIONS OF SUBHOMOGENEOUS ALGEBRAS

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Abstract

Let n be a positive integer. A C^*-algebra is said to be n-subhomogeneous if all its irreducible representations have dimension at most n. We give various approximation properties characterising n-subhomogeneous C^*-algebras.

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1. Introduction

Let A and B be C^*-algebras and let $\phi\colon A \to B$ be a bounded linear map. For each integer $n \geq 1$, we can define maps

$$\phi \otimes \mathrm{id}_{\mathbb{M}_n} : A \otimes \mathbb{M}_n \to B \otimes \mathbb{M}_n,$$

where $\mathbb{M}_n$ denotes the C^*-algebra of $n \times n$ matrices. We say that ϕ is *n-positive* if $\phi \otimes \mathrm{id}_{\mathbb{M}_n}$ is positive and *n-contractive* if $\phi \otimes \mathrm{id}_{\mathbb{M}_n}$ is contractive. We say that a map is *completely positive (completely contractive)* if it is n-positive (n-contractive) for all $n \geq 1$. As usual, we abbreviate *completely positive* by c.p., *contractive and completely positive* by c.c.p., *unital and completely positive* by u.c.p. and *completely contractive* by c.c. Note that u.c.p. maps are c.c.p. and c.c.p. maps are c.c. by the Stinespring dilation theorem [10, Theorem 1].

Finite-dimensional approximation properties of maps and C^*-algebras play an important role in the study of C^*-algebras (see [2] for a comprehensive treatment).

Definition 1.1. A c.c.p. map $\theta\colon A \to B$ is said to be *nuclear* if there exist *finite-dimensional C^*-algebras F_α* and nets of c.c.p. maps $\phi_\alpha : A \to F_\alpha$ and $\psi_\alpha : F_\alpha \to B$ such that for all $x \in A$,

$$\|(\theta - \psi_\alpha \circ \phi_\alpha)(x)\| \to 0 \quad \text{as } \alpha \to \infty.$$

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Definition 1.2. A C^*-algebra A is said to be *nuclear* if the identity map $\mathrm{id}_A\colon A \to A$ is nuclear and *exact* if there exists a faithful representation $\pi\colon A \to B(H)$ which is nuclear.

The following is the standard example.

Example 1.3. Let Γ be a countable discrete group. Then the *reduced group C^*-algebra* $C^*_\lambda(\Gamma)$ is nuclear if and only if Γ is amenable. In particular, the reduced group C^*-algebra $C^*_\lambda(F_2)$ of a free group on two generators is non-nuclear (see [2, Section 2.6]).

It is well known that a C^*-algebra is nuclear if and only if the identity map is a point-norm limit of finite-rank c.c.p. maps. On the other hand, it was shown by De Cannière and Haagerup [4, Corollary 3.11] that the identity map on $C^*_\lambda(F_2)$ is a point-norm limit of finite-rank c.c. maps. This is in contrast to the following theorem of Smith, which says that we recover nuclearity if we insist that the finite-rank c.c. maps factor through finite-dimensional C^*-algebras.

Theorem 1.4 (Smith [9]). *A C^*-algebra A is nuclear if and only if there exist* finite-dimensional C^*-algebras F_α *and nets of c.c. maps $\phi_\alpha\colon A \to F_\alpha$ and $\psi_\alpha\colon F_\alpha \to A$ such that for all $x \in A$,*

$$\|(\mathrm{id}_A - \psi_\alpha \circ \phi_\alpha)(x)\| \to 0 \quad \textit{as } \alpha \to \infty.$$

All *abelian* C^*-algebras are nuclear. In fact, the standard proof based on partition of unities shows that one can take the finite-dimensional C^*-algebras F_α to be abelian and the c.c.p. maps ϕ_α to be $*$-homomorphisms (see [2, Proposition 2.4.2]).

Our investigation grew out of the following simple question.

Question 1.5. Suppose that the there exist finite-dimensional *abelian* C^*-algebras F_α and nets of c.c.p. maps $\phi_\alpha\colon A \to F_\alpha$ and $\psi_\alpha\colon F_\alpha \to A$ such that for all $x \in A$,

$$\|(\mathrm{id}_A - \psi_\alpha \circ \phi_\alpha)(x)\| \to 0 \quad \text{as } \alpha \to \infty.$$

Can we conclude that A is abelian? Can we still conclude that A is abelian if we assume that the maps ϕ_α and ψ_α are only c.c.?

Not surprisingly, the answer is positive. In this paper we prove the following result.

Definition 1.6. Let $n \geq 1$. A C^*-algebra is said to be *n-subhomogeneous* if all of its irreducible representations have dimension $\leq n$.

Clearly, a C^*-algebra is abelian if and only if it is 1-subhomogeneous. A finite-dimensional C^*-algebra is n-subhomogeneous if and only if it is a finite product of matrix algebras $\mathbb{M}_k$ of size $k \leq n$.

Theorem 1.7. *Let A be a C^*-algebra and let $n \geq 1$ be an integer. Then the following are equivalent.*

(i) *The C^*-algebra A is n-subhomogeneous.*

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(ii) *There exist nets of* $*$-homomorphisms $\phi_\alpha\colon A \to F_\alpha$ *and c.c.p. maps* $\psi_\alpha\colon F_\alpha \to A$, *with* F_α *finite dimensional and n-subhomogeneous, such that for all* $x \in A$,

$$\|x - \psi_\alpha \circ \phi_\alpha(x)\| \to 0 \quad \textit{as } \alpha \to \infty.$$

(iii) *There exist nets of c.c.p. maps* $\phi_\alpha\colon A \to F_\alpha$ *and* $\psi_\alpha\colon F_\alpha \to A$, *with* F_α *(finite dimensional and) n-subhomogeneous, such that for all* $x \in A$,

$$\|x - \psi_\alpha \circ \phi_\alpha(x)\| \to 0 \quad \textit{as } \alpha \to \infty.$$

(iv) *There exist nets of c.c. maps* $\phi_\alpha\colon A \to F_\alpha$ *and* $\psi_\alpha\colon F_\alpha \to A$, *with* F_α *(finite dimensional and) n-subhomogeneous, such that for all* $x \in A$,

$$\|x - \psi_\alpha \circ \phi_\alpha(x)\| \to 0 \quad \textit{as } \alpha \to \infty.$$

Proof. The nontrivial implications are (i) ⇒ (ii), (iii) ⇒ (i) and (iv) ⇒ (i). See Theorem 1.8 below. □

Our proof is based on the solution of the Choi conjecture [3], due to Tomiyama [12] and Smith [8], and a contractive analogue of the Choi conjecture (see Theorem 2.13). See also [5, 6].

The following is a summary of the results.

Theorem 1.8. *Let A be a* C^**-algebra and let* $n \geq 1$ *be an integer. Then the following are equivalent.*

(i) *The* C^**-algebra A is n-subhomogeneous.*

(ii) *There exist nets of* $*$-homomorphisms $\phi_\alpha\colon A \to F_\alpha$ *and c.c.p. maps* $\psi_\alpha\colon F_\alpha \to A$, *with* F_α *finite dimensional and n-subhomogeneous, such that for all* $x \in A$,

$$\|x - \psi_\alpha \circ \phi_\alpha(x)\| \to 0 \quad \textit{as } \alpha \to \infty.$$

(iii) *There exist nets of n-positive maps* $\phi_\alpha\colon A \to F_\alpha$ *and* $\psi_\alpha\colon F_\alpha \to A$, *with* F_α *finite dimensional and n-subhomogeneous, such that for all* $x \in A$,

$$\|x - \psi_\alpha \circ \phi_\alpha(x)\| \to 0 \quad \textit{as } \alpha \to \infty.$$

(iv) *All n-positive maps with domain and/or range A are completely positive.*

(v) *There exist nets of n-contractive maps* $\phi_\alpha\colon A \to F_\alpha$ *and* $(n+1)$*-contractive maps* $\psi_\alpha\colon F_\alpha \to A$, *with* F_α *finite dimensional and n-subhomogeneous, such that for all* $x \in A$,

$$\|x - \psi_\alpha \circ \phi_\alpha(x)\| \to 0 \quad \textit{as } \alpha \to \infty.$$

(vi) *All n-contractive maps with range A are completely contractive.*

Proof. See Theorems 2.7, 2.9 and 2.14. □

In Section 3, we show that an even weaker approximation property characterises abelianness. See Theorem 3.2.

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2. Subhomogeneous algebras

In the following proposition, we summarise some well-known properties of n-subhomogeneous C^*-algebras (see also [1, Subsection IV.1.4]).

PROPOSITION 2.1. *Let $n \geq 1$ be an integer. The following statements hold.*

(i) *A C^*-subalgebra of an n-subhomogeneous algebra is n-subhomogeneous.*
(ii) *A C^*-algebra A is n-subhomogeneous if and only if $A \subseteq \mathbb{M}_n(B)$ for some abelian C^*-algebra B.*
(iii) *A C^*-algebra A is n-subhomogeneous if and only if its bidual A^{**} is n-subhomogeneous as a C^*-algebra.*
(iv) *The product/sum of C^*-algebras A_i, $i \in I$, is n-subhomogeneous if and only if each A_i, $i \in I$, is n-subhomogeneous.*
(v) *Let $0 \to I \to A \to B \to 0$ be an extension of C^*-algebras. Then A is n-subhomogeneous if and only if I and B are n-subhomogeneous.*

PROOF. (i) Follows from [7, Proposition 4.1.8].
(ii) If A is n-subhomogeneous, then $A \subseteq \mathbb{M}_n(l^\infty(\widehat{A}))$, where $\widehat{A}$ denotes the set of unitary equivalence classes of irreducible representations of A. The other direction follows from (i).
(iii) Since $A \subseteq A^{**}$, if A^{**} is n-subhomogeneous, then so is A by (i). Conversely, if A is n-subhomogeneous, then writing $A \subseteq \mathbb{M}_n(B)$ with B abelian and using (ii), we see that $A^{**} \subseteq \mathbb{M}_n(B^{**})$. The assertion follows from (ii), since B^{**} is abelian.
(iv) Follows from (ii).
(v) Follows from (iii) and (iv), since $A^{**} \cong I^{**} \oplus B^{**}$. □

The structure of n-subhomogeneous C^*-algebras can be rather complicated (see, for instance, [13]). However, the situation for von Neumann algebras is well known to be very simple.

LEMMA 2.2. *Suppose that a von Neumann algebra M is n-subhomogeneous as a C^*-algebra. Then*

$$M \cong \prod_{k \leq n} \mathbb{M}_k(B_k),$$

where B_k, $k \leq n$, are abelian *von Neumann algebras.*

PROOF. Since exactness passes to C^*-subalgebras, n-subhomogeneous algebras are exact by Proposition 2.1(ii). Now [2, Proposition 2.4.9] completes the proof. □

Subhomogeneous algebras are type I and hence nuclear (see [2, Proposition 2.7.7]). Scrutinising the proof, we see that the following slightly stronger approximation property holds. We consider the unital case first.

THEOREM 2.3. *Let $n \geq 1$ and let A be a unital n-subhomogeneous C^*-algebra. Then there exist finite-dimensional n-subhomogeneous C^*-algebras F_α and nets of unital* $*$-homomorphisms *$\phi_\alpha \colon A \to F_\alpha$ and u.c.p. maps $\psi_\alpha \colon F_\alpha \to A$, such that for all $x \in A$,*

$$\|x - \psi_\alpha \circ \phi_\alpha(x)\| \to 0 \quad \text{as } \alpha \to \infty.$$

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Definition 2.4. Let A and B be unital C^*-algebras. We say that a u.c.p. map $\theta\colon A \to B$ is *n-factorable* if it can be expressed as a composition $\theta = \psi \circ \phi$, where $\phi\colon A \to F$ is a unital $*$-homomorphism, $\psi\colon F \to B$ a u.c.p. map and F a finite-dimensional n-subhomogeneous C^*-algebra.

Lemma 2.5. *For any unital C^*-algebras A and B, the set of n-factorable maps $A \to B$ is convex.*

Proof. The proof of [2, Lemma 2.3.6] applies. □

Lemma 2.6. *Let F be a finite-dimensional C^*-algebra and let A be a unital C^*-algebra. Then u.c.p. maps $F \to A^{**}$ can be approximated by u.c.p. maps $F \to A$ in the point-ultraweak topology.*

Proof. We claim that c.p. maps $F \to A$ correspond bijectively to positive elements in $F \otimes A$. Indeed, for matrix algebras this is a well-known result of Arveson (cf. [2, Proposition 1.5.12]). The general case follows, since F is a finite product of matrix algebras and for c.p. maps finite products and finite coproducts coincide. Since positive elements in $F \otimes A$ are ultraweakly dense in the positive elements in $F \otimes A^{**} \cong (F \otimes A)^{**}$, we see that c.p. maps $F \to A^{**}$ can be approximated by c.p. maps $F \to A$ in the point-ultraweak topology.

Let $\psi\colon F \to A^{**}$ be a u.c.p. map and let $\psi_\lambda\colon F \to A$ be a net of c.p. maps converging to ψ in the point-ultraweak topology. Since $\psi_\lambda(1_F) \in A$ is a net converging to 1_A weakly, by passing to convex linear combinations, we may assume that $\psi_\lambda(1_F)$ converges to 1_A in norm and passing to a subnet we may assume that $\psi_\lambda(1_F)$ is invertible. Then $\tilde{\psi}_\lambda(x) := \psi_\lambda(1_F)^{-1/2}\psi_\lambda(x)\psi_\lambda(1_F)^{-1/2}$, $x \in F$, gives the required approximation. □

Proof of Theorem 2.3. For $n = 1$ and A unital abelian, the claim follows from the classical proof of nuclearity for abelian algebras (see [2, Proposition 2.4.2]).

For general n, first assume that A is of the form

$$\prod_{k\leq n} \mathbb{M}_k(A_k), \tag{2.1}$$

where A_k, $k \leq n$, are unital abelian C^*-algebras. Then the claim is easily deduced from the case $n = 1$.

Now we consider a general n-subhomogeneous A. By Proposition 2.1(iii) and Lemma 2.2, the bidual A^{**} is of the form (2.1) and hence $\mathrm{id}_{A^{**}}$ can be approximated by n-factorable maps $A^{**} \to A^{**}$ in the point-norm topology. Then, by Lemma 2.6, id_A can be approximated by n-factorable maps in the point-weak topology. Now Lemma 2.5 and [2, Lemma 2.3.4] complete the proof. □

As a corollary, we obtain the following result.

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THEOREM 2.7. *Let A be a C^*-algebra and let $n \geq 1$ be an integer. Then A is n-subhomogeneous if and only if there exist nets of* $*$-homomorphisms $\phi_\alpha : A \to F_\alpha$ *and c.c.p. maps* $\psi_\alpha : F_\alpha \to A$*, with F_α finite-dimensional n-subhomogeneous, such that for all $x \in A$,*

$$\|x - \psi_\alpha \circ \phi_\alpha(x)\| \to 0 \quad \text{as } \alpha \to \infty.$$

PROOF. ($\Rightarrow$) The unitisation A^+ is n-subhomogeneous, so id_{A^+} can be approximated as in Theorem 2.3. Now restrict ϕ_α to A and replace ψ_α by $e_\beta \psi_\alpha e_\beta$, where e_β is an approximate unit in A (see [2, Exercise 2.3.4]).

($\Leftarrow$) Clearly A is a C^*-subalgebra of $\prod_\alpha F_\alpha$. By Proposition 2.1(iv) and (i), A is n-subhomogeneous. □

It turns out that much weaker approximation properties imply n-subhomogeneity. Our first result depends on the following theorem.

THEOREM 2.8 (Choi, Tomyama, Smith). *Let A and B be C^*-algebras and let $n \geq 1$ be an integer. Then all n-positive maps $A \to B$ are completely positive if and only if A or B is n-subhomogeneous.*

PROOF. Choi proved the sufficiency ($\Leftarrow$) for $A = \mathbb{M}_n(D)$ (see [3, Theorem 8]) and $B = \mathbb{M}_n(D)$ (see [3, Theorem 7]) with D abelian and conjectured the necessity ($\Rightarrow$). A complete proof was obtained by Tomiyama (see [12, Theorem 1.2]). The necessity was also proved by Smith (see [8, Theorem 3.1]). □

THEOREM 2.9. *Let A be a C^*-algebra and let $n \geq 1$ be an integer. Then the following are equivalent.*

(i) *There exist nets of n-positive maps $\phi_\alpha : A \to F_\alpha$ and $\psi_\alpha : F_\alpha \to A$, with F_α finite-dimensional n-subhomogeneous, such that for all $x \in A$,*

$$\|x - \psi_\alpha \circ \phi_\alpha(x)\| \to 0 \quad \text{as } \alpha \to \infty.$$

(ii) *All n-positive maps with domain A are completely positive.*
(iii) *All n-positive maps with range A are completely positive.*
(iv) *All n-positive maps $A \to A$ are completely positive.*
(v) *The C^*-algebra A is n-subhomogeneous.*

PROOF. Let $\phi_\alpha : A \to F_\alpha$ and $\psi_\alpha : F_\alpha \to A$ be an n-positive approximation of id_A in the point-norm topology, with F_α (finite-dimensional) n-subhomogeneous. Let $\theta : A \to B$ be an n-positive map. Then $\theta \circ \psi_\alpha : F_\alpha \to B$ is an n-positive map with n-subhomogeneous domain and hence a c.p. map by Theorem 2.8 and $\phi_\alpha : A \to F_\alpha$ is an n-positive map with n-subhomogeneous range and hence also c.p. Since θ is the point-norm limit of $(\theta \circ \psi_\alpha) \circ \phi_\alpha$, we see that θ is c.p. Hence, (i) $\Rightarrow$ (ii). Similarly, (i) $\Rightarrow$ (iii).

The implications (ii) $\Rightarrow$ (iv) and (iii) $\Rightarrow$ (iv) are clear and the implication (iv) $\Rightarrow$ (v) is immediate from Theorem 2.8. Finally, the implication (v) $\Rightarrow$ (i) follows from Theorem 2.7. □

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REMARK 2.10. The sufficiency in Theorem 2.8 can be deduced from the cases $A = \mathbb{M}_n$ (see [3, Theorem 6]) and $B = \mathbb{M}_n$ (see [3, Theorem 5]) using Theorem 2.9.

Now we consider the contractive analogue.

LEMMA 2.11. *Let $\tau_n : \mathbb{M}_n \to \mathbb{M}_n$, $n \geq 1$, denote the transpose map and let $m \geq 1$. Then*

$$\|\tau_n \otimes \mathrm{id}_{\mathbb{M}_m} : \mathbb{M}_n \otimes \mathbb{M}_m \to \mathbb{M}_n \otimes \mathbb{M}_m\| = \min\{m, n\}.$$

PROOF. For $n \leq m$, this is well known. The general case follows from the identity

$$(\tau_n \otimes \tau_m) \circ (\tau_n \otimes \mathrm{id}_{\mathbb{M}_m}) = \mathrm{id}_{\mathbb{M}_n} \otimes \tau_m,$$

since $\tau_n \otimes \tau_m$ can be identified with τ_{nm} and hence is an isometry. □

COROLLARY 2.12. *Let $n \geq 2$ be an integer. Then the map*

$$\frac{1}{n-1}\tau_n : \mathbb{M}_n \to \mathbb{M}_n$$

is $(n-1)$-contractive, but not n-contractive.

As a corollary, we obtain the following contractive analogue of Theorem 2.8. Note that we have only one of the directions (see [6, Theorem C]).

THEOREM 2.13. *Let A and B be C^*-algebras and let $n \geq 1$ be an integer. If A and B both admit irreducible representations of dimension $\geq (n+1)$, then there exists an n-contractive map $A \to B$ which is not $(n+1)$-contractive.*

PROOF. The proof of [8, Theorem 3.1] applies. See also [12, Lemma 1.1 and Theorem 1.2]. □

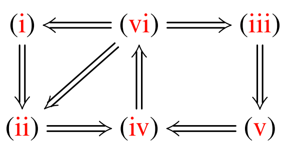
THEOREM 2.14. *Let A be a C^*-algebra and let $n \geq 1$ be an integer. Then the following are equivalent.*

(i) *There exist nets of n-contractive maps $\phi_\alpha : A \to F_\alpha$ and $(n+1)$-contractive maps $\psi_\alpha : F_\alpha \to A$, with F_α finite-dimensional n-subhomogeneous, such that for all $x \in A$,*

$$\|x - \psi_\alpha \circ \phi_\alpha(x)\| \to 0 \quad \text{as } \alpha \to \infty.$$

(ii) *All n-contractive maps with range A are $(n+1)$-contractive.*
(iii) *All n-contractive maps with range A are completely contractive.*
(iv) *All n-contractive maps $A \to A$ are $(n+1)$-contractive.*
(v) *All n-contractive maps $A \to A$ are completely contractive.*
(vi) *The C^*-algebra A is n-subhomogeneous.*

PROOF. We prove the implications



The implications (iii) $\Rightarrow$ (v), (v) $\Rightarrow$ (iv) and (ii) $\Rightarrow$ (iv) are clear. The implication (iv) $\Rightarrow$ (vi) follows from Theorem 2.13 and the implication (vi) $\Rightarrow$ (i) follows from Theorem 2.7. The implication (vi) $\Rightarrow$ (iii) follows from [8, Theorem 2.10]. Since (iii) $\Rightarrow$ (ii) is clear, we also have (vi) $\Rightarrow$ (ii).

Finally, the implication (i) $\Rightarrow$ (ii) is analogous to the proof of Theorem 2.9((i) $\Rightarrow$ (iii)). □

Compare with the Loebl conjecture [6], solved affirmatively by Huruya and Tomiyama [5] and Smith [8].

Remark 2.15. Note that the statement

(vii) All n-contractive maps with *domain* A are $(n+1)$-contractive

is *not* equivalent to the conditions in Theorem 2.14 in general (see [6, Theorem C]).

3. The abelian case

Specialising to $n = 1$ in Theorem 2.9, we obtain the following result.

Theorem 3.1. *Let A be a C^*-algebra. Suppose that there exist nets of contractive positive maps $\phi_\alpha\colon A \to F_\alpha$ and $\psi_\alpha\colon F_\alpha \to A$, with F_α* abelian, *such that for all $x \in A$,*

$$\|x - \psi_\alpha \circ \phi_\alpha(x)\| \to 0 \quad \text{as } \alpha \to \infty.$$

Then A is abelian.

We give an alternative proof.

Proof. First note that ϕ_α and ψ_α are c.c.p. (see [10, Theorems 3 and 4]).

Unitising if necessary, we may assume that A is unital. Let A^{opp} denote the opposite algebra of A. Then the canonical map $\iota\colon A \to A^{\mathrm{opp}}$ is a pointwise limit of c.c.p. maps $\psi_\alpha^{\mathrm{opp}} \circ \phi_\alpha\colon A \to F_\alpha \cong F_\alpha^{\mathrm{opp}} \to A^{\mathrm{opp}}$ and hence a c.c.p. map. Moreover, since ι sends unitaries to unitaries, its multiplicative domain is the whole of A. It follows that ι is a $*$-homomorphism and A is abelian. Alternatively, we may use Walter's 3×3 trick to conclude that A is abelian (cf. [14]). □

In fact, the following is true.

Theorem 3.2. *Let $\theta\colon A \to B$ be an* injective *$*$-homomorphism. Suppose that there exist nets of contractive maps $\phi_\alpha\colon A \to F_\alpha$ and 2-contractive maps $\psi_\alpha\colon F_\alpha \to B$, with F_α* abelian, *such that for all $x \in A$,*

$$\|(\theta - \psi_\alpha \circ \phi_\alpha)(x)\| \to 0 \quad \text{as } \alpha \to \infty.$$

Then A is abelian.

Our main tool is the following beautiful theorem of Takesaki. Let A_1 and A_2 be C^*-algebras. The *injective cross-norm* of A_1 and A_2 is defined by

$$\|x\|_\lambda := \sup |(\varphi_1 \otimes \varphi_2)(x)|,$$

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where φ_1 and φ_2 run over all contractive linear functionals of A_1 and A_2, respectively. The *injective C^*-cross-norm* of A_1 and A_2 is defined by

$$\|x\|_{\min} := \sup \|(\pi_1 \otimes \pi_2)(x)\|,$$

where π_1 and π_2 run over all unitary representations of A_1 and A_2, respectively.

Note that we always have $\|\cdot\|_\lambda \le \|\cdot\|_{\min}$ (see [11, Section IV.4, Inequality (12)]).

Theorem 3.3 (Takesaki [11, Theorem IV.4.14]). *Let A_1 and A_2 be C^*-algebras. Then the norms $\|\cdot\|_{\min}$ and $\|\cdot\|_\lambda$ on $A_1 \otimes A_2$ are equal if and only if A_1 or A_2 is abelian.* □

Equipped with Takesaki's theorem, we can now mimic the proof that nuclear C^*-algebras are tensor-nuclear (see [2, Proposition 3.6.12]).

Proof of Theorem 3.2. We show that for any $x \in A \otimes \mathbb{M}_2$, we have $\|x\|_{\min} \le \|x\|_\lambda$. Then Theorem 3.3 completes the proof.

Let $x \in A \otimes \mathbb{M}_2$. The map

$$\theta \otimes_{\min} \mathrm{id}_{\mathbb{M}_2} : A \otimes_{\min} \mathbb{M}_2 \to B \otimes_{\min} \mathbb{M}_2$$

is an injective $*$-homomorphism and hence an isometry. Thus,

$$\|x\|_{A\otimes_{\min}\mathbb{M}_2} = \|\theta \otimes \mathrm{id}_{\mathbb{M}_2}(x)\|_{B\otimes_{\min}\mathbb{M}_2}.$$

Writing x as the sum of elementary tensors,

$$\|(\theta - \psi_\alpha \circ \phi_\alpha) \otimes \mathrm{id}_{\mathbb{M}_2}(x)\|_{B\otimes_{\min}\mathbb{M}_2} \to 0 \quad \text{as } \alpha \to \infty.$$

Hence,

$$\|x\|_{A\otimes_{\min}\mathbb{M}_2} = \lim_{n\to\infty} \|(\psi_\alpha \circ \phi_\alpha) \otimes \mathrm{id}_{\mathbb{M}_2}(x)\|_{B\otimes_{\min}\mathbb{M}_2}.$$

On the other hand, it follows from the assumptions that the maps

$$\phi_\alpha \otimes_\lambda \mathrm{id}_{\mathbb{M}_2} : A \otimes_\lambda \mathbb{M}_2 \to F_\alpha \otimes_\lambda \mathbb{M}_2,$$
$$\psi_\alpha \otimes_{\min} \mathrm{id}_{\mathbb{M}_2} : F_\alpha \otimes_{\min} \mathbb{M}_2 \to B \otimes_{\min} \mathbb{M}_2$$

are contractions and, since F_α is abelian, the canonical map

$$F_\alpha \otimes_{\min} \mathbb{M}_2 \to F_\alpha \otimes_\lambda \mathbb{M}_2$$

is an isometry by Theorem 3.3. Hence,

$$\begin{aligned}\|(\psi_\alpha \circ \phi_\alpha) \otimes \mathrm{id}_{\mathbb{M}_2}(x)\|_{B\otimes_{\min}\mathbb{M}_2} &\le \|\phi_\alpha \otimes \mathrm{id}_{\mathbb{M}_2}(x)\|_{F_\alpha\otimes_{\min}\mathbb{M}_2}\\ &= \|\phi_\alpha \otimes \mathrm{id}_{\mathbb{M}_2}(x)\|_{F_\alpha\otimes_\lambda\mathbb{M}_2}\\ &\le \|x\|_{A\otimes_\lambda\mathbb{M}_2}.\end{aligned}$$

It follows that $\|x\|_{\min} \le \|x\|_\lambda$. □

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