GAPS IN TAYLOR SERIES OF ALGEBRAIC FUNCTIONS

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Abstract

Let f be a rational function on an algebraic curve over the complex numbers. For a point p and local parameter x we can consider the Taylor series for f in the variable x. In this paper we give an upper bound on the frequency with which the terms in the Taylor series have 0 as their coefficient.

1. Introduction

Let f be an algebraic function over the complex numbers and suppose that in local coordinates f can be represented by a Taylor series of the form

$$f = \sum_{n=0}^{\infty} \alpha_n x^{a_n},$$

where the a_n are a strictly increasing sequence of non-negative integers and the α_n are nonzero complex numbers. Our goal is to understand how large the average gap between consecutive a_n can be.

In [2], Christol *et al*. prove an equivalency between formal algebraic power series over finite fields and automatic sequences. An overview of this theorem and some of its generalisations can be found in [5]. These results have been extended to arbitrary fields of positive characteristic by Adamczewski and Bell in [1]. For algebraic functions over the complex numbers there are some classic results from the field of complex analysis. For instance, the Fabry gap theorem [6] states that if

$$\lim_{n \to \infty} \frac{a_n}{n} = \infty$$

then f cannot be analytically continued to any point beyond the domain of convergence. In particular, f is not algebraic. This is in contrast to the positive characteristic case wherein the a_n can grow exponentially fast.

In [3] Chudnovsky and Chudnovsky prove a version of Roth's theorem for functions satisfying a linear differential equation with rational function coefficients. A consequence of their version of Roth's theorem is that there exists an effectively computable constant C such that $a_{n+1} - a_n < C$ for all n. Since every algebraic function satisfies a linear differential equation with rational function coefficients (see [4]), the same result holds for algebraic functions. However, in order to establish this bound one must first determine the linear differential equation satisfied by f, or bound the height of the rational function coefficients and the order of the differential equation. The constant C is then determined through the examination of a Wronskian. In practice this bound can be quite large.

On the other hand, if $f = p(x)/q(x)$ is rational, the gap sequences are fully understood. For sufficiently large n the coefficients of the Taylor series of f satisfy a linear homogeneous recurrence relation of degree $\deg(q(x))$. Consequently, the length of the gaps in the Taylor series is eventually bounded by $\deg(q(x)) - 1$. With a little work it can be shown that the gaps are bounded by the number of distinct zeros of $q(x)$. An even stronger conclusion is given by the Skolem–Mahler–Lech theorem [7], which states that the zero terms of the Taylor series of a rational function consist of finitely many arithmetic progressions plus a finite set. Therefore $\lim_{n\to\infty} a_n/n$ exists and is effectively bounded by the number of poles of f.

Our goal in this paper is to establish a bound for algebraic functions that is analogous in its simplicity to the rational function case. In particular, we show that $\limsup a_n/n$ is effectively bounded by the number of distinct poles of f plus a term corresponding to the choice of local coordinates. This is a consequence of Theorem 2.1 which provides a bound for each a_n in terms of similar data.

2. Main theorem

Throughout this section we will let C be a smooth complete algebraic curve of genus g over $\mathbb{C}$ and K be the function field of C. At each point $q \in C$ we will denote by v_q the unique surjective discrete valuation at q, $v_q : K^\times \to \mathbb{Z}$. Recall that for $f \in K^\times$ the height of f is defined to be

$$h(f) = -\sum_{q\in C} \min\{v_q(f), 0\}$$

and has the property that $v_q(f) \le h(f)$ for any $q \in C$. We now state our main theorem.

THEOREM 2.1 (Main theorem). *Let $f \in K$ be a rational function and $x \in K$ a local parameter at $p \in C$ such that $f \notin \mathbb{C}[x]$ and $v_p(f) = 0$. In the completion of the local ring at p we can uniquely write*

$$f = \sum_{n=0}^{\infty} \alpha_n x^{a_n},$$

where $\alpha_n \neq 0$ for all n and $\{a_n\}$ is a sequence of strictly increasing integers with $a_0 = 0$. Then for all positive integers n,

$$a_n \leq h(f) + (n-1)\Big(\#S_1 + \sum_{q \in S_2}\Big(v_q\Big(\frac{dx}{dx_q}\Big) + 1\Big)\Big), \tag{2.1}$$

where

$$S_1 = \{q \in C : v_q(f) < 0 \text{ and } v_q(dx/dx_q) = 0\},$$
$$S_2 = \{q \in C : v_q(x) = 0 \text{ and } v_q(dx/dx_q) > 0\},$$

and x_q is a local parameter at q.

As an immediate consequence of the above theorem we get an effective bound on $\limsup_{n \to \infty} a_n/n$, namely

$$\limsup_{n \to \infty} \frac{a_n}{n} \leq \#S_1 + \sum_{q \in S_2}\Big(v_q\Big(\frac{dx}{dx_q}\Big) + 1\Big).$$

Since S_1 is a subset of the set of poles of f, the limit superior is bounded by the number of poles of f plus a term corresponding to the choice of local coordinates.

In the case where C has genus 0, Theorem 2.1 is sharp.

EXAMPLE 2.2. Let $C = \mathbb{P}^1_{\mathbb{C}}$ with $K = \mathbb{C}(x)$ and x our local parameter. Under this hypothesis S_2 is empty and (2.1) simplifies to

$$a_n \leq h(f) + (n-1)\#S_1.$$

For each pair of positive integers k, m with $k > m$ we define the rational function

$$f = 1 + \frac{x^k}{1 - x^m} = 1 + \sum_{n=1}^{\infty} x^{(n-1)m+k}.$$

By construction $a_n = k + (n-1)m$ for all $n \geq 1$. However, $\#S_1 = m$ and $h(f) = k$, therefore $a_n = h(f) + (n-1)\#S_1$ as desired.

In the case where the genus is positive, it is not clear whether Theorem 2.1 is sharp. No examples demonstrating sharpness are known to the author. In particular, the term corresponding to the choice of local coordinates may be too large.

To prove Theorem 2.1 we begin by constructing auxiliary rational functions that vanish to order a_n for each positive integer n.

LEMMA 2.3. *Let $x, f \in K$ and $p \in C$ be as in Theorem 2.1. Then for every positive integer n there exist constants $c_i \in \mathbb{C}$, for $i = 0, \ldots, n$, such that*

$$v_p(c_0 + c_1 f + c_2 x f' + c_3 x^2 f'' + \cdots + c_n x^{n-1} f^{(n-1)}) = a_n,$$

where all derivatives are taken with respect to x.

PROOF. We begin by considering the $(n + 1) \times (n + 1)$ matrix

$$A = \begin{pmatrix} 1 & \alpha_0 & 0 & \cdots & 0 \\ 0 & \alpha_1 x^{a_1} & \alpha_1(a_1)_1 x^{a_1} & \cdots & \alpha_1(a_1)_{n-1} x^{a_1} \\ 0 & \alpha_2 x^{a_2} & \alpha_2(a_2)_1 x^{a_2} & \cdots & \alpha_2(a_2)_{n-1} x^{a_2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \alpha_n x^{a_n} & \alpha_n(a_n)_1 x^{a_n} & \cdots & \alpha_n(a_n)_{n-1} x^{a_n} \end{pmatrix},$$

where the ith column, for $i = 2, \ldots, n + 1$, is the first $n + 1$ monomials of the power series expansion of $x^{i-1} f^{(i-1)}$. Similarly, we define B to be the $n \times (n + 1)$ matrix consisting of the first n rows of A. Since B represents an underdetermined system there exists some nonzero vector c such that $Bc = 0$. We claim that the components of c are the c_i in the conclusion of our theorem. All that remains to be shown is that $Ac \neq 0$, guaranteeing that the x^{a_n} term does not vanish in this sum. Let $A_{1,1}$ be the submatrix of A after removing the first row and column. It suffices to show that $\det(A_{1,1}) \neq 0$. After factoring out x^{i-1} from the ith column of $A_{1,1}$, for $i = 1, \ldots, n$,

$$\det(A_{1,1}) = x^{n(n-1)/2} \det \begin{pmatrix} \alpha_1 x^{a_1} & \alpha_1(a_1)_1 x^{a_1-1} & \cdots & \alpha_1(a_1)_{n-1} x^{a_1-(n-1)} \\ \alpha_2 x^{a_2} & \alpha_2(a_2)_1 x^{a_2-1} & \cdots & \alpha_2(a_2)_{n-1} x^{a_2-(n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_n x^{a_n} & \alpha_n(a_n)_1 x^{a_n-1} & \cdots & \alpha_n(a_n)_{n-1} x^{a_n-(n-1)} \end{pmatrix}.$$

However, the latter determinant is the Wronskian $W(\alpha_1 x^{a_1}, \alpha_2 x^{a_2}, \ldots, \alpha_n x^{a_n})$. Since the entries of this Wronskian are monomials of strictly increasing degree they must be linearly independent. Therefore we conclude that $\det(A_{1,1}) \neq 0$ and $Ac \neq 0$. □

We now proceed to bound from below the valuation of the derivatives of f with respect to x.

PROPOSITION 2.4. *Let $f, x \in K$ be nonconstant and $q \in C$ be arbitrary. Then for any non-negative integer n,*

$$v_q\left(\frac{d^n f}{dx^n}\right) \geq v_q(f) - n\left(v_q\left(\frac{dx}{dx_q}\right) + 1\right),$$

where x_q is a local parameter at q.

PROOF. If $d^n f / dx^n$ is identically 0 then the inequality trivially holds. Without further mention we will assume that all valuations are finite. We proceed by induction. If $n = 0$ the result is immediate. Suppose now that $v_q(d^n f/dx^n) \geq v_q(f) - n(v_q(dx/dx_q) + 1)$ for some $n \geq 0$. By the chain rule

$$v_q\left(\frac{d^{n+1} f}{dx^{n+1}}\right) = v_q\left(\frac{d}{dx_q} \frac{df^n}{dx^n}\right) - v_q\left(\frac{dx}{dx_q}\right).$$

Since x_q is a local parameter at q, differentiating with respect to x_q drops the order by at most 1. Combining this observation with our induction hypothesis gives the inequality

$$v_q\left(\frac{d^{n+1}f}{dx^{n+1}}\right) \geq \left(v_q(f) - n\left(v_q\left(\frac{dx}{dx_q}\right) + 1\right) - 1\right) - v_q\left(\frac{dx}{dx_q}\right)$$

$$= v_q(f) - (n+1)\left(v_q\left(\frac{dx}{dx_q}\right) + 1\right),$$

which, by the principle of mathematical induction, completes the proof. □

REMARK 2.5. If $v_q(x) \neq 0$, then $v_q(dx/dx_q) = v_q(x) - 1$. Under this additional hypothesis Proposition 2.4 simplifies to $v_q(d^n f/dx^n) \geq v_q(f) - n v_q(x)$.

We now proceed to prove Theorem 2.1.

PROOF OF THEOREM 2.1. Let n be a positive integer. By Lemma 2.3 we can construct a rational function

$$F = c_0 + c_1 f + c_2 x f' + c_3 x^2 f'' + \cdots + c_n x^{n-1} f^{(n-1)}$$

such that $v_p(F) = a_n$. Since $a_n \leq h(F)$ it suffices to bound $h(F)$. To do so we consider four cases.

Case 1. Let $q \in S_1 = \{q \in C : v_q(f) < 0 \text{ and } v_q(dx/dx_q) = 0\}$. It follows that $v_q(x) \geq 0$, and by Proposition 2.4, for each $i = 0, \ldots, n-1$,

$$v_q\left(x^i \frac{d^i f}{dx^i}\right) \geq v_q\left(\frac{d^i f}{dx^i}\right) \geq v_q(f) - i.$$

We next bound the height of F over the set S_1:

$$-\sum_{q \in S_1} \min\{v_q(F), 0\} \leq -\sum_{q \in S_1} \min\{v_q(f), 0\} + (n-1)\#S_1.$$

Case 2. Let $q \in S_2 = \{q \in C : v_q(dx/dx_q) > 0 \text{ and } v_q(x) = 0\}$. Then by Proposition 2.4, for each $i = 0, \ldots, n-1$,

$$v_q\left(x^i \frac{d^i f}{dx^i}\right) \geq v_q(f) - i\left(v_q\left(\frac{dx}{dx_q}\right) + 1\right).$$

The above inequality gives us the following bound on the height of F over S_2:

$$-\sum_{q \in S_2} \min\{v_q(F), 0\} \leq -\sum_{q \in S_2} \min\{v_q(f), 0\} + (n-1)\sum_{q \in S_2}\left(v_q\left(\frac{dx}{dx_q}\right) + 1\right).$$

Case 3. Let $q \in S_3 = \{q \in C : v_q(f) \geq 0 \text{ and } v_q(dx/dx_q) = 0\}$. By Proposition 2.4, for each $i = 0, \ldots, n-1$,

$$v_q\left(x^i \frac{d^i f}{dx^i}\right) \geq 0.$$

Therefore the height of F over S_3 is 0.

Case 4. Let $q \in S_4 = \{q \in C : v_q(dx/dx_q) \neq 0 \text{ and } v_q(x) \neq 0\}$. By Remark 2.5, for each $i = 0, \ldots, n-1$,

$$v_q\Big(x^i \frac{d^i f}{dx^i}\Big) \geq v_q(f).$$

Finally, we bound the height of F over S_4:

$$-\sum_{q \in S_4} \min\{v_q(F), 0\} \leq -\sum_{q \in S_4} \min\{v_q(f), 0\}.$$

Combining the four cases, we obtain

$$-\sum_{q \in C} \min\{v_q(F), 0\} \leq -\sum_{q \in C} \min\{v_q(f), 0\}$$
$$+ (n-1)\#S_1 + (n-1) \sum_{q \in S_2} \Big(v_q\Big(\frac{dx}{dx_q}\Big) + 1\Big).$$

Therefore the desired inequality holds. □

The summation over S_2 in the conclusion of our theorem can be bounded above in terms of the genus and the support of x. To do so we need the following lemma.

LEMMA 2.6. *For any nonconstant $x \in K$,*

$$\sum_{q \notin \mathrm{Supp}\{x\}} v_q\Big(\frac{dx}{dx_q}\Big) = \#\mathrm{Supp}\{x\} + 2g - 2.$$

PROOF. By Riemann–Roch,

$$2g - 2 = \sum_{q \in C} v_q\Big(\frac{1}{x}\frac{dx}{dx_q}\Big)$$
$$= \sum_{q \in \mathrm{Supp}\{x\}} v_q\Big(\frac{1}{x}\frac{dx}{dx_q}\Big) + \sum_{q \notin \mathrm{Supp}\{x\}} v_q\Big(\frac{dx}{dx_q}\Big)$$
$$= \sum_{q \in \mathrm{Supp}\{x\}} -1 + \sum_{q \notin \mathrm{Supp}\{x\}} v_q\Big(\frac{dx}{dx_q}\Big),$$

which, after rearranging terms, completes the proof. □

COROLLARY 2.7. *Let $f, x \in K$ and $p \in C$ be as in the statement of Theorem 2.1. Then for all positive integers n,*

$$a_n \leq h(f) + (n-1)(\#S_1 + 2(\#\mathrm{Supp}\{x\} + 2g - 2)).$$

PROOF. Recall that $S_2 = \{q \in C : v_q(x) = 0 \text{ and } v_q(dx/dx_q) > 0\}$. Then

$$\sum_{q \in S_2} \Big(v_q\Big(\frac{dx}{dx_q}\Big) + 1\Big) \leq 2\sum_{q \in S_2} v_q\Big(\frac{dx}{dx_q}\Big)$$
$$= 2\sum_{q \notin \mathrm{Supp}\{x\}} v_q\Big(\frac{dx}{dx_q}\Big).$$

By Lemma 2.6 the right-hand side is precisely $2(\# \operatorname{Supp}\{x\} + 2g - 2)$. Substituting this value into (2.1) gives the desired result. □

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