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HOPF ALGEBRAS FROM BRANCHING RULES

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Below we work out the algebra structure of some Hopf algebras which arise concretely in restricting representations of the symmetric group to certain subgroups. The basic idea generalizes that used by Adams [**1**] for $H_*(\mathrm{BSU})$. The question arose in discussions with H. K. Farahat. I would like to thank him for his interest in the work and to acknowledge the usefulness of several stimulating conversations with him.

1. Review and statement of results. A homogeneous element of a graded abelian group will have its gradation referred to as its *dimension*. In all such groups below there will be no non-zero elements with negative or odd dimension. A graded algebra (resp. coalgebra) will be associative (resp. coassociative), strictly commutative (resp. co-commutative) and in dimension zero will be isomorphic to the ground ring F, providing the unit (resp. counit). We shall deal amost entirely with $F = \mathbf{Z}$ or $F = \mathbf{Z}/p$ for a prime p; the cases $F = \mathbf{Q}$ or a localization of $\mathbf{Z}$ will occur briefly. In every case, the component in each dimension will be a finitely generated free F-module, so dualization works simply.

The basic example is $S = \mathbf{Z}[h_1, h_2, \ldots]$ as a $\mathbf{Z}$-algebra, graded by $\dim h_n = 2n$, and made into a Hopf algebra with coalgebra structure

$$h_n \mapsto \sum_{i+j=n} h_i \otimes h_j.$$

Here $h_0 = 1$. For each $n \geqq 1$, the primitives $P_{2n}(S) \cong Z$, with generator

$$s_n = \sum (-1)^l t_0 h_{t_0} h_{t_1} \ldots h_{t_l},$$

summation over all finite sequences $(t_0, \ldots, t_l)$ with $\sum_\nu t_\nu = n$ (see [**3**, 5.12]). The basis $\{h_\alpha | \alpha \vdash n\}$ of monomials in the h_i, indexed by partitions α of n, has a dual basis in S^*, where the dual is defined by $(S^*)_{2n} = \mathrm{Hom}\,(S_{2n}, \mathbf{Z})$ as a group. Let a_i denote the member of this dual basis corresponding to $(h_1)^i$. Then the algebra map $S \to S^*$ determined by $h_i \mapsto a_i$ is in fact an isomorphism of Hopf algebras [**4**]. Our main aim is to describe the algebra structure of some Hopf algebras derived from S.

Let $|h_1, \ldots, h_{k-1}|$ be the ideal in S generated by $\{h_1, \ldots, h_{k-1}\}$. It is a Hopf ideal, so $S/|h_1, \ldots, h_{k-1}|$ becomes a Hopf algebra (which is of course isomorphic as an algebra to $\mathbf{Z}[h_k, h_{k+1}, \ldots])$.

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THEOREM A. *As an algebra, $(S/|h_1, \ldots, h_{k-1}|)^*$ is also a polynomial algebra on generators, one in each even dimension $2k, 2k + 2, \ldots$.*

The subalgebra $\mathbf{Z}[h_1, \ldots, h_l]$ of S is clearly a sub Hopf algebra. By abuse of notation we shall also denote by $a_i \in \mathbf{Z}[h_1, \ldots, h_l]^*$ the "dual" of $(h_1)^i$ in the basis dual to the monomials in the h_i, and we shorten $a_i \otimes 1$ to a_i in $\mathbf{Z}[h_1, \ldots, h_l]^* \otimes \mathbf{Z}/p$. For given $l \geqq 1$ and a prime p, let $I = I(l, p)$ be the set of those positive integers which can be written in at least one way as ip^f for some $f \geqq 0$ and some i satisfying $1 \leqq i \leqq l$.

THEOREM B. *As an algebra*

$$\mathbf{Z}[h_1, \ldots, h_l]^* \otimes \mathbf{Z}/p = \frac{\mathbf{Z}/p[a_n | \, n \in I]}{\langle r_n | \, n \in I \text{ and } n > l \rangle},$$

where the denominator on the right is the ideal generated by certain elements r_n, exactly one in each dimension $2n$ as indicated.

In Section 2 we give the proof of B. For $l > 2$ the relations r_n seem quite complicated. In particular, the set $\{a_n | n \in I\}$ is seldom a set of Borel generators; that is, usually $r_n \neq (a_{n/p^c})^{p^c}$ for any c. At the end of Section 2 we record some computations. For all p, the a_n are Borel generators when $l = 1$ and $l = 2$, but this fails for $p = 2$ when $l = 3$.

In Section 3 we show how A follows from B. This is a straightforward generalization of Adams' argument [**1**, pp. 258–9] which is the case $k = 2$ in mild topological disguise.

In Section 4 we show how to construct specific polynomial generators for the algebra in A. The original motivation for this work was the work of Kochman [**5**] who gives formulae over $\mathbf{Z}/p$ and over $\mathbf{Z}_{(p)}$ in the case $k = 2$.

In Section 5 we discuss the interpretations of the above Hopf algebras in topology and in the representation theory of the symmetric group.

2. Proof of theorem B. We denote as usual by Q_{2n} (resp. P_{2n}) the indecomposable quotient (resp. primitive subgroup) of an algebra (resp. coalgebra) in dimension $2n$. Fix l and p and let

$$A = \mathbf{Z}[h_1, \ldots, h_l]^* \otimes \mathbf{Z}/p.$$

Since $Q_{2n}(A)^* \cong P_{2n}(A^*)$, we obtain, for all $n \geqq 1$

$$Q_{2n}(A) \cong P_{2n}(\mathbf{Z}/p[h_1, \ldots, h_l]) \hookrightarrow P_{2n}(S \otimes \mathbf{Z}/p)$$
$$\cong Q_{2n}(S \otimes \mathbf{Z}/p) \cong \mathbf{Z}/p.$$

The second last isomorphism follows from the self duality of S. A generator $\bar{s}_n$ of $P_{2n}(S \otimes \mathbf{Z}/p)$ is the mod p reduction of s_n, which is non-zero in $S \otimes \mathbf{Z}/p$ since $(h_1)^n$ occurs with coefficient $(-1)^{n+1}$ in s_n. We show $Q_{2n}(A) = 0$ for $n \notin I$ by showing that $\bar{s}_n \notin \mathbf{Z}/p[h_1, \ldots, h_l]$ for these

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values of n. This follows by writing $n = qp^f$ with q prime to p. Then $q > l$ since $n \notin I$, and the coefficient of $(h_q)^{p^f}$ in s_n is $(-1)^{p+1}q$, so $s_n \notin \mathbf{Z}[h_1, \ldots, h_l] + p \cdot S$, as required.

The dual of $\mathbf{Z}[h_1, \ldots, h_l] \hookrightarrow S$ is an epimorphism $S^* \rightarrow \mathbf{Z}[h_1, \ldots, h_l]^*$, and the elements a_n generate S^* as an algebra. Combining this with the previous paragraph, we see that A is generated as an algebra by $\{a_n | n \in I\}$. What we must now prove is that the algebra map

$$\begin{aligned} &\mathbf{Z}/p[x_n | n \in I] \rightarrow A \\ &x_n \mapsto a_n \end{aligned}$$

has kernel which can be generated by a set of elements, exactly one in each dimension $\{n | n \in I \text{ and } n > l\}$.

For this, it suffices to find an epimorphism

$$\mathbf{Z}/p[y_n | n \in I] \rightarrow A$$

of algebras whose kernel is generated as above, since one can easily complete the commutative diagram

$$\begin{tikzcd} \mathbf{Z}[x_n \mid n \in I] \arrow[dr] \arrow[dd, dashed, "\theta", "\cong"'] & \\ & A \\ \mathbf{Z}[y_n \mid n \in I] \arrow[ur] & \end{tikzcd}$$

with an algebra isomorphism θ (not necessarily unique).

But A is a commutative algebra of finite type admitting a Hopf algebra structure, so by Borel's theorem [**2**], we may find a surjective algebra map

$$\mathbf{Z}/p[y_\alpha | \alpha \in \Gamma] \rightarrow A$$

with kernel generated by $\{(y_\alpha)^{p^{\mu_\alpha}} | \alpha \in \Gamma\}$ for suitable μ_α with $1 \leqq \mu_\alpha \leqq \infty$. But from our knowledge of $Q(A)$, we can take $\Gamma = I$, with $\dim y_n = 2n$ for each $n \in I$. (In fact it is easy to see $a^{p^\mu} = 0$ for all $a \in A$ for any μ with $p^\mu > l$, so each $\mu_n \leqq \log_p l$, but we don't need this.) It remains only to show that the elements $\{(y_n)^{p^{\mu_n}} | n \in I\}$ lie in all dimensions $\{2n | n \in I \text{ and } n > l\}$ with only one in each dimension. This is combinatorially obvious to anyone who has worked with tensor products of truncated polynomial algebras (note that A has the same rank in each dimension as a polynomial algebra with one generator in each dimension 2, 4, 6, $\ldots$, $2l$, since it is the dual of $\mathbf{Z}/p[h_1, \ldots, h_l]$). However, we shall state the relevant combinatorial result below and give a formal proof.

LEMMA. *Let $0 < d_1 < d_2 < \ldots$ be an infinite sequence of positive integers. Suppose A is a graded $\mathbf{Z}/p$ algebra satisfying:*

a) rank A_{2n} = *# of partitions of* n *with all parts* $\in \{d_1, d_2, \ldots, d_l\}$;

$$(b)\ A \simeq \bigotimes_{i=1}^{\infty} \left[\frac{\mathbf{Z}/p[y_i]}{(y_i)^{f_i} = 0}\right]$$

with $\dim y_i = 2d_i$ *for some* f_i *which are positive integers or* ∞.

Then the numbers $\{f_i d_i | i \geqq 1, f_i < \infty\}$ *take exactly the values* $\{d_j | j > l\}$, *each exactly once.*

Proof. Assume the result fails, let $e_1 \leqq e_2 \leqq \ldots$ be the set $\{d_i f_i | i \geqq 1,\ f_i < \infty\}$ written in non-decreasing order, and pick the smallest t with $e_t \neq d_{l+t}$.

Case i). If $e_t < d_{l+t}$, we shall derive a contradiction to a) by showing that

$$1 + \text{rank } A_{2e_t} = \#\text{ partitions of } e_t \text{ with parts } \in \{d_1, \ldots, d_l\}.$$

To this end, for each i, $1 \leqq i \leqq l$ define (inductively on j) finite sequences y_{ij}, d_{ij} and f_{ij} by:

$$y_{i0} = y_i, \quad d_{i0} = d_i, \quad f_{i0} = f_i.$$

Given y_{ij}, d_{ij} and f_{ij} such that for some k,

$$y_{ij} = y_k, \quad d_{ij} = d_k \quad \text{and} \quad f_{ij} = f_k,$$

then either

(I) $f_{ij} d_{ij} \geqq e_t$:

Here define $s_i = j$, and terminate the sequences. By the inductive hypothesis, this case will occur with equality exactly once. Denote the corresponding value of i as i_0.

Or

(II) $f_{ij} d_{ij} < e_t$.

In case (II), choose the unique $k' > l$ such that $f_{ij} d_{ij} = d_{k'}$ (using the inductive hypothesis). Define $y_{i,j+1} = y_{k'}$, $d_{i,j+1} = d_{k'}$ and $f_{i,j+1} = f_{k'}$. Then

$$\{y_{ij} | 1 \leqq i \leqq l, 0 \leqq j \leqq s_i\} = \{y_k | 1 \leqq k \leqq l + t - 1\}$$

and $y_{ij} \neq y_{i'j'}$ if $(i, j) \neq (i', j')$. Then A_{2e_t} has basis

$$\left\{ \prod_{\substack{1 \leqq i \leqq l \\ 0 \leqq i \leqq s_i}} y_{ij}^{a_{ij}} \,\middle|\, 0 \leqq a_{ij} < f_{ij}; \sum_{i,j} a_{ij} d_{ij} = e_t \right\}.$$

The last sum can be written

$$e_t = \sum_{i=1}^{l} d_i \left[a_{i0} + f_{i0} a_{i1} + f_{i1} f_{i0} a_{i2} + \ldots + \left(\prod_{j=0}^{s_i - 1} f_{ij} \right) a_{is_i} \right].$$

The number multiplied into d_i in the sum is the unique representation of an integer a_i satisfying

$$0 \leqq a_i < \prod_{j=0}^{s_i} f_{ij}$$

in which the base sequence is $f_{i0}, f_{i0}f_{i1}, f_{i0}f_{i1}f_{i2}, \ldots$ and digit range $0 \leqq a_{ij} < f_{ij}$. Every representation of e_t as $\sum_{i=1}^{l} d_i a_i$ will occur exactly once, except the representation

$$e_t = d_{t0}\left(\prod_{j=0}^{s_{t0}} f_{t0,j}\right),$$

which fails to occur. This completes Case i).

Case ii). If $e_t > d_{l+t}$, the contradiction is that $A_{2d_{l+t}}$ has rank $1 + \#\text{part}^{ns}$ d_{l+t} with parts $\in \{d_1, \ldots, d_l\}$. Define y_{ij}, d_{ij}, f_{ij} inductively on j for $1 \leqq i \leqq l$ exactly as in Case i), except that we note that we cannot have $d_j f_k = d_{l+t}$, and we terminate at that j for which $d_{ij} f_{ij} > d_{l+t}$. The basis for $A_{2d_{l+t}}$ leading to the above equation is

$$\{y_{l+t}\} \cup \left\{ \prod_{\substack{1 \leqq i \leqq l \\ 0 \leqq j \leqq s_i}} (y_{ij})^{a_{ij}} \,\middle|\, 0 \leqq a_{ij} < f_{ij}, \sum a_{ij} d_{ij} = d_{l+t} \right\},$$

the last summation in this case giving all partitions of d_{l+t} with parts $\in \{d_1, \ldots, d_l\}$ by the same digital representation of the "coefficient" a_i of d_i as in case i). This completes the proofs of the lemma and of Theorem B.

For small l, the relations r_n in Theorem B can be calculated with a bit of elbow grease.

When $l = 1$, it is well known that

$$\mathbf{Z}[h_1]^* = \frac{\mathbf{Z}[a_1, a_2, \ldots]}{\left\langle a_i a_j = \dfrac{i+j!}{i!j!} a_{i+j} \right\rangle} \quad \text{and}$$

$$\mathbf{Z}[h_1]^* \otimes \mathbf{Z}/p = \frac{\mathbf{Z}[a_1, a_p, a_{p^2}, \ldots]}{(a_{p^k})^p = 0} \quad \text{for all } p;$$

see [**1**, 2.1].

When $l = 2$, one can compute

$$\mathbf{Z}[h_1, h_2]^* \otimes \mathbf{Z}/2 = \frac{\mathbf{Z}/2[a_1, a_2, a_4, a_8, \ldots]}{\langle (a_{2^k})^4 = 0 \rangle} \quad \text{and}$$

$$\mathbf{Z}[h_1, h_2]^* \otimes \mathbf{Z}/p = \frac{\mathbf{Z}/p[a_1, a_2, a_p, a_{2p}, a_{p^2}, a_{2p^2}, \ldots]}{(a_n)^p = 0 \quad \text{for} \quad n = p^k \quad \text{or} \quad 2p^k} \quad \text{for } p > 2,$$

so that in this case the a_n are Borel generators.

Finally, when $l = 3$ and $p = 2$, one has

$$\mathbf{Z}[h_1, h_2, h_3]^* \otimes \mathbf{Z}/2 = \frac{\mathbf{Z}/2[a_{2^k}; a_{2^k.3} | k \geqq 0]}{(a_{2^k})^4 = 0;\ (a_{2^k}a_{2^{k+1}})^2 = (a_{2^k.3})^2},$$

so in this case, the a_n are not Borel generators.

The following is exactly what is needed for the proof of Theorem A in the next section.

COROLLARY. *The kernel of*

$$S^* \otimes \mathbf{Z}/p \xrightarrow{\varphi} \mathbf{Z}[h_1, \dots, h_l]^* \otimes \mathbf{Z}/p$$

can be generated by a set of elements with exactly one in each dimension $2n$ *for each* $n > l$.

Proof. For clarity, let us momentarily abuse notation by calling in $S^* \otimes \mathbf{Z}/p$ by a_n' the element previously called a_n. Then the given map is

$$\mathbf{Z}/p[a_1', a_2', \dots] \to \frac{\mathbf{Z}/p[a_n | n \in I(l, p)]}{\langle r_n : n \in I(l, p), n > l \rangle}$$
$$a_n' \mapsto a_n.$$

A suitable set of generators for the kernel is then

$$\{r_n' | n \in I(l, p); n > l\} \cup \{a_n' - f_n(a_1', a_2', \dots) | n \notin I(l, p)\}$$

where, if $r_n = g_n(a_1, a_2, \dots)$ for polynomial g_n, we define r_n' to be $g_n(a_1', a_2', \dots)$; and where the polynomial f_n is chosen so that $a_n = f_n(a_1, a_2, \dots)$ for $n \notin I(l, p)$.

3. Proof of theorem A. As indicated in the introduction, we need only make a simple algebraic generalization of Adams' argument [**1**] to deduce A from the previous corollary. The definitions below (particularly α) are inspired by the corresponding diagram in [**1**], whose properties are noted by Adams to follow trivially at the topological level.

Construct a diagram

$$\begin{tikzcd} \dfrac{S}{|h_1, \dots, h_{k-1}|} \otimes_Z \mathbf{Z}[h_1, \dots, h_{k-1}] & S \arrow[l, "\alpha"'] \\ \mathbf{Z} \otimes_{\mathbf{Z}} \mathbf{Z}[h_1, \dots, h_{k-1}] \arrow[u, "\delta \otimes 1"] & \mathbf{Z}[h_1, \dots, h_{k-1}] \arrow[l, "\gamma"'] \arrow[u, "\beta"'] \end{tikzcd}$$

by requiring all maps to be maps of graded algebras and setting

$$\delta(1) = 1; \gamma(x) = 1 \otimes x; \beta(h_i) = h_i;$$

and

$$\alpha(h_i) = \begin{cases} 1 \otimes h_i & \text{for } 1 \leqq i \leqq k-1 \\ \sum_{t=0}^{k-1} h_{i-t} \otimes h_t & \text{for } i \geqq k. \end{cases}$$

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LEMMA. i) *The diagram is commutative;*

ii) α *and* γ *are bijective;*

iii) α *is a map of comodules over the coalgebra*

$$S/|h_1, \ldots, h_{k-1}|.$$

Proof. i) Both maps are algebra maps, and both send h_i to $1 \otimes h_i$ for $1 \leqq i \leqq k - 1$.

ii) For γ this is obvious. Note that, as an algebra, the codomain of α is isomorphic to $\mathbf{Z}[x_1, x_2, \ldots]$ under

$$x_i \mapsto \begin{cases} 1 \otimes h_i & \text{for } i < k \\ h_i \otimes 1 & \text{for } i \geqq k. \end{cases}$$

Via this identification

$$\alpha(h_i) = \begin{cases} x_i, & \text{for } i < k \\ x_i + \text{products}, & \text{for } i \geqq k, \end{cases}$$

so α is an algebra isomorphism.

iii) We must show the diagram below commutes, where Δ' and Δ'' are induced by the coproduct $\Delta : S \to S \otimes S$.

$$\begin{tikzcd} & S \arrow[r, "\alpha"] \arrow[d, "\Delta'"'] & S/|h_1, \ldots, h_{k-1}| \otimes \mathbf{Z}[h_1, \ldots, h_{k-1}] \arrow[d, "\Delta'' \otimes 1"] \\ S/|h_1, \ldots, h_{k-1}| \otimes & S \arrow[r, "1 \otimes \alpha"] & S/|h_1, \ldots, h_{k-1}| \otimes S/|h_1, \ldots, h_{k-1}| \otimes \mathbf{Z}[h_1, \ldots, h_{k-1}] \end{tikzcd}$$

But all maps are algebra maps, so this follows from the computation of both composites on h_i:

$$h_i \mapsto \begin{cases} 1 \otimes 1 \otimes h_i & \text{for } i < k \\ \displaystyle\sum_{t=0}^{k-1} (h_{i-t} \otimes 1 \otimes h_t + 1 \otimes h_{i-t} \otimes h_t) & \\ \displaystyle + \sum \left\{ h_a \otimes h_b \otimes h_c \,\middle|\, \begin{matrix} a \geqq k; b \geqq k; 0 \leqq c < k \\ a + b + c = i \end{matrix} \right\} & \text{for } i \geqq k. \end{cases}$$

Now fix a prime p, and define Hopf algebras as follows:

Definition. $A = \mathbf{Z}[h_1, \ldots, h_{k-1}]^* \otimes_{\mathbf{Z}} \mathbf{Z}/p$ (i.e., let $l = k - 1$ in Section 2)

$$B = S^* \otimes_{\mathbf{Z}} \mathbf{Z}/p$$

$$C = \left(\frac{S}{|h_1 \ldots h_{k-1}|}\right)^* \otimes_{\mathbf{Z}} \mathbf{Z}/p.$$

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Dualizing the previous diagram, and tensoring with $\mathbf{Z}/p$ we obtain a commutative diagram:

$$\begin{tikzcd} C \otimes_{\mathbf{Z}/p} A \arrow[r, "\alpha'", "\cong"'] \arrow[d, "\psi \otimes 1"'] & B \arrow[d, "\varphi"] \\ \mathbf{Z}/p \otimes_{\mathbf{Z}/p} A \arrow[r, "\cong"] & A \end{tikzcd}$$

in which α' is a map of C-modules by iii) of the lemma;
φ is an algebra map since β is a coalgebra map; and

$$\psi_0 : C_0 \overset{\cong}{\rightarrow} \mathbf{Z}/p.$$

Let $\tilde{C} = \bigoplus_{n>0} C_{2n}$. Then, as $\mathbf{Z}/p$ modules,

$$\tilde{C} \otimes_C \mathbf{Z}/p \cong [\tilde{C} \otimes_C B] \otimes_B \mathbf{Z}/p$$
$$\cong [\tilde{C} \otimes_C (C \otimes_{\mathbf{Z}/p} A)] \otimes_B \mathbf{Z}/p$$

by the diagram

$$\cong [\tilde{C} \otimes_{\mathbf{Z}/p} A] \otimes_B \mathbf{Z}/p$$
$$\cong \operatorname{Ker} \varphi \otimes_B \mathbf{Z}/p,$$

by the diagram, since $\operatorname{Ker} \psi = \tilde{C}$.

But by the corollary at the end of Section 2, the ideal $\operatorname{Ker} \varphi$ is generated by elements g_n, one in each dimension $n > k - 1$. The elements $g_n \otimes 1$ then generate $\operatorname{Ker} \varphi \otimes_B \mathbf{Z}/p$ as a $\mathbf{Z}/p$-module, so $\tilde{C} \otimes_C \mathbf{Z}/p$ can be generated by one element in each dimension $2n$, for each $n \geqq k$. But

$$\tilde{C} \otimes_C \mathbf{Z}/p \cong Q(C) = Q\left[\left(\frac{S}{|h_1 \ldots h_{k-1}|}\right)^* \otimes \mathbf{Z}/p\right]$$
$$\cong Q\left[\left(\frac{S}{|h_1 \ldots h_{k-1}|}\right)^*\right] \otimes \mathbf{Z}/p.$$

Thus for

$$n \geqq k,\ Q_{2n}\left[\left(\frac{S}{|h_1 \ldots h_{k-1}|}\right)^*\right]$$

is a finitely generated abelian group G such that, for all p, $G \otimes \mathbf{Z}/p$ i $\mathbf{Z}/p$ or 0. Hence

$$Q_{2n}\left[\left(\frac{S}{|h_1 \ldots h_{k-1}|}\right)^*\right] \quad \text{is a cyclic group for each } n \geqq k.$$

But then if $\mathbf{Z}[x_k, x_{k+1}, \ldots]$ is polynomial ring with one generator in each even dimension $\geqq 2k$, the algebra map

$$\mathbf{Z}[x_k, x_{k+1}, \ldots] \xrightarrow{\theta} (S/|h_1 \ldots h_{k-1}|)^*$$

mapping each x_n to any element which projects to a generator of $Q_{2n}[(S/|h_1 \ldots h_{k-1}|)^*]$, is an epimorphism. But for all d, [domain $\theta]_{2d}$ and [codomain $\theta]_{2d}$ are both free abelian groups whose rank is the number of partitions of d all of whose parts are $\geqq k$, so θ is an isomorphism. This completes the proof of Theorem A.

We finish this section with a few comments about self duality. By Theorem A, $S/|h_1 \ldots h_{k-1}|$ is a Hopf algebra H such that

$$H \cong H^* \quad \text{as algebras.}$$

Thus, as coalgebras, $H^* \cong H^{**} \cong H$. But I do not know whether $H \cong H^*$ as Hopf algebras. It would be interesting to try to classify graded connected $\mathbf{Z}$-Hopf algebras of finite type which are self dual. Zelevinski [**8**, 2.2] has a classification analogous to the classical theorems over fields using positivity conditions as well as self duality. On the other hand, an apparently related condition, "bipolynomiality", allows for classification over $\mathbf{Z}_{(p)}$ [**6**], but counterexamples exist [**7**] to the obvious analogue over $\mathbf{Z}$.

4. Construction of generators for $(S/|h_1 \ldots h_{k-1}|)^*$**.** The algebra of the title may be identified with a subalgebra of S via

$$\Phi^{(k)}: (S/|h_1 \ldots h_{k-1}|)^* \overset{\lambda}{\hookrightarrow} S^* \underset{\cong}{\overset{\nu}{\rightarrow}} S.$$

Here λ is the dual of the projection, and ν is the inverse of the isomorphism of Section 1 (i.e., $a_n \overset{\nu}{\mapsto} h_n$). This can be made more specific.

PROPOSITION 4.1. $[\text{Im } \Phi^{(k)}]_{2n} = \bigcap_{i=1}^{k-1} \text{Ker } [\Delta_{i,n-i}: S_{2n} \to S_{2i} \otimes S_{2n-2i}]$.

Proof. To prove $\subseteq$, given $f \in S_{2n}{}^*$ with $f(x) = 0$ for all $x \in |h_1 \ldots h_{k-1}|$, and given i with $0 < i < k$, we must show

$$\Delta_{i,n-i}(\nu_n f) = 0.$$

It suffices to show

$$(g \otimes h)[\Delta_{i,n-i}(\nu_n f)] = 0 \quad \text{for all } g \in S_{2i}{}^* \text{ and } h \in S^*_{2n-2i}.$$

But the left side is $f[\nu_n(g \cdot h)]$, since the multiplication $\cdot$ in S^* comes from Δ in S. But

$$\nu_n(g \cdot h) = (\nu_i g)(\nu_{n-i} h) \quad \text{and} \quad \nu_i g \in S_{2i} \subset |h_1 \ldots h_{k-1}|$$

since $1 \leqq i \leqq k - 1$, so

$$(\nu g)(\nu h) \in |h_1 \ldots h_{k-1}|.$$

Hence $f[(\nu g)(\nu h)] = 0$, as required.

To prove $\supseteq$, we must show that if $x \in S_{2n}$ and $\Delta_{i,n-i}(x) = 0$ for $0 < i < k$ then

$$(\nu_n^{-1}x)(y) = 0 \quad \text{for all } y \in |h_1, \ldots, h_{k-1}|.$$

It suffices to take $y = h_i z$ where $0 < i < k$. But

$$(\nu_n^{-1}x)(h_i \cdot z) = [\Delta^*_{i,n-i}(\nu_n^{-1}x)](h_i \otimes z),$$

by the definition of Δ^*, the coproduct on S^*. But

$$\Delta^*_{i,n-i}(\nu_n^{-1}x) = (\nu_i^{-1} \otimes \nu_{n-i}^{-1})(\Delta_{i,n-i}(x))$$

since ν^{-1} is a map of coalgebras. Since $\Delta_{i,n-i}(x) = 0$, the proof is complete.

Note that we did not use the specific definition of ν. The embedding $\Phi^{(k)}$ depends on the choice of Hopf isomorphism ν (of which there are exactly four, since Hopf Aut $S \cong \mathbf{Z}/2 \oplus \mathbf{Z}/2$ [**4**]). But the subalgebra Im $\Phi^{(k)}$ is independent of ν. Note also that, by (4.1), Im $\Phi^{(k)}$ may be characterized as the unique maximal sub Hopf algebra of S of connectivity $2k - 1$.

Now we shall construct a family of elements $C_{d,n}$ which have quite probably occurred elsewhere in the context of symmetric function theory.

Definitions. Let $P = \prod_{n=0}^{\infty} S_{2n}$. Then $P \otimes \mathbf{C}$ is canonically isomorphic to $\prod_{n=0}^{\infty} (S_{2n} \otimes \mathbf{C})$. If $\omega \in \mathbf{C}$ is a primitive dth-root of 1 and $i \geqq 1$, let

$$\zeta(w, i) = \sum_{i=0}^{\infty} w^{ij} h_j \in P \otimes \mathbf{C},$$

and let

$$\text{(4.2)} \quad \zeta_d = \prod_{i-1}^{d} \zeta(w, i).$$

Then

$$\text{(4.3)} \quad \zeta_d = \sum_{\substack{n \geqq 0 \\ d|n}} C_{d,n} \in \prod_{\substack{n \geqq 0 \\ d|n}} S_{2n}$$

where

$$C_{d,dl} = \sum_{\alpha \vdash dl} \mu_\alpha h_\alpha$$

for the unique μ_α for which

$$\sigma_l(t_1^d, t_2^d, \ldots) = (-1)^l \sum_{\substack{s_1 \geqq s_2 \geqq \ldots \\ \Sigma s_i = ld}} \mu_{s_1, s_2, \ldots, s_k} \sigma_{s_1}(t_1, t_2, \ldots) \ldots \sigma_{s_k}(t_1, t_2, \ldots).$$

Here σ_s is the sth-elementary symmetric function, and the above identity takes place in the ring of symmetric polynomials in a sufficiently large

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set $\{t_1, t_2, \ldots\}$ of variables. The proof of (4.3) is a simple manipulation with the identity defining σ_s.

Applying the definition (4.2) directly, we get

(4.4) $C_{d,n} \equiv dh_n$ mod decomposables in S for all $d|n$.

Particular cases are

(4.5) $C_{1,n} = h_n$

and

(4.6) $C_{n,n} = s_n$, the generator of $P_{2n}(S)$.

The verification of (4.6) is immediate from (4.4) and the fact that $C_{n,n}$ is primitive. In fact,

(4.7) $\Delta(C_{d,dl}) = \sum_{s+t=l} C_{d,ds} \otimes C_{d,dt}$

where $C_{d,0} = h_0 = 1$. This last formula is obtained by taking the obvious extension of the coproduct $\Delta : S \rightarrow S \otimes S$ to

$$\Delta : P \rightarrow \prod_{i,j} (S_{2i} \otimes S_{2j}),$$

then to

$$\Delta : P \otimes \mathbf{C} \rightarrow \prod_{i,j} (S_{2i} \otimes \mathbf{C}) \otimes (S_{2j} \otimes \mathbf{C}).$$

Now

$$\Delta\zeta(w, i) = \sum_{i=0}^{\infty} w^{ij} \sum_{a+b=j} h_a \otimes h_b$$
$$= \sum_{a,b} w^{ia}h_a \otimes w^{ib}h_b = \zeta(w, i) \otimes \zeta(w, i).$$

But Δ preserves products, so by (4.2) we get

(4.8) $\Delta\zeta_d = \zeta_d \otimes \zeta_d$

which immediately yields (4.7). In particular, combining (4.1), (4.3) and (4.7), we obtain

(4.9) $C_{d,n} \in [\mathrm{Im}\ \Phi^k]_{2n}$ for all $d \geqq k$, $d|n$.

THEOREM 4.10. *A set of polynomial generators $\{h_{k,n}|n \geqq k\}$ for the subring* Im $\Phi^{(k)}$ *of S may be obtained as follows. Define*

$$g(k, n) = \text{g.c.d.}\ \{d|d \geqq k \text{ and } d|n\}.$$

Choose integers u_i and divisors d_i of n, $d_i \geqq k$ such that $\sum u_i d_i = g(k, n)$.

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Then set

$$h_{k,n} = \sum_i u_i C_{d_i,n}.$$

For example, when $k = 2$, we have

$$g(2, n) = \begin{cases} p & \text{if } n = p^u \text{ for a prime } p, \\ 1 & \text{otherwise.} \end{cases}$$

Then we may take $h_{2,p^u} = C_{p,p^u}$, but if n is divisible by two primes p_1 and p_2, take $h_{2,n} = u_1 C_{p_1,n} + u_2 C_{p_2,n}$ for any integers u_1 and u_2 with $u_1 p_1 + u_2 p_2 = 1$. Modulo using the Euclidean algorithm to obtain (u_1, u_2), this gives a simple formula for the $h_{2,n}$ in terms of the h_n. One can simplify this by replacing $\mathbf{Z}$-coefficients by $\mathbf{Z}_{(p)}$, the localization at a fixed prime p, and one rederives Kochman's formulae [**5**]. Note that the formula for $g(2, n)$ agrees with his result that

$$Q_{2n}(\text{Im } \Phi^{(2)}) \rightarrow Q_{2n}(S)$$

has cokernel of order p when $n = p^u$, and of order 1 otherwise.

It may be worth mentioning that the family $\{C_{d,n} \in S_{2n}\}$ is characterized by properties (4.4) and (4.7). The proof is not difficult; one proceeds by induction on n/d. The proof of Theorem 4.10 is combinatorial.

LEMMA 4.11. *For each partition* $\alpha = 1^{f_1}2^{f_2} \dots$ *of* n *with length* $l = \sum f_i$, *define*

$$\beta_\alpha = \frac{(l-1)!n}{\pi(f_i!)} \in \mathbf{Z}.$$

Then

$$s_n = \sum_{\alpha \vdash n} (-1)^{l+1} \beta_\alpha h_\alpha.$$

Proof. We have

$$s_n = \sum (-1)^l t_0 h_{t_0} h_{t_1} \dots h_{t_l},$$

summation being taken over all sequences $(t_0, \dots, t_l)$ with $\sum t_i = n$. We must show

$$l \cdot \sum \{t_0 | (t_0, \dots, t_l) \in \alpha\} = n \cdot \frac{l!}{\pi(f_i!)}.$$

But listing the $(t_0, \dots, t_l)$ which give the same partition α as a column of sequences, the sum of all entries in the array is the left side above if we first add along each column, and is the right side if we first add along each row.

LEMMA 4.12. *For each* $\alpha \vdash n$ *such that* $\alpha = 1^{f_1}2^{f_2} \dots$ *with* $f_i = 0$ *for* $1 \leqq i < k$, *we have that* $g(k, n)$ *divides* β_α.

Proof. Let p be any prime. We show that $p^\nu | g(k, n)$ implies $p^\nu | \beta_\alpha$. Let $n = p^m q$ with q prime to p. Since $p^\nu | d$ for all $d \geqq k$ with $d|n$, we must have $k > p^{\nu-1}q$ (otherwise $d = p^{\nu-1}q$ gives a contradiction). But no parts of α are smaller than k, so $n \geqq kl$. (Recall $l = \sum f_i$.) Hence

$$l \leqq \frac{n}{k} < \frac{p^m q}{p^{\nu-1}q} = p^{m-\nu+1}.$$

Write $l = p^j s$ with s prime to p. Then $j \leqq m - \nu$. Thus

$$\beta_\alpha = \frac{n}{l} \cdot \frac{l!}{\pi(f_i!)} = \frac{p^m q}{p^j s} \frac{l!}{\pi(f_i!)}$$

is an integer multiple of p^{m-j}/s. But s is prime to p and $\nu \leqq m - j$, so p^ν divides β_α.

PROPOSITION 4.13. *Given integers k and n such that* $0 \leqq k \leqq n \geqq 1$, *the following four integers are all equal:*

$$g'''(k, n) = \text{g.c.d. } \{\beta_\alpha | \alpha \vdash n \textit{ and all parts of } \alpha \textit{ are} \geqq k\}$$
$$g''(k, n) = \textit{order of } P_{2n}(S/|h_1 \dots h_{k-1}|)/\text{Im } P_{2n}(S)$$
$$g'(k, n) = \textit{order of } Q_{2n}(S)/Q_{2n}(\text{Im } \Phi^{(k)})$$
$$g(k, n) = \text{g.c.d. } \{d | d \geqq k \textit{ and } d \textit{ divides } n\}.$$

Proof. We show

$$g'''(k, n) | g''(k, n) = g'(k, n) | g(k, n) | g'''(k, n).$$

The first divisibility relation follows from the fact that $s_n/g'''(k, n)$ has integer coefficients modulo $|h_1, \dots, h_{k-1}|$ by 4.11, therefore gives a primitive in $S/|h_1 \dots h_{k-1}|$, and projects to an element of order $g'''(k, n)$ in the group defining $g''(k, n)$. The equality is deduced below from the usual connection between $P_{2m}(A^*)$ and $Q_{2m}(A)^*$.

$$g'(k, n) := \text{order} \left[\frac{Q_{2n}(S)}{Q_{2n}(\text{Im } \Phi^{(k)})}\right] = \text{order} \frac{Q_{2n}(S^*)}{Q_{2n}\left[\text{Im}\left(\frac{S}{|h_1 \dots h_{k-1}|}\right)^*\right]}$$

$$= \text{order} \frac{Q_{2n}(S^*)}{\text{Im } Q_{2n}\left[\left(\frac{S}{|h_1 \dots h_{k-1}|}\right)^*\right]}$$

$$= \text{order} \frac{Q_{2n}\left[\left(\frac{S}{|h_1 \dots h_{k-1}|}\right)^*\right]^*}{\text{Im } [Q_{2n}(S^*)]^*}$$

$$= \text{order} \left[\frac{P_{2n}\left(\frac{S}{|h_1 \dots h_{k-1}|}\right)}{\text{Im } P_{2n}(S)}\right] := g''(k, n),$$

as required, the second last equality following from the diagram

$$\begin{array}{ccccc}
Q_{2n}\left[\left(\dfrac{S}{|h_1 \ldots h_{k-1}|}\right)^*\right]^* & \xrightarrow{\cong} & P_{2n}\left[\left(\dfrac{S}{|h_1 \ldots h_{k-1}|}\right)^{**}\right] & \cong & P_{2n}\left(\dfrac{S}{|h_1 \ldots h_{k-1}|}\right) \\
\uparrow & & & & \uparrow \\
[Q_{2n}(S^*)]^* & \xrightarrow{\cong} & P_{2n}(S^{**}) & \cong & P_{2n}(S)
\end{array}$$

The relation $g'|g$ follows by considering the element

$$h_{k,n} = \sum_i u_i C_{d_i,n},$$

where $\sum u_i d_i = g(k, n)$ with $d_i \geqq k$ and $d_i|n$ for all i. By (4.4),

$$h_{k,n} \equiv g(k, n)h_n(\text{mod decomposables}).$$

But $h_{k,n}$ projects to 0 in the group defining $g'(k, n)$. This group is cyclic generated by the image of h_n, hence $g'(k, n)|g(k, n)$. The last relation is exactly Lemma 4.12.

Proof of Theorem 4.10. We must check that $h_{k,n}$ projects to a generator of $Q_{2n}(\text{Im } \Phi^{(k)})$. But the equality $g'(k, n) = g(k, n)$ of (4.13) means that we need only check that

$$h_{k,n} \equiv \pm g(k, n)h_n \text{ (modulo decomposables in } S).$$

This is immediate from (4.4), as already noted in the proof of (4.13).

5. Concrete interpretations in the representation theory of the symmetric group and topology. If we set $S_{2n} = R(\Sigma_n)$, the underlying group of the representation ring of the symmetric group, then the product and coproduct in the graded ring S are derived from inducing and restricting, respectively, with respect to the Young subgroup embeddings $\Sigma_i \times \Sigma_j \rightarrow \Sigma_{i+j}$. By (4.1), the subobject Im $\Phi^{(k)}$ becomes identified with those virtual representations of Σ_n which restrict to zero in $R(\Sigma_{i_1} \times \Sigma_{i_2} \times \ldots)$ for all Young subgroups (with $\sum i_\nu = n$) for which $i_\nu < k$ for at least one ν. Note in particular that

$$\text{Im } \Phi^1 \cong \bigoplus_{n=0}^{\infty} \text{Ker } [R(\Sigma_n) \rightarrow R(\Sigma_{n-1})].$$

Because Aut $S \cong \mathbf{Z}/2 \oplus \mathbf{Z}/2$, there are four choices for the interpretation of h_n; viz.

$$h_n = e_n \text{ or } f_n \text{ or } (-1)^n e_n \text{ or } (-1)^n f_n$$

where e_n and f_n are respectively the trivial and the sign representations of Σ_n on $\mathbf{C}^1$. One must choose one of these four and stick to it for all n.

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There are two alternate descriptions of Im $\Phi^{(k)}$. The primitive generator s_m becomes identified with that virtual representation whose character is $\pm n$ on the n-cycle, and zero on all other conjugacy classes of Σ_n. These are often useful because $S \otimes \mathbf{Q} = \mathbf{Q}[s_1, s_2, \ldots]$.

THEOREM 5.1. Im $\Phi^{(k)} = \mathbf{Q}[s_k, s_{k+1}, \ldots] \cap S$ (*the intersection is in* $S \otimes \mathbf{Q}$ *where we identify* $a \in S$ *with* $a \otimes 1 \in S \otimes \mathbf{Q}$).

Proof. Using (4.1) and that s_n is primitive, we easily see the inclusion $\supseteq$. Conversely, if we write an element of Im $\Phi^{(k)}$ as a polynomial $p(s_1, s_2, \ldots)$ with rational coefficients and suppose that the least i for which s_i occurs is less than k, one immediately gets a contradiction to (4.1) by applying $\Delta_{i,n-i}$: We have

$$p(s_1, s_2, \ldots) = s_i^f x + y$$

where $\Delta_{i,n-fi-i}(x) = 0$, $\Delta_{i,n-i}(y) = 0$, the exponent $f > 0$ and $x \neq 0$. But

$$\Delta(s_i^f x + y) = \Delta(s_i^f)\Delta(x) + \Delta(y),$$

so

$$\Delta_{i,n-i}(s_i^f x + y) = fs_i \otimes s_i^{f-1}x \neq 0.$$

The other description involves the canonical inner product $<\ ,\ >$ on $R(\Sigma_n)$.

THEOREM 5.2. Im $\Phi^{(k)} = |h_1, \ldots, h_{k-1}|^\perp$, *where by* $\perp$ *we denote the orthogonal complement in each dimension with respect to* $<\ ,\ >$ *above.*

Proof. Both inclusions are easy enough, but it suffices to prove $\supseteq$, since both sides have the same rank in each dimension, and the right side is a direct summand. So given $x \in S_{2n}$ with $\langle x, y\rangle = 0$ for all $y \in |h_1 \ldots h_{k-1}|$, we must show $\Delta_{i,n-i}(x) = 0$ for $0 < i < k$. Write

$$\Delta_{i,n-i}(x) = \sum_{\alpha \vdash i} h_\alpha \otimes x_\alpha$$

where h_α is the monomial in h_i and $x_\alpha \in S_{2n-2i}$. But if we let $\{\tilde{h}_\alpha\}$ be the dual basis to $\{h_\alpha\}$, then for all $z \in S_{2n-2i}$ and all $\beta \vdash i$,

$$0 = \langle x, \tilde{h}_\beta z\rangle = \langle\langle \Delta_{i,n-i}(x), \tilde{h}_\beta \otimes z\rangle\rangle = \sum_\alpha \langle h_\alpha, \tilde{h}_\beta\rangle\langle x_\alpha, z\rangle = \langle x_\beta, z\rangle.$$

Thus $x_\beta = 0$ for all β and $\Delta_{i,n-i}(x) = 0$, as required. (Recall that $\langle\langle\ ,\ \rangle\rangle$ is the inner product induced on $S \otimes S$.)

The "reciprocity" property used above follows in the present interpretation from Frobenius reciprocity applied to $\Sigma_i \times \Sigma_j \hookrightarrow \Sigma_{i+j}$. Note that we appear to have two possible subrings $|e_1 \ldots e_{k-1}|^\perp$ and $|f_1 \ldots f_{k-1}|^\perp$ depending on our choice for interpreting h_n. This is not so, as it would contradict (5.1); the primitives s_n depend only up to sign on the interpretation of the h_n. But the real explanation is simpler: in fact $|e_1 \ldots e_{k-1}| = |f_1 \ldots f_{k-1}|$ since both are the ideal generated by $R(\Sigma_1) \cup R(\Sigma_2) \cup \ldots \cup R(\Sigma_{k-1})$.

An interpretation of S well known to topologists is

$$S \cong H_*(BU) \cong H^*(BU).$$

Then

$$\mathrm{Im}\ \Phi^{(2)} \cong \mathrm{Im}\ [H_*(BSU) \to H_*(BU)]$$

(see [**1**]). But $BSU = BU(4, \ldots, \infty)$. A more subtle question is the structure of

$$\mathrm{Im}\ [H_*(BU(2k, \ldots, \infty)) \to H_*(BU)].$$

For $k > 2$, this is strictly smaller than $\mathrm{Im}\ \Phi^{(k)}$.

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