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AMITSUR COHOMOLOGY IN ADDITIVE FUNCTORS

BY
DAVID E. DOBBS([1])

§1. **Introduction and notation.** Let L/k be a Galois extension of fields with group G and let **A** be the category of k-algebras isomorphic to finite products of finite field subextensions of L/k. It is known that, with appropriately defined covers, **A** is dual to the underlying category of a Grothendieck topology T [5, Ch. I, Theorem 4.2] and that (strict) cohomological dimension of G may be characterized via T-Cech cohomology with coefficients in either additive (product-preserving) functors or sheaves [5, Ch. I, Theorems 4.3 and 5.9]. Indeed for any (torsion) additive functor $F:\mathbf{A}\to Ab$, there is a (torsion) T-sheaf S giving isomorphisms of Cech groups $\check{H}^n_T(k, F)\cong\check{H}^n_T(k, S)$ for all n.

In this paper we show, i.a., that analogous isomorphisms do not, in general, hold for Amitsur cohomology at corresponding "layers" of the Cech groups. More precisely, §3 presents examples of finite field subextensions K/k of L/k and additive functors F with Amitsur cohomology $H^1(K/k, F)$ failing to satisfy certain properties (e.g., torsion, monomorphic inflation) known to hold for Amitsur cohomology in sheaves. We begin in §2 by indicating some relationships (e.g., the adjoint functor pair in Theorem 2.2) between the categories **P** of T-presheaves (functors $\mathbf{A}\to$Ab) and **Ad** of additive T-presheaves.

§2. **Additive functors.** It is well-known [1, p. 14] that **P** is an abelian category in which a sequence $P_1\to P_2\to P_3$ is exact if, and only if, $P_1(A)\to P_2(A)\to P_3(A)$ is exact for every object A of **A**. We now prove a similar result for **Ad**.

PROPOSITION 2.1. **Ad** *is an abelian category with arbitrary products and coproducts. The canonical fully faithful functor* $j:\mathbf{Ad}\to\mathbf{P}$ *is exact.*

Proof. We shall establish that **Ad** is an abelian category by verifying the axioms in [6, p. 35]. It follows from the five lemma that the kernel and cokernel in **P** of any morphism in **Ad** are themselves in **Ad**. Since **P** and **Ad** have the same zero object, any kernel (resp. cokernel) in **P** of a morphism in **Ad** is thus a kernel (resp. cokernel) in **Ad**. Hence **Ad** has kernels and cokernels.

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To show any monomorphism $m:F_1 \to F_2$ in **Ad** is a kernel, let $i:F_3 \to F_1$ be ker(m) in **P**. By the preceding remark, i is also ker(m) in **Ad**, hence $F_3 \simeq 0$ and there is an exact sequence $0 \to F_1 \xrightarrow{m} F_2 \xrightarrow{n} F_4 \to 0$ in **P**. Then m is ker(n) in **P** and hence is also ker(n) in **Ad**. Dually, any epimorphism in **Ad** is a cokernel.

A simple diagram chase shows that the product (resp. coproduct) in **P** of an arbitrary family of objects of **Ad** is itself additive, and hence is the product (resp. coproduct) in **Ad**. Thus **Ad** is abelian with arbitrary products and coproducts. Since we have seen that kernels and cokernels in **Ad** may be computed in **P**, it also follows that j is exact, to complete the proof.

Further information about **Ad** will be obtained in Corollaries 2.3 and 2.4 with the aid of the following analogue of sheafification (Cf. [1, Ch. II, Theorem 1.1]).

THEOREM 2.2. *There exists an exact left adjoint* $*:\mathbf{P} \to \mathbf{Ad}$ *to* j.

Proof. We recall from [5, Ch. I, Proposition 3.5] that **A** has a skeletal full subcategory **B** such that the inclusion $i:\mathbf{B} \to \mathbf{A}$ is a categorical equivalence. Indeed there is an additive functor $\theta:\mathbf{A} \to \mathbf{B}$ such that θi is the identity on **B** and $i\theta$ is naturally equivalent to the identity on **A**. Since the objects of **B** are easily described (they are cartesian products of the form $K_1 \times \cdots \times K_m$, for certain finite field subextensions K_r of L/k), it is convenient to find an equivalent statement of the theorem involving **B**.

If **Q** is the category of Ab-valued additive functors defined on **B**, then inverse categorical equivalences $\phi:\mathbf{Ad} \to \mathbf{Q}$ and $\psi:\mathbf{Q} \to \mathbf{Ad}$ are clearly induced by composition with i and θ respectively. Hence the adjointness assertion of the theorem will follow from the existence of a left adjoint $\alpha:\mathbf{P} \to \mathbf{Q}$ to $j\psi$. In fact, with $* = \psi\alpha$, there will correspond to any objects P of **P** and F of **Ad** the following sequence of natural bijections of morphism classes:

$$({}^*P, F) = (\psi\alpha P, F) \simeq (\phi(\psi\alpha P), \phi F) = (\alpha P, \phi F) \simeq (P, (j\psi)(\phi F)) \simeq (P, jF).$$

For any object P of **P**, define αP on a typical object $K_1 \times \cdots \times K_m$ of **B** as $P(K_1) \oplus \cdots \oplus P(K_m)$. If $f:K_1 \times \cdots \times K_m \to L_1 \times \cdots \times L_n$ is a morphism in **B** and $1 \leq s \leq n$, there is a unique factoring

$$\begin{array}{ccc} K_1 \times \cdots \times K_m & \longrightarrow & K_{\pi(s)} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_s} \\ L_1 \times \cdots \times L_n & \longrightarrow & L_s \end{array}$$

where the horizontal maps are projections. Then $(\alpha P)f$ is defined by requiring commutativity of the diagrams

$$\begin{array}{ccc} (\alpha P)(K_1 \times \cdots \times K_m) & \longrightarrow & P(K_{\pi(s)}) \\ \downarrow{\scriptstyle (\alpha P)f} & & \downarrow{\scriptstyle P(f_s)} \\ (\alpha P)(L_1 \times \cdots \times L_n) & \longrightarrow & P(L_s) \end{array}$$

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in which the horizontal maps are projections. It is clear that αP is an additive functor.

To define a natural transformation $t_P:P\to(j\psi)(\alpha P)$ on an object A, let $\lambda_A:A\to \theta A=K_1\times\cdots\times K_m$ be the canonical isomorphism and $q_r:K_1\times\cdots\times K_m\to K_r$ the r-th projection. Then $(t_P)_A:P(A)\to(j\psi)(\alpha P)(A)=(\alpha P)(\theta A)=P(K_1)\oplus\cdots\oplus P(K_m)$ is defined by requiring that its composition with the projection into $P(K_r)$ be $P(q_r\lambda_A)$ for all r. To check naturality of t_P with respect to a morphism $f:A\to A'$, let $\theta A'=L_1\times\cdots\times L_n$ and consider commutative diagrams

$$\begin{array}{ccc} \theta A & \xrightarrow{\theta f} & \theta A' \\ {\scriptstyle q_{\pi(s)}}\downarrow & & \downarrow{\scriptstyle p_s} \\ K_{\pi(s)} & \xrightarrow{(\theta f)_s} & L_s \end{array}$$

as above. Naturality of t_P, i.e. checking commutativity of

$$\begin{array}{ccc} PA & \longrightarrow & (\alpha P)(\theta A) \\ \downarrow & & \downarrow \\ PA' & & (\alpha P)(\theta A') \\ \downarrow & & \downarrow \\ (\alpha P)(\theta A') & \longrightarrow & P(L_s) \end{array}$$

amounts to showing $P(p_s\lambda_{A'}f)=P((\theta f)_s q_{\pi(s)}\lambda_A)$, which follows since $(\theta f)_s q_{\pi(s)}=p_s(\theta f)$ and $\lambda_{A'}f=(\theta f)\lambda_A$.

The question of adjointness now reduces ([2, Proposition 1.14]) to the following universal mapping problem. Given objects P of **P** and Q of **Q** with a natural transformation $u:P\to(j\psi)Q$, we must show there is a unique natural transformation $v:\alpha P\to Q$, such that $u=(j\psi v)t_P$.

For any object $B=K_1\times\cdots\times K_m$ of **B**, it is clear that $(t_P)_B:PB\to(j\psi)(\alpha P)(B)=P(K_1)\oplus\cdots\oplus P(K_m)$ is given by applying P to the projection maps $q_r:B\to K_r$, since $\theta B=B$. Similarly, any natural transformation $v:\alpha P\to Q$ satisfies $j\psi v=v$. If v satisfies the required mapping property, it follows that $v_K=u_K$ for any field object K of **B**. Naturality of v with respect to the q_r gives a commutative diagram

$$\begin{array}{ccc} (\alpha P)(B) & \longrightarrow & \oplus(\alpha P)(K_r) \\ {\scriptstyle v_B}\downarrow & & \downarrow \\ QB & \longrightarrow & \oplus Q(K_r) \end{array}$$

whose top arrow is the identity, whose bottom arrow is an isomorphism (since Q is additive), and whose right vertical arrow is $\oplus v_{K_r}=\oplus u_{K_r}$. Hence v, if it exists, is uniquely determined.

Define v on an object $B=K_1\times\cdots\times K_r$ by requiring that $v_B:(\alpha P)(B)=\oplus P(K_r)\to QB$ be the composite of $\oplus u_{K_r}:\oplus P(K_r)\to\oplus(j\psi Q)(K_r)=\oplus Q(K_r)$ with the inverse of the isomorphism $QB\to\oplus Q(K_r)$. To show v is a natural transformation, consider

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any morphism $f: B=K_1\times\cdots\times K_m\to B'=L_1\times\cdots\times L_n$ in **B**; next observe that naturality of u implies that the juxtaposition of the diagram

$$\begin{array}{ccc} \oplus P(K_r)=(\alpha P)(B) & \longrightarrow & QB \\ \downarrow & & \downarrow \\ \oplus P(L_s)=(\alpha P)(B') & \longrightarrow & Q(B') \end{array}$$

whose commutativity is in question, with the commutative diagram

$$\begin{array}{ccc} QB & \xrightarrow{\cong} & \oplus Q(K_r) \\ \downarrow & & \downarrow \\ Q(B') & \xrightarrow{\cong} & \oplus Q(L_s) \end{array}$$

produces a diagram which is commutative along its outer edges. Finally, it is straightforward to check that naturality of u implies v satisfies the mapping property, and so the required adjointness holds.

It remains only to verify that $*=\psi\alpha$ is exact. If $P_1\to P_2\to P_3$ is exact in **P**, then exactness of j (Proposition 2.1) shows we need only prove $(*P_1)A\to(*P_2)A\to(*P_3)A$ is exact for any object A of **A**. Additivity of the $*P_v$ shows that it suffices to consider a field A, in which case the sequence in question becomes $P_1(\theta A)\to P_2(\theta A)\to P_3(\theta A)$ and is clearly exact. This completes the proof of the theorem.

COROLLARY 2.3. **Ad** *is a Grothendieck category* (*Definition in* [6, p. 111]) *with a generator.*

Proof. The functor * of the theorem exhibits **Ad** as a quotient category of **P** in the sense of [2, p. 121]. Since **P** has a family of generators [1, Ch. I, (2.6)], an application of [2, Proposition 5.39] shows the same is true of **Ad**. The existence of arbitrary coproducts in **Ad** now implies **Ad** has a generator [2, Proposition 5.33] and is well-powered [2, Proposition 5.35]. It remains only to establish that **Ad** inherits the property (AB5) from **P**, and this also follows by [2, Proposition 5.39], to complete the proof.

We now use dimension-shifting techniques to find another similarity in the behaviour of **P** and **Ad**. The following result justifies, to some extent, the concentration of one-dimensional cohomology in §3.

COROLLARY 2.4. *For every object F of* **Ad** *and every positive integer n, there exists an object Q of* **Ad** *and natural isomorphisms of Amitsur cohomology $H^n(B/A, F)\cong H^1(B/A, Q)$ for any morphism $A\to B$ in* **A**.

Proof. Since * is exact, [2, Proposition 6.3] shows that any injective object I of **Ad** is also injective in **P**. Then [1, Ch. I, Theorem 3.1] implies $H^m(B/A, I)=0$ for all morphisms $A\to B$ and all $m\geq 1$.

Since **Ad** is a Grothendieck category with a generator, any object F of **Ad** has an injective envelope, say I, in **Ad** ([7, Théorème 1.10.1], [6, Theorem 6.25]). Take $O \to F \to I \to Q \to 0$ exact in **Ad**; it is also exact in **P** by Proposition 2.1. The resulting long exact sequence of Amitsur cohomology [4, p. 48] gives, in view of the above remarks, isomorphisms $H^m(B/A, F) \cong H^{m-1}(B/A, Q)$ for all $m \geq 2$, from which the result is evident.

We close this section with an observation that will be helpful in constructing additive functors in §3.

REMARK 2.5. Let **C** be a full subcategory of **A** whose objects form a set of k-algebra isomorphism class representatives for the finite field subextensions of L/k. (For example, one could take the "chosen fields" in [5, p. 19] as the objects of **C**.) Then **Ad** is equivalent to $\mathbf{Ab}^{\mathbf{C}}$, the category of Ab-valued functors defined on **C**.

For a proof, let **Q** be as in the proof of Theorem 2.2. Since we have seen **Q** is equivalent to **Ad**, it is enough to construct an equivalence $\mu: Ab^{\mathbf{C}} \to \mathbf{Q}$. Assuming for convenience that **C** is obtained from the chosen fields, we can define μ by aping the construction of α in the proof of Theorem 2.2. For example, if F is an object of $Ab^{\mathbf{C}}$, put $(\mu F)(K_1 \times \cdots \times K_m) = F(K_1) \oplus \cdots \oplus F(K_m)$. It is straightforward to check that μ, so defined, is fully faithful and essentially surjective, and hence is an equivalence.

§3. **Cohomological examples.**

In this section, we construct examples of additive functors whose Amitsur cohomology groups fail to satisfy certain properties of cohomology in sheaves. The usual notation and terminology concerning Amitsur cohomology [3, §5] (as well as the notation of the Introduction) will be in effect.

We begin with a result that applies to sheaves.

THEOREM 3.1. *Let M be the normal closure (in L) of a finite field subextension K/k of L/k. Let n be a positive integer. Let F be an object of* **Ad** *such that the canonical homomorphism $F(K^\nu) \to H^0(K^\nu \otimes_k M/K^\nu, F)$ is surjective for $\nu = n$ and injective for $\nu = n+1$. Then $H^n(K/k, F)$ is a torsion group.*

Proof. For each $q \geq 0$, let F^q be the object of **Ad** given by $F^q(A) = H^q(A \otimes_k M/A, F)$. Then [4, Theorem 7.2] provides a first quadrant spectral sequence

$$H^p(K/k, F^q) \Rightarrow H^{p+q}(M/k, F)$$

which we shall use to prove $H^\lambda(K/k, F^0)$ is torsion for all $\lambda \geq 1$.

Let $q \geq 1$. Since [3, Lemma 1.7] shows $K^\mu \otimes_k M$ is a Galois extension of K^μ with group $\Pi = \text{gal}(M/k)$, [3, Theorem 5.4] identifies $F^q(K^\mu)$ with group cohomology $H^q(\Pi, F(K^\mu \otimes_k M))$, which is annihilated by the order of Π. Hence $H^p(K/k, F^q)$ is torsion for all p. As it also reduces to group cohomology, $H^p(M/k, F)$ is torsion for all $p \geq 1$.

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The above spectral sequence yields the exact sequence of low terms

$$0 \to H^1(K/k, F^0) \to H^1(M/k, F)$$

whose end terms are torsion, by the preceding remarks. Hence $H^1(K/k, F^0)$ is torsion, and the above claim is established in case $\lambda = 1$.

Let $\lambda > 1$. If the above spectral sequence is denoted by $E_2^{pq} \Rightarrow E^{p+q}$, then with the standard notation of spectral sequences [8, Chap. VI], there are exact sequences

$$E_j^{\lambda-j, j-1} \to E_j^{\lambda, 0} \to E_{j+1}^{\lambda, 0} \to 0 \qquad (\text{for } 2 \leq j \leq \lambda)$$

and $0 \to E_{\lambda+1}^{\lambda, 0} \to E^\lambda$. As we have seen, each $E_2^{\lambda-j, j-1}$ is torsion; hence $E_j^{\lambda-j, j-1}$ is torsion. Since E^λ is torsion, it is now easy to check $E_2^{\lambda, 0} = H^\lambda(K/k, F^0)$ is torsion, as claimed.

The natural transformation $F \to F^0$ induces a commutative diagram

$$\begin{array}{ccccc} F(K^n) & \to & F(K^{n+1}) & \to & F(K^{n+2}) \\ \downarrow & & \downarrow & & \downarrow \\ F^0(K^n) & \to & F^0(K^{n+1}) & \to & F^0(K^{n+2}) \end{array}$$

whose rows form part of the Amitsur complexes $C(K/k, F)$ and $C(K/k, F^0)$. By hypotheses, the left vertical map is surjective and the middle vertical is injective. A diagram chase shows the induced map $H^n(K/k, F) \to H^n(K/k, F^0)$ is injective, from which the conclusion is immediate.

The next result shows the conclusion of Theorem 3.1 is not true in general for additive functors.

THEOREM 3.2. *Let K/k be a cubic nonnormal field subextension of a separable closure L/k. (For example, let k be the rational field Q and let K be generated by a root of an irreducible polynomial $X^3 + aX + b \in Z[X]$ such that $(-4a^3 - 27b^2)^{1/2} \notin Q$.) Then there exists an object F of* **Ad** *such that $F(K) = 0$ and $H^1(K/k, F)$ is the additive group of $Q(\omega)$, where ω is a primitive cube root of unity.*

Proof. We can write $K = k(\alpha)$, where α is a root of some irreducible cubic polynomial $f \in k[X]$; let α_2, α_3 be the other roots of f in L. Then $M = K(\alpha_2)$, the normal closure of K/k inside L, is a Galois extension of k with group S_3, generated by the cycles $\sigma = (1\ 2\ 3)$ and $\tau = (1\ 2)$.

By Remark 2.5, an additive functor F can be described by its restriction to the full subcategory **C** of **A** whose objects are the "chosen" fields. Let $F(M) = Q(\omega)$, for a primitive cube root ω of unity, and let $F(N) = 0$ for any chosen field $N \neq M$. To define F on morphisms, i.e. on elements of S_3, we let $F\tau$ and $F\sigma$ be the automorphisms of $Q(\omega)$ given by multiplication by -1 and ω, respectively; then F on the other elements of S_3 is determined by functoriality. It is clear that F, so defined, is an object of $Ab^{\mathbf{C}}$, and hence can be viewed in **Ad**.

To compute $H^1(K/k, F)$, we first describe the tensor powers K^2 and K^3 as products of fields. The composition of isomorphisms $K^2 \cong k[X]/(f) \otimes_k K \cong K[X]/(f) \cong$

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$K[X]/(X-\alpha)\times K[X]/(X^2-(\alpha_2+\alpha_3)X+\alpha_2\alpha_3)\cong K\times M$ sends $p(\alpha)\otimes q(\alpha)$ to $(p(\alpha)q(\alpha), p(\alpha_2)q(\alpha))$, for p and q in $k[X]$. Similarly $K\otimes_k M\xrightarrow{\cong} M\times M\times M$ via $p(\alpha)\otimes m\mapsto (mp(\alpha), mp(\alpha_2), mp(\alpha_3))$. Then the composition of isomorphisms $K^3\cong K\otimes_k(K\times M)\cong K^2\times(K\otimes_k M)\cong K\times M\times M\times M\times M$ sends $p(\alpha)\otimes q(\alpha)\otimes r(\alpha)$ to $(p(\alpha)q(\alpha)r(\alpha), p(\alpha_2)q(\alpha)r(\alpha), p(\alpha)q(\alpha_2)r(\alpha), p(\alpha_2)q(\alpha_2)r(\alpha), p(\alpha_3)q(\alpha_2)r(\alpha))$. It is straightforward to check that, viewed as k-algebra homomorphisms from $K\times M$ to $K\times M\times M\times M\times M$, the face maps $\varepsilon_i:K^2\to K^3$ send $(p(\alpha)q(\alpha), p(\alpha_2)q(\alpha))$ to $(p(\alpha)q(\alpha), p(\alpha)q(\alpha), p(\alpha_2)q(\alpha), p(\alpha_2)q(\alpha), p(\alpha_2)q(\alpha))$, $(p(\alpha)q(\alpha), p(\alpha_2)q(\alpha), p(\alpha)q(\alpha), p(\alpha_2)q(\alpha), p(\alpha_3)q(\alpha))$, and $(p(\alpha)q(\alpha), p(\alpha_2)q(\alpha), p(\alpha)q(\alpha_2), p(\alpha_2)q(\alpha_2), p(\alpha_3)q(\alpha_2))$ respectively.

By means of the identifications $C^1(K/k, F)=F(K^2)=F(K\times M)=F(K)\oplus F(M)=\mathcal{Q}(\omega)$ and $C^2(K/k, F)=\mathcal{Q}(\omega)\oplus\mathcal{Q}(\omega)\oplus\mathcal{Q}(\omega)\oplus\mathcal{Q}(\omega)$, the maps $F(\varepsilon_i):F(K^2)\to F(K^3)$ therefore send $y\in\mathcal{Q}(\omega)$ to $(0, y, y, y)$, $(y, 0, y, F(\sigma^2\tau)y)$, and $(y, (F\tau)y, 0, (F\sigma)y)$ respectively. Finally, the equation $\omega^2+\omega+1=0$ implies the coboundary map $d^1:F(K^2)\to F(K^3)$, which is the alternating sum of the $F(\varepsilon_i)$, is identically zero. As $F(K)=0$, $H^1(K/k, F)$ is identified with the group $F(K^2)=\mathcal{Q}(\omega)$ of 1-cocycles, to complete the proof.

COROLLARY 3.3. *Let K/k be a cubic nonnormal subextension of a Galois field extension L/k. Let M be the normal closure of K/k (in L). Let F be an object of* **Ad** *such that $F(\alpha)=1$ for all $\alpha\in$ gal(M/k). Then $H^1(K/k, F)=0$.*

Proof. As in the preceding proof, we can compute the maps $F(\varepsilon_i):F(K^2)\to F(K^3)$. Consequently, if $i:K\to M$ is the inclusion and σ, τ are as above, then $d^1:F(K^2)\to F(K^3)$ is identified with the map $FK\oplus FM\to FK\oplus FM\oplus FM\oplus FM\oplus FM$ given by $d^1(a, b)=(a, (Fi)a, b-(Fi)a+(F\tau)b, (F\sigma i)a, b-(F\sigma^2\tau)b+(F\sigma)b)=(a, (Fi)a, 2b-(Fi)a, (F\sigma i)a, b)$. Since the group of 1-cocycles is clearly trivial, the proof is complete.

COROLLARY 3.4. *Let K/k be a cubic nonnormal field subextension of a separable closure L of a local (resp. global) field k. Let J be the unit functor (resp., the T-sheaf assigning to any finite separable field extension of k its idèle class group). Then $H^3(K/k, J)=0$.*

Proof. In the terminology of [9, p. 53], class field theory shows J is a formation sheaf, and hence [9, Theorem 5.3.4] supplies a reciprocity isomorphism $H^3(K/k, J)\cong H^1(K/k, z)$. As z, defined in [9, pp. 31–33], clearly satisfies the conditions of Corollary 3.3, the proof is complete.

REMARK. Let L be the separable closure of the field k of p-adic numbers, for some prime p. Since $G=$gal(L/k) has strict cohomological dimension 2 [11, Chap. II, Proposition 15], it follows from [5, Ch. I, Theorem 4.3] that the unit functor U

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satisfies $\check{H}^3_T(k, U)=0$. It would be interesting to use this fact to find a proof of the local case of Corollary 3.4 by analyzing inflation of three-dimensional Amitsur cohomology.

Morris [9, Theorem 3.4.1] has used his inflation-restriction result [10, Theorem 3.2] to deduce that the vanishing of Amitsur cohomology in a sheaf for a separable field extension is guaranteed by its vanishing for all finite Galois field extensions. The next result shows that such a cohomological triviality criterion is not true in general for additive functors.

COROLLARY 3.5. *In the situation of Theorem* 3.2, *let K_2/K_1 be any Galois extension of finite subextensions K_i of L/k. Then $H^1(K_2/K_1, F)=0$, although $H^1(K/k, F)\neq 0$.*

Proof. As [3, Theorem 5.4] shows $H^1(K_2/K_1, F)\cong H^1(\Pi, F(K_2))$, where $\Pi=\text{gal}(K_2/K_1)$, the construction of F shows we need only consider the case $K_2=M$. The possibilities for K_1 are then K, the conjugates $k(\alpha_2)$ and $k(\alpha_3)$ of K, the unique quadratic subextension Q of M/k, and k itself; corresponding values of Π are $\langle\sigma^2\tau\rangle$, $\langle\sigma\tau\rangle$, $\langle\tau\rangle$, $\langle\sigma\rangle$ and S_3, respectively. Since $y-\omega^2 y\neq 0$ for $0\neq y\in F(M)=\mathcal{Q}(\omega)$, the usual formula for cohomology of finite cyclic groups shows $H^1(\langle\sigma^2\tau\rangle, Q(w))=0$. The other three cases of cyclic Π are equally straightforward. To handle the case $\Pi=S_3$, apply the exact sequence of low terms of the Hochschild-Serre spectral sequence [8, p. 160, Théorème 1]

$$0\to H^1(S_3/\langle\sigma\rangle, \mathcal{Q}(\omega)^\sigma)\to H^1(S_3, \mathcal{Q}(\omega))\to H^1(\langle\sigma\rangle, \mathcal{Q}(\omega)).$$

As $\mathcal{Q}(\omega)^\sigma=0$, we conclude $H^1(S_3, \mathcal{Q}(\omega))=0$, to complete the proof.

By way of a positive criterion for cohomological triviality of additive functors, we have only the following proposition and example.

PROPOSITION 3.6. *Let $k\subset K\subset M\subset L$ be a chain of fields with K/k nonnormal, M/k finite Galois and L/k Galois. Assume $K=k(\alpha)$, where α is the only root in K of some irreducible polynomial over k. Let P be an object of* **Ad** *such that $P(N)=0$ for all fields N satisfying $K \underset{\neq}{\subset} N\subset M$. Then $H^0(K/k, P)=P(K)$ and $H^n(K/k, P)=0$ for all $n\geq 1$.*

Proof. As in the proof of Theorem 3.2, we may compute the tensor powers K^m by the Chinese remainder theorem. The upshot is that, for all $m\geq 2$, K^m is the product of K with finitely many fields each of which properly contains K and is contained in M. By the hypotheseses, the Amitsur cochains $C^n(K/k, P)$ can be identified with $P(K)$ and every $P(\varepsilon_i)=1$, from which the conclusion is evident.

EXAMPLE 3.7. Let k, K, and L be as in Theorem 3.2. Use Remark 2.5 to define an object P in **Ad** as follows. Arrange that K be a chosen field in [5, p. 19], set $P(K)=Z/2Z$ and let $P(N)=0$ for any chosen field $N\neq K$. Then P is a torsion additive functor, the Amitsur cochains $C^m(K/k, P)=Z/2Z$ for all m, and Proposition 3.6 applies to show $H^n(K/k, P)=0$ for all $n\geq 1$. Note this cannot be deduced from the cohomological triviality criterion of Morris referred to above, since P is not a T-sheaf.

We pause to record the following, a special case of an inflation result essentially proved by Morris in [10, Theorem 3.2].

PROPOSITION 3.8. *Let $A\to B\to C$ be morphisms in* **A** *and P an object of* **P** *such that $P(B\otimes_A B)\to P(C\otimes_A C)$ is injective and $P(B)\to H^0(C/B, P)$ is surjective. Then the inflation homomorphism $H^1(B/A, P)\to H^1(C/A, P)$ is injective.*

We next show that the conclusion of Proposition 3.8 is not true for arbitrary $A\to B\to C$ in **A** and torsion additive functor P.

EXAMPLE 3.9. Let K/k be a quadratic subextension of a separable closure L/k. Then there exists a torsion object F of **Ad** such that for all $n\geq 0$, $H^n(K/k, F)\simeq Z/2Z$ and $H^n(K_2/K_1, F)=0$ for any extension K_2/K_1, other than K/k, of finite subextensions K_i of L/k.

Proof. Use Remark 2.5 to construct an object F in **Ad** as follows. Set $F(k)=Z/2Z=F(K)$, let $F(N)=0$ for any chosen field $N\neq k$ or K, and adopt either of the possibilities for $F(k\to K)$. As $\Pi=\text{gal}(K/k)$ is cyclic, periodicity implies $H^{2n+1}(K/k, F)\simeq H^{2n+1}(\Pi, Z/2Z)\simeq H^1(\Pi, Z/2Z)\simeq \text{Hom}(\Pi, Z/2Z)\simeq Z/2Z$ for all $n\geq 0$. Since the norm map $Z/2Z\to Z/2Z$ is trivial, we have $H^{2n}(K/k, F)\simeq H^{2n}(\Pi, Z/2Z)\simeq (Z/2Z)^{\Pi}=Z/2Z$. The final assertion follows by observing $C(K_2/K_1, F)=0$.

Let P be an object of **P** and $P\to {}^*P$ the natural transformation given by the adjointness in Theorem 2.2. We close by suggesting the need to study the induced map $H^n(K/k, P)\to H^n(K/k, {}^*P)$, if one is to consider Amitsur cohomology for arbitrary presheaves.

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Rutgers University,
New Brunswick, New Jersey